Universidad de Burgos
Departamento de Física

(A)dS Drinfel'd doubles and quantum gravity with cosmological constant

Angel Ballesteros

F.J. Herranz, C. Meusburger, F. Musso, P. Naranjo

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Quantum groups and the Planck scale

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Quantization of gravity should imply the introduction of a **'quantum' space-time** in which time and/or space would exhibit a 'quantum' structure that would be governed by a **parameter related to the Planck scale**.

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- The **Hopf algebra structure** of the quantum symmetries generates **space-times** whose **noncommutativity** is governed by the deformation parameter and could account for Planck scale uncertainty relations between space and time coordinates.
- **Curved momentum spaces** arise in a natural way in these quantum Hopf algebras as a consequence of the **non-cocommutativity** of momenta (non-abelian addition law for momenta).

Quantum groups in (2+1) gravity

Quantum group symmetries in (3+1) gravity are introduced heuristically and the full coalgebra structure is not often invoked.

However, for **(2+1)-gravity** it was stated in [1] that the **perturbations of the vacuum state of a Chern-Simons quantum gravity theory with cosmological constant** Λ, **are invariant under transformations that close a quantum (Anti) de Sitter algebra**.

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- The **low energy regime/zero-curvature limit** was found to be the known **κ-Poincaré quantum algebra**.[2]
- The **κ-Poincaré quantum Casimirs are** (here $z = 1/\kappa$):

$$\mathcal{C}_z = 4\frac{\sinh^2(\frac{z}{2}P_0)}{z^2} - \mathbf{P}^2 \qquad \longrightarrow \text{deformed dispersion relation}$$

$$\mathcal{W}_z = -\frac{\sinh(zP_0)}{z} + (K_1P_2 - K_2P_1).$$

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This is consistent with the fact that, in **(2+1)-gravity**, the **classical limit of quantum groups (Poisson–Lie groups)** arise in a natural way:

- Poisson-Lie (PL) structures on the isometry groups of (2+1) spaces with constant curvature play a relevant role as **phase spaces when (2+1) gravity coupled to point particles is considered as a Chern-Simons gauge theory.**[3] [4] [5]

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- The **admissible classical r-matrices defining such Poisson-Lie groups** are such that their **symmetric component coincides with a tensorized Casimir element** (Fock–Rosly condition).
- The **corresponding quantum (Anti) de Sitter and Poincaré groups should be meaningful ones in a quantum gravity context.**

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For a given Lie algebra/group, there are **many possible quantum deformations** (for (2+1) (A)dS see [6]).

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It can be proven that:

All the classical r-matrices coming from a Drinfel'd double structure of the isometry group -(A)dS and Poincaré- fulfill the Fock-Rosly condition and are compatible with the CS formalism. Thus:

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- All the **possible DD structures for the de Sitter Lie algebra** $so(3,1)$ and the **Anti de Sitter** one $so(2,2)$ can be explicitly found. [7]
- **Two main candidates** for quantum deformations of the (A)dS symmetries that would be appropriate in a (2+1) setting are obtained. [8] [9]

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Plan of the talk

2. (A)dS algebras as Drinfel'd Doubles

Drinfel'd doubles

A $2d$-dimensional Lie algebra $\mathfrak{a}$ has the structure of a (classical) Drinfel'd double if there exists a basis $\{X_1, \ldots, X_d, x^1, \ldots, x^d\}$ of $\mathfrak{a}$ in which the Lie bracket takes the form

$$[X_i, X_j] = c_{ij}^k X_k \qquad [x^i, x^j] = f_k^{ij} x^k \qquad [x^i, X_j] = c_{jk}^i x^k - f_j^{ik} X_k .$$

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- This implies that the two sets of generators $\{X_1,\ldots,X_d\}$ and $\{x^1,\ldots,x^d\}$ form **two Lie subalgebras** with structure constants c_{ij}^k and f_k^{ij}, respectively.
- Moreover, the expression for the crossed brackets $[x^i,X_j]$ implies that an **Ad-invariant symmetric bilinear form on $\mathfrak{a}$** is given by

$$\langle X_i,X_j\rangle=0 \qquad \langle x^i,x^j\rangle=0 \qquad \langle x^i,X_j\rangle=\delta_j^i \qquad \forall i,j.$$

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- And a **quadratic Casimir operator for $\mathfrak{a}$** is always given by

$$C = \tfrac{1}{2} \sum_i (x^i X_i + X_i x^i).$$

The DD – Fock/Rosly correspondence

Moreover, if $\mathfrak{a}$ is a DD Lie algebra, **its corresponding Lie group can be always endowed with a PL structure** generated by the **canonical classical r-matrix**

$$r = \sum_i x^i \otimes X_i$$

which is a (constant) solution of the Classical Yang-Baxter equation $[[r, r]] = 0$.

- The **skew-symmetric component** of the r-matrix is

$$r' = \tfrac{1}{2} \sum_i x^i \wedge X_i .$$

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- And the **symmetric component** Ω coincides with the **tensorized form of the canonical quadratic Casimir element in $\mathfrak{a}$**

$$\Omega = r - r' = \tfrac{1}{2} \sum_i (x^i \otimes X_i + X_i \otimes x^i),$$

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Therefore, in Lorentzian (2+1) gravity with nonvanishing Λ, any DD structure on $so(3,1)$ and $so(2,2)$ will provide an admissible r-matrix.

Lie algebras of (2+1) Lorentzian gravity

- The **Lie algebras** of the (A)dS and Poincaré groups can be written in a common **kinematical basis** in terms of generators J_a, P_a, $a = 0, 1, 2$.

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- In this basis the cosmological constant Λ and the signature of the metric arise as **parameters** in the Lie bracket: [10] [11]

$$[J_a, J_b] = \epsilon_{abc} J^c \qquad [J_a, P_b] = \epsilon_{abc} P^c \qquad [P_a, P_b] = \chi\, \epsilon_{abc} J^c$$

$$\text{where } \chi = \begin{cases} \Lambda & \text{for Euclidean signature;} \\ -\Lambda & \text{for Lorentzian signature.} \end{cases}$$

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- If $g = \text{diag}(\alpha, 1, 1)$ with $\alpha = \pm 1$ denotes the Euclidean / Minkowski metric and $\Lambda = \alpha\chi$, we have

$$\begin{array}{lll}
[J_0, J_1] = J_2, & [J_0, J_2] = -J_1, & [J_1, J_2] = \alpha\, J_0, \\
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The basis $\{J_a, P_a\}_{a=0,1,2}$ have a **direct geometrical interpretation**

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For all values of the parameters α, χ we have **two quadratic Casimir elements**

$$C_1 = \alpha\, P_0^2 + P_1^2 + P_2^2 + \chi\,(\alpha\, J_0^2 + J_1^2 + J_2^2),$$
$$C_2 = \tfrac{1}{2}\,(\alpha\,(J_0\,P_0 + P_0\,J_0) + J_1\,P_1 + P_1\,J_1 + J_2\,P_2 + P_2\,J_2)\,.$$

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If the duals of J_a and P_a are identified with, respectively, P_a and J_a, the **symmetric bilinear forms** associated to C_1 and C_2 are

$$\langle J_a, P_b\rangle_s = 0, \qquad \langle J_a, J_b\rangle_s = g_{ab}, \qquad \langle P_a, P_b\rangle_s = \chi\, g_{ab}.$$
$$\langle J_a, P_b\rangle_t = g_{ab}, \qquad \langle J_a, J_b\rangle_t = 0, \qquad \langle P_a, P_b\rangle_t = 0,$$

with $g = \text{diag}(\alpha, 1, 1)$.

$so(3,1)$ and $so(2,2)$ as Drinfel'd double Lie algebras

The **complete classification** of the six-dimensional DD Lie algebras is known [12] and is equivalent to the classification of three-dimensional real Lie bialgebras. [13]

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The **de Sitter Lie algebra** $so(3,1)$ admits four families of DD structures [14]

$(c^i_{jk}|f^{ij}_k|\eta):\quad [X_i,X_j]=c^k_{ij}X_k \qquad [x^i,x^j]=f^{ij}_k x^k \qquad [x^i,X_j]=c^i_{jk}x^k - f^{ik}_j X_k.$

- A: $(8|5.ii|\eta)\equiv(so(2,1)|\mathfrak{an}(2)''|\eta)$
- B: $(9|5|\eta)\equiv(so(3)|\mathfrak{an}(2)|\eta)$
- C: $(7_0|5.ii|\eta)\equiv(iso(2)|\mathfrak{an}(2)''|\eta)$
- D: $(7_\mu|7_{1/\mu}|\eta)$

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- D: $(7_\mu|7_{1/\mu}|\eta)$

While the **Anti de Sitter Lie algebra** $so(2,2)$ admits three:

- E: $(8|5.i|\eta) \equiv (so(2,1)|\mathfrak{an}(2)'|\eta)$
- F: $(6_0|5.iii|\eta) \equiv (iso(1,1)|\mathfrak{an}(2)'''|\eta)$
- G: $(6_a|6_{1/a}.i|\eta)$

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Summary: DD r-matrices for $so(3,1)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
A	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \eta J_1\wedge J_2 + \frac{1}{2}(-P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \frac{1}{2}(-P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{M}^{2+1}$
B	$(1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = -\eta J_1\wedge J_2 + \frac{1}{2}(P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{H}^3$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = \frac{1}{2}(P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{E}^3$
C	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1\wedge P_0 - J_0\wedge P_1 + J_2\wedge P_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1\wedge P_0 - J_0\wedge P_1 + J_2\wedge P_2)$	$\mathbf{M}^{2+1}$
D	$(1,1,1)$	$-\eta^2$	$\frac{\mu(\mu^2-1)}{(1+\mu^2)^2}\langle\ ,\ \rangle_t - \frac{2\mu^2}{\eta(1+\mu^2)^2}\langle\ ,\ \rangle_s$	$r'_{\rm D} = J_0\wedge P_1 - J_1\wedge P_0 + \frac{(1+\mu^2)}{2\mu} P_2\wedge J_2 + \frac{(\mu^2-1)}{2\eta\mu}(\eta^2 J_0\wedge J_1 - P_0\wedge P_1)$	$\mathbf{H}^3$
		0	None	$r'_{\rm D} = J_0\wedge P_1 - J_1\wedge P_0 + P_2\wedge J_2 \quad (\mu=1)$	$\mathbf{E}^3$

Summary: DD r-matrices for $so(3,1)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
A	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \eta J_1 \wedge J_2 + \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{M}^{2+1}$
B	$(1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = -\eta J_1 \wedge J_2 + \frac{1}{2}(P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{H}^3$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = \frac{1}{2}(P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	$\mathbf{E}^3$
C	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	$\mathbf{M}^{2+1}$
D	$(1,1,1)$	$-\eta^2$	$\frac{\mu(\mu^2-1)}{(1+\mu^2)^2}\langle\ ,\ \rangle_t - \frac{2\mu^2}{\eta(1+\mu^2)^2}\langle\ ,\ \rangle_s$	$r'_{\rm D} = J_0 \wedge P_1 - J_1 \wedge P_0 + \frac{(1+\mu^2)}{2\mu} P_2 \wedge J_2 + \frac{(\mu^2-1)}{2\eta\mu}(\eta^2 J_0 \wedge J_1 - P_0 \wedge P_1)$	$\mathbf{H}^3$
		0	None	$r'_{\rm D} = J_0 \wedge P_1 - J_1 \wedge P_0 + P_2 \wedge J_2 \quad (\mu = 1)$	$\mathbf{E}^3$

- The **κ-deformation is generated by** $J_0 \wedge P_1 - J_1 \wedge P_0$.

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		0	$\langle\ ,\ \rangle_t$	$r'_{\rm A} = \frac{1}{2}(-P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{M}^{2+1}$
B	$(1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = -\eta J_1\wedge J_2 + \frac{1}{2}(P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{H}^3$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm B} = \frac{1}{2}(P_0\wedge J_0 + P_1\wedge J_1 + P_2\wedge J_2)$	$\mathbf{E}^3$
C	$(-1,1,1)$	η^2	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1\wedge P_0 - J_0\wedge P_1 + J_2\wedge P_2)$	$\mathbf{dS}^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm C} = \frac{1}{2}(J_1\wedge P_0 - J_0\wedge P_1 + J_2\wedge P_2)$	$\mathbf{M}^{2+1}$
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		0	None	$r'_{\rm D} = J_0\wedge P_1 - J_1\wedge P_0 + P_2\wedge J_2 \quad (\mu=1)$	$\mathbf{E}^3$

- The **κ-deformation is generated by** $J_0\wedge P_1 - J_1\wedge P_0$.
- Case A–B corresponds to a deformation that has not been considered so far.

Summary: DD r-matrices for $so(2,2)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
E	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \eta J_0 \wedge J_2 + \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**AdS**$^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**M**$^{2+1}$
F	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm F} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	**AdS**$^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm F} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	**M**$^{2+1}$
G	$(-1,1,1)$	$-\eta^2$	$\frac{(1+\rho^2)}{2\rho^2}\langle\cdot,\cdot\rangle_t + \frac{(1-\rho^2)}{2\eta\rho^2}\langle\cdot,\cdot\rangle_s$	$r'_{\rm G} = \frac{(1+\rho^2)}{4}(J_1 \wedge P_0 - J_0 \wedge P_1) + \frac{\rho}{2} J_2 \wedge P_2 + \frac{(1-\rho^2)}{4\eta}(\eta^2 J_0 \wedge J_1 + P_0 \wedge P_1)$	**AdS**$^{2+1}$
		0	None	None	**M**$^{2+1}$

Summary: DD r-matrices for $so(2,2)$

#	Metric	Λ	Pairing	Skew-symmetric r-matrix	Space
E	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \eta J_0 \wedge J_2 + \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**AdS**$^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**M**$^{2+1}$
F	$(-1,1,1)$	$-\eta^2$	$\langle\ ,\ \rangle_t$	$r'_{\rm F} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	**AdS**$^{2+1}$
		0	$\langle\ ,\ \rangle_t$	$r'_{\rm F} = \frac{1}{2}(J_1 \wedge P_0 - J_0 \wedge P_1 + J_2 \wedge P_2)$	**M**$^{2+1}$
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		0	$\langle\ ,\ \rangle_t$	$r'_{\rm E} = \frac{1}{2}(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2)$	**M**$^{2+1}$
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Since $\Lambda = \pm\eta^2$, the flat (Poincaré) limit is obtained when $\eta \to 0$.

3. The twisted κ–AdS$_\omega$ algebra in (2+1) dimensions

The AdS$_\omega$ algebra in (2+1) dimensions

The **6D Lie algebra AdS$_\omega$** of the three relativistic spacetimes of constant curvature is given in terms of the generators $\{J, P_0, P_i, K_i\}$ as

$$
\begin{array}{lll}
[J, P_i] = \epsilon_{ij} P_j, & [J, K_i] = \epsilon_{ij} K_j, & [J, P_0] = 0, \\
[P_i, K_j] = -\delta_{ij} P_0, & [P_0, K_i] = -P_i, & [K_1, K_2] = -J, \\
[P_0, P_i] = \omega K_i, & [P_1, P_2] = -\omega J, &
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where $\omega = -\Lambda$, $i, j = 1, 2$ and $\epsilon_{12} = 1$.

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According to the sign of ω we find that these Lie brackets reproduce:

- The **AdS** algebra, $so(2,2)$, when $\omega = +1/R^2 > 0$.
- The **dS** algebra, $so(3,1)$, when $\omega = -1/R^2 < 0$.
- And the **Poincaré** algebra, $iso(2,1)$, when $\omega = 0$; it corresponds to the flat limit/contraction $R \to \infty$ such that $so(2,2) \to iso(2,1) \leftarrow so(3,1)$.

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The two **Casimir invariants** of AdS$_\omega$ are given by

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 + \omega(J^2 - \mathbf{K}^2) \qquad \mathcal{W} = -JP_0 + K_1 P_2 - K_2 P_1$$

$\mathcal{C}$ comes from the Killing–Cartan form, and $\mathcal{W}$ is the Pauli–Lubanski vector.

The kappa-AdS$_\omega$ quantum group: first order relations

Let us consider the following **classical r-matrix** on AdS$_\omega$

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2) + \vartheta J \wedge P_0$$

where $z = 1/\kappa = \ln q$.

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- The **first order deformation** of the coproduct is given by the cocommutator δ through the relation $\delta(Y_i) = [1 \otimes Y_i + Y_i \otimes 1, r]$:

$$\begin{aligned}
\delta(P_0) &= \delta(J) = 0,\\
\delta(P_1) &= z(P_1 \wedge P_0 - \omega K_2 \wedge J) + \vartheta(P_0 \wedge P_2 + \omega K_1 \wedge J),\\
\delta(P_2) &= z(P_2 \wedge P_0 + \omega K_1 \wedge J) - \vartheta(P_0 \wedge P_1 - \omega K_2 \wedge J),\\
\delta(K_1) &= z(K_1 \wedge P_0 + P_2 \wedge J) + \vartheta(P_0 \wedge K_2 - P_1 \wedge J),\\
\delta(K_2) &= z(K_2 \wedge P_0 - P_1 \wedge J) - \vartheta(P_0 \wedge K_1 + P_2 \wedge J).
\end{aligned}$$

Twisted κ-AdS$_\omega$ quantum group: first order relations

We denote by $\{\hat{\theta}, \hat{x}_\mu, \hat{\xi}_i\}$ the **dual non-commutative coordinates** of the generators $\{J, P_\mu, K_i\}$, respectively.

The dual of the cocommutator map gives the **first order quantum group**:

$$[\hat{x}_0, \hat{x}_1] = -z\hat{x}_1 - \vartheta\hat{x}_2, \qquad [\hat{x}_0, \hat{x}_2] = -z\hat{x}_2 + \vartheta\hat{x}_1, \qquad [\hat{x}_1, \hat{x}_2] = 0,$$

as well as

$$[\hat{\theta}, \hat{x}_i] = z\epsilon_{ij}\,\hat{\xi}_j + \vartheta\hat{\xi}_i \qquad [\hat{\theta}, \hat{\xi}_i] = -\omega\,(z\epsilon_{ij}\,\hat{x}_j + \vartheta\hat{x}_i)\,, \qquad [\hat{\theta}, \hat{x}_0] = 0,$$
$$[\hat{x}_0, \hat{\xi}_i] = -z\hat{\xi}_i - \vartheta\epsilon_{ij}\,\hat{\xi}_j, \qquad [\hat{\xi}_1, \hat{\xi}_2] = 0, \qquad [\hat{x}_i, \hat{\xi}_j] = 0, \qquad i, j = 1, 2.$$

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[15] P. Maslanka, J. Phys. A 26 (1993) L1251

[16] S. Majid, H. Ruegg, Phys. Lett. B 334 (1994) 348

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The well-known **κ-Minkowski spacetime** [15] [16] [17] is given by

$$[\hat{x}_0, \hat{x}_1] = -z\hat{x}_1, \qquad [\hat{x}_0, \hat{x}_2] = -z\hat{x}_2, \qquad [\hat{x}_1, \hat{x}_2] = 0, \qquad z = 1/\kappa.$$

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Twisted κ-Minkowski spacetime

The **'quantum' time and space translation parameters do not commute**:

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- These relations **do not depend on** ω, so the three first order (A)dS and Minkowskian non-commutative spacetimes coincide.
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- These relations **do not depend on ω**, so the three first order (A)dS and Minkowskian non-commutative spacetimes coincide.
- **Higher order corrections depending on ω** will appear when the full quantum (A)dS groups are considered.
- Other 'quantum' coordinates (rotation angle, velocities) are also non-commuting objects.

The κ-AdS$_\omega$ Poisson-Lie group

The **quantization of the PL group associated to the previous r matrix** will give rise to the **all–orders twisted quantum AdS$_\omega$ group.**

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Therefore, we have to compute:

- The group element

$$T = \exp(x_0 P_0) \exp(x_1 P_1) \exp(x_2 P_2) \exp(\xi_1 K_1) \exp(\xi_2 K_2) \exp(\theta J)$$

- Left and right invariant vector fields, Y^L and Y^R
- The Sklyanin bracket:

$$\{f, g\} = r^{ij}(Y_i^L f \, Y_j^L g - Y_i^R f \, Y_j^R g)$$

where

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2) + \vartheta J \wedge P_0$$

In this way we obtain the **fundamental Poisson–Lie brackets** between the six *commutative* group coordinates $\{\theta, x_\mu, \xi_i\}$.

Fundamental Poisson brackets I

Relations involving spacetime x_μ group coordinates:

$$\{x_0, x_1\} = -z\,\frac{\tanh\sqrt{\omega}x_1}{\sqrt{\omega}\cosh^2\sqrt{\omega}x_2} - \vartheta\cosh\sqrt{\omega}x_1\,\frac{\tanh\sqrt{\omega}x_2}{\sqrt{\omega}}$$

$$\{x_0, x_2\} = -z\,\frac{\tanh\sqrt{\omega}x_2}{\sqrt{\omega}} + \vartheta\,\frac{\sinh\sqrt{\omega}x_1}{\sqrt{\omega}}$$

$$\{x_1, x_2\} = 0$$

Fundamental Poisson brackets II

$$\{x_1,\xi_1\} = \frac{z}{\cosh\sqrt{\omega}x_2}\left(\frac{\cosh\sqrt{\omega}x_2}{\cosh\sqrt{\omega}x_1} - \frac{\cosh\xi_1}{\cosh\xi_2} + \tanh\sqrt{\omega}x_1 \sinh\sqrt{\omega}x_2\, A\right),$$

$$\{x_1,\xi_2\} = -z\cosh\xi_2\, B, \qquad \{x_2,\xi_2\} = z\left(\frac{\cosh\sqrt{\omega}x_1}{\cosh\sqrt{\omega}x_2}\cosh\xi_1 - \cosh\xi_2\right),$$

$$\{x_2,\xi_1\} = -zA, \qquad \{\xi_1,\xi_2\} = z\sqrt{\omega}\sinh\sqrt{\omega}x_1\left(C - \frac{\tanh\xi_2}{\cosh^2\sqrt{\omega}x_2}\right),$$

$$\{x_0,\theta\} = -\frac{zB}{\cosh\sqrt{\omega}x_1} + \frac{\vartheta}{2}\,\frac{\cosh\xi_1\left(\cosh 2\sqrt{\omega}x_1 - \cosh 2\xi_2\right)}{\cosh\sqrt{\omega}x_1\cosh\sqrt{\omega}x_2\cosh\xi_2},$$

$$\{x_0,\xi_1\} = z\left(\frac{\sinh\xi_2}{\cosh\sqrt{\omega}x_1}B - \frac{\sinh\xi_1\cosh\xi_2}{\cosh\sqrt{\omega}x_1\cosh\sqrt{\omega}x_2}\right) - \vartheta\,\frac{\cosh\sqrt{\omega}x_1\cosh\xi_1\tanh\xi_2}{\cosh\sqrt{\omega}x_2},$$

$$\{x_0,\xi_2\} = -zC + \vartheta\,\frac{\cosh\sqrt{\omega}x_1\sinh\xi_1}{\cosh\sqrt{\omega}x_2}, \qquad \{\theta,x_1\} = z\,\frac{\cosh\sqrt{\omega}x_1}{\cosh\xi_2}C + \vartheta\,\frac{\sinh\xi_1\cosh\xi_2}{\cosh\sqrt{\omega}x_2},$$

$$\{\theta,x_2\} = -z\,\frac{\cosh\sqrt{\omega}x_1\sinh\xi_1}{\cosh\sqrt{\omega}x_2\cosh\xi_2} + \vartheta\sinh\xi_2,$$

$$\{\theta,\xi_1\} = -z\sqrt{\omega}\left(\tanh\sqrt{\omega}x_2 + \tanh\sqrt{\omega}x_1\, B\right) - \vartheta\,\frac{\sqrt{\omega}\tanh\sqrt{\omega}x_1\cosh\xi_1\cosh\xi_2}{\cosh\sqrt{\omega}x_2},$$

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$$A = \frac{\sinh\sqrt{\omega}x_1\sinh\sqrt{\omega}x_2 + \cosh\sqrt{\omega}x_1\sinh\xi_1\tanh\xi_2}{\cosh\sqrt{\omega}x_2}, \qquad B = \frac{\sinh\sqrt{\omega}x_1\tanh\sqrt{\omega}x_2\cosh\xi_1 + \sinh\xi_1\sinh\xi_2}{\cosh\sqrt{\omega}x_2\cosh\xi_2},$$

$$C = \frac{\sinh\sqrt{\omega}x_1\tanh\sqrt{\omega}x_2\sinh\xi_1 + \cosh\xi_1\sinh\xi_2}{\cosh\sqrt{\omega}x_1\cosh\sqrt{\omega}x_2}.$$

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$$[\hat{x}_0, \hat{x}_1] = -z\,\frac{\tanh\sqrt{\omega}\hat{x}_1}{\sqrt{\omega}\cosh^2\sqrt{\omega}\hat{x}_2} - \vartheta\cosh\sqrt{\omega}\hat{x}_1\,\frac{\tanh\sqrt{\omega}\hat{x}_2}{\sqrt{\omega}}$$
$$= -z\left(\hat{x}_1 - \frac{1}{3}\omega\hat{x}_1^3 - \omega\hat{x}_1\hat{x}_2^2\right) - \vartheta\left(\hat{x}_2 + \frac{1}{2}\omega\hat{x}_1^2\hat{x}_2 - \frac{1}{3}\omega\hat{x}_2^3\right) + \mathcal{O}(\omega^2)$$
$$[\hat{x}_0, \hat{x}_2] = -z\,\frac{\tanh\sqrt{\omega}\hat{x}_2}{\sqrt{\omega}} + \vartheta\,\frac{\sinh\sqrt{\omega}\hat{x}_1}{\sqrt{\omega}}$$
$$= -z\left(\hat{x}_2 - \frac{1}{3}\omega\hat{x}_2^3\right) + \vartheta\left(\hat{x}_1 + \frac{1}{6}\omega\hat{x}_1^3\right) + \mathcal{O}(\omega^2),$$
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$$[\hat{x}_1, \hat{x}_2] = 0.$$

- The twisted κ-Minkowski space $\mathbf{M}_z^{2+1}$ is the **first-order noncommutative spacetime for all the AdS$_\omega$ groups**.

Quantum κ-AdS$_\omega$ algebra in (2+1)

The **AdS$_\omega$ universal enveloping algebra** has the following cocommutative Hopf algebra structure

$$\Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1, \qquad \Delta(J) = 1 \otimes J + J \otimes 1,$$
$$\Delta(P_i) = 1 \otimes P_i + P_i \otimes 1, \qquad \Delta(K_i) = 1 \otimes K_i + K_i \otimes 1$$

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The κ-AdS$_\omega$ r-matrix

$$r = z(K_1 \wedge P_1 + K_2 \wedge P_2)$$

provides the **first order deformation of the coproduct**

$$\Delta = \sum_{k=0}^{\infty} \Delta_{(k)} = \sum_{k=0}^{\infty} \eta^k \delta_{(k)} = \Delta_0 + z\,\delta_{(1)} + o[z^2]$$

$$\delta(P_0) = 0 \qquad \delta(J) = 0$$
$$\delta(P_i) = z(P_i \wedge P_0 - \omega \epsilon_{ij} K_j \wedge J)$$
$$\delta(K_i) = z(K_i \wedge P_0 + \epsilon_{ij} P_j \wedge J).$$

Quantum κ-AdS$_\omega$ algebra in (2+1)

The full (all orders in z) **quantum universal enveloping algebra** of the κ-deformation of AdS$_\omega$ can be constructed[18] and reads

$$\Delta(P_0) = 1 \otimes P_0 + P_0 \otimes 1, \qquad \Delta(J) = 1 \otimes J + J \otimes 1,$$

$$\begin{aligned}\Delta(P_i) &= \mathrm{e}^{-\frac{z}{2}P_0} \cosh(\tfrac{z}{2}\sqrt{\omega}J) \otimes P_i + P_i \otimes \mathrm{e}^{\frac{z}{2}P_0} \cosh(\tfrac{z}{2}\sqrt{\omega}J) \\ &\quad + \sqrt{\omega}\, \mathrm{e}^{-\frac{z}{2}P_0} \sinh(\tfrac{z}{2}\sqrt{\omega}J) \otimes \epsilon_{ij} K_j - \sqrt{\omega}\, \epsilon_{ij} K_j \otimes \mathrm{e}^{\frac{z}{2}P_0} \sinh(\tfrac{z}{2}\sqrt{\omega}J),\end{aligned}$$

$$\begin{aligned}\Delta(K_i) &= \mathrm{e}^{-\frac{z}{2}P_0} \cosh(\tfrac{z}{2}\sqrt{\omega}J) \otimes K_i + K_i \otimes \mathrm{e}^{\frac{z}{2}P_0} \cosh(\tfrac{z}{2}\sqrt{\omega}J) \\ &\quad - \mathrm{e}^{-\frac{z}{2}P_0} \frac{\sinh(\tfrac{z}{2}\sqrt{\omega}J)}{\sqrt{\omega}} \otimes \epsilon_{ij} P_j + \epsilon_{ij} P_j \otimes \mathrm{e}^{\frac{z}{2}P_0} \frac{\sinh(\tfrac{z}{2}\sqrt{\omega}J)}{\sqrt{\omega}},\end{aligned}$$

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18 A.B., F.J. Herranz, M.A. del Olmo, M. Santander, J. Phys. A **27** (1994) 1283.

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$$[J, P_i] = \epsilon_{ij}P_j, \qquad [J, K_i] = \epsilon_{ij}K_j, \qquad [J, P_0] = 0,$$

$$[P_i, K_j] = -\delta_{ij}\frac{\sinh(zP_0)}{z}\cosh(z\sqrt{\omega}J), \qquad [P_0, K_i] = -P_i,$$

$$[K_1, K_2] = -\cosh(zP_0)\frac{\sinh(z\sqrt{\omega}J)}{z\sqrt{\omega}}, \qquad [P_0, P_i] = \omega K_i,$$

$$[P_1, P_2] = -\omega\cosh(zP_0)\frac{\sinh(z\sqrt{\omega}J)}{z\sqrt{\omega}},$$

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- **This is exactly the quantum (A)dS algebra** introduced in [19] as the symmetry algebra of the vacuum excitations in (2+1) quantum gravity.
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$$\mathcal{C}_z = 4\cos(z\sqrt{\omega})\left\{\frac{\sinh^2(\frac{z}{2}P_0)}{z^2}\cosh^2\left(\tfrac{z}{2}\sqrt{\omega}J\right) + \frac{\sinh^2(\frac{z}{2}\sqrt{\omega}J)}{z^2}\cosh^2\left(\tfrac{z}{2}P_0\right)\right\} - \frac{\sin(z\sqrt{\omega})}{z\sqrt{\omega}}\left(\mathbf{P}^2 + \omega\mathbf{K}^2\right)$$

$$\mathcal{W}_z = -\cos(z\sqrt{\omega})\,\frac{\sinh(z\sqrt{\omega}J)}{z\sqrt{\omega}}\,\frac{\sinh(zP_0)}{z} + \frac{\sin(z\sqrt{\omega})}{z\sqrt{\omega}}(K_1P_2 - K_2P_1).$$

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- The AdS$_\omega$ **dispersion relation** coming from $\mathcal{C}_z$ would also **include the Lorentz sector.**

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- The AdS$_\omega$ **dispersion relation** coming from $\mathcal{C}_z$ would also **include the Lorentz sector.**
- The **coproduct (addition) of momenta involves rotation and boosts.**

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Adding the twist induced by the DD

The **twisted coproduct $\Delta_{\vartheta,z}$** is obtained by twisting the κ–AdS$_\omega$ coproduct through an element $\mathcal{F}_\vartheta \in \kappa$-AdS$_\omega \otimes \kappa$-AdS$_\omega$:

$$\Delta_{\vartheta,z}(Y) = \mathcal{F}_\vartheta \Delta_z(Y) \mathcal{F}_\vartheta^{-1}, \quad \forall Y \in \kappa\text{-AdS}_\omega,$$

where

$$\mathcal{F}_\vartheta = \exp(-\vartheta J \wedge P_0).$$

[20] A.B. , F.J. Herranz, C. Meusburger, P. Naranjo, SIGMA 10 (2014) 052

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The twist $\mathcal{F}_\vartheta$ satisfies the so-called twisting co-cycle and normalisation conditions

$$\mathcal{F}_{\vartheta,12}(\Delta_z \otimes \mathrm{id})\mathcal{F}_\vartheta = \mathcal{F}_{\vartheta,23}(\mathrm{id} \otimes \Delta_z)\mathcal{F}_\vartheta \,, \qquad (\epsilon \otimes \mathrm{id})\mathcal{F}_\vartheta = 1 = (\mathrm{id} \otimes \epsilon)\mathcal{F}_\vartheta \,.$$

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In this way we obtain (full expressions can be found in [20]):

[20] A.B. , F.J. Herranz, C. Meusburger, P. Naranjo, SIGMA 10 (2014) 052

Adding the twist induced by the DD

$$\Delta_{\vartheta,z}(P_0) = 1 \otimes P_0 + P_0 \otimes 1\,, \qquad \Delta_{\vartheta,z}(J) = 1 \otimes J + J \otimes 1\,,$$

$$\begin{aligned}
\Delta_{\vartheta,z}(P_i) = {} & \Delta_z(P_i) + \mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\sqrt{\omega}J)\left[\cos(\vartheta\sqrt{\omega}J)\cos(\vartheta P_0) - 1\right] \otimes P_i \\
& + \mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\sqrt{\omega}J)\sin(\vartheta P_0)\cos(\vartheta\sqrt{\omega}J) \otimes \epsilon_{ij}P_j - \sqrt{\omega}\,\mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\sqrt{\omega}J)\sin(\vartheta\sqrt{\omega}J)\cos(\vartheta P_0) \otimes K_i \\
& - \sqrt{\omega}\,\mathrm{e}^{-\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\sqrt{\omega}J)\sin(\vartheta\sqrt{\omega}J)\,\sin(\vartheta P_0) \otimes \epsilon_{ij}K_j + P_i \otimes \mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\sqrt{\omega}J)\left[\cos(\vartheta\sqrt{\omega}J)\,\cos(\vartheta P_0) - 1\right] \\
& - \epsilon_{ij}P_j \otimes \mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\sqrt{\omega}J)\,\sin(\vartheta P_0)\,\cos(\vartheta\sqrt{\omega}J) + \sqrt{\omega}\,K_i \otimes \mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\sqrt{\omega}J)\sin(\vartheta\sqrt{\omega}J)\cos(\vartheta P_0) \\
& - \sqrt{\omega}\,\epsilon_{ij}K_j \otimes \mathrm{e}^{\frac{z}{2}P_0}\cosh(\tfrac{z}{2}\sqrt{\omega}J)\sin(\vartheta\sqrt{\omega}J)\sin(\vartheta P_0) - \mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\sqrt{\omega}J)\sin(\vartheta\sqrt{\omega}J)\sin(\vartheta P_0) \otimes P_i \\
& + \mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\sqrt{\omega}J)\sin(\vartheta\sqrt{\omega}J)\cos(\vartheta P_0) \otimes \epsilon_{ij}P_j - \sqrt{\omega}\,\mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\sqrt{\omega}J)\sin(\vartheta P_0)\cos(\vartheta\sqrt{\omega}J) \otimes K_i \\
& + \sqrt{\omega}\,\mathrm{e}^{-\frac{z}{2}P_0}\sinh(\tfrac{z}{2}\sqrt{\omega}J)[\cos(\vartheta\sqrt{\omega}J)\cos(\vartheta P_0) - 1] \otimes \epsilon_{ij}K_j \\
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\end{aligned}$$

But **commutation rules are left unchanged**.

4. The Snyder–type deformation

First order deformation

The **canonical classical r-matrix** is

$$r' = \eta J_0 \wedge J_2 + \tfrac{1}{2}\left(-P_0 \wedge J_0 + P_1 \wedge J_1 + P_2 \wedge J_2\right).$$

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Again, we will multiply r' by the quantum double deformation parameter z and

$$\delta_z(J_0) = \eta z J_1 \wedge J_0, \qquad \delta_z(J_1) = 0, \qquad \delta_z(J_2) = \eta z J_1 \wedge J_2,$$

$$\delta_z(P_0) = z\left(P_1 \wedge P_2 + \eta P_1 \wedge J_0 + \eta^2 J_2 \wedge J_1\right),$$

$$\delta_z(P_1) = z\left(P_0 \wedge P_2 + \eta P_0 \wedge J_0 - \eta P_2 \wedge J_2 + \eta^2 J_2 \wedge J_0\right),$$

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$$\delta_z(P_2) = z\left(P_1 \wedge P_0 + \eta P_1 \wedge J_2 + \eta^2 J_0 \wedge J_1\right),$$

- The cosmological constant is $\Lambda = -\eta^2$.
- The $\eta \to 0$ limit gives a (simpler) twisted Poincaré algebra.

First order non–commutative space–time

In terms of the dual basis $(\hat{x}_a, \hat{\theta}_a)$ $(a = 0, 1, 2)$, we find that the first-order dual Lie brackets among the spacetime coordinates are given by

$$[\hat{x}_0, \hat{x}_1] = -z\hat{x}_2, \qquad [\hat{x}_0, \hat{x}_2] = z\hat{x}_1, \qquad [\hat{x}_1, \hat{x}_2] = z\hat{x}_0.$$

This is a noncommutative spacetime of **Snyder** type.

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This is a noncommutative spacetime of **Snyder** type.

The remaining first-order non-commutative relations between the quantum spacetime and Lorentz parameters are

$$\begin{aligned}
&[\hat{\theta}_0, \hat{\theta}_1] = -\eta z(\hat{\theta}_0 - \eta\hat{x}_2), && [\hat{\theta}_0, \hat{\theta}_2] = -\eta^2 z\hat{x}_1, && [\hat{\theta}_1, \hat{\theta}_2] = \eta z(\hat{\theta}_2 - \eta\hat{x}_0),\\
&[\hat{\theta}_0, \hat{x}_0] = -\eta z\hat{x}_1, && [\hat{\theta}_0, \hat{x}_1] = -\eta z\hat{x}_0, && [\hat{\theta}_0, \hat{x}_2] = 0,\\
&[\hat{\theta}_1, \hat{x}_0] = 0, && [\hat{\theta}_1, \hat{x}_1] = 0, && [\hat{\theta}_1, \hat{x}_2] = 0,\\
&[\hat{\theta}_2, \hat{x}_0] = 0, && [\hat{\theta}_2, \hat{x}_1] = -\eta z\hat{x}_2, && [\hat{\theta}_2, \hat{x}_2] = \eta z\hat{x}_1.
\end{aligned}$$

Note that in the Poincaré limit all these relations vanish.

All–orders Snyder nc spacetime deformation

From the Sklyanin bracket we get the PL brackets for the x_a coordinates [21]

$$\{x_0, x_1\} = -z\frac{\tanh \eta x_2}{\eta}\,\Upsilon,$$

$$\{x_0, x_2\} = z\frac{\tanh \eta x_1}{\eta}\,\Upsilon,$$

$$\{x_1, x_2\} = z\frac{\tan \eta x_0}{\eta}\,\Upsilon,$$

$$\text{where} \qquad \Upsilon(x_0, x_1) = \cos \eta x_0 (\cos \eta x_0 \cosh \eta x_1 + \sinh \eta x_1).$$

[21] A.B., F.J. Herranz, C. Meusburger, Phys. Lett. B 732 (2014) 201

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where $\qquad \Upsilon(x_0, x_1) = \cos \eta x_0 (\cos \eta x_0 \cosh \eta x_1 + \sinh \eta x_1).$

Therefore, we have a **cosmological constant deformation of a 'Snyder' $so(2,1)$ nc spacetime**, whose quantization is by no means trivial:

$$\{x_0, x_1\} = -z\, x_2 - \eta z\, x_1 x_2 + \eta^2 z \left(x_0^2 x_2 - \tfrac{1}{2}\, x_1^2 x_2 + \tfrac{1}{3}\, x_2^3 \right) + o[\eta^3],$$

$$\{x_0, x_2\} = z\, x_1 + \eta z x_1^2 - \eta^2 z \left(x_0^2 x_1 - \tfrac{1}{6}\, x_1^3 \right) + o[\eta^3],$$

$$\{x_1, x_2\} = z\, x_0 + \eta z x_0 x_1 - \eta^2 z \left(\tfrac{2}{3}\, x_0^3 - \tfrac{1}{2}\, x_1^2 x_0 \right) + o[\eta^3].$$

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5. Quantum AdS$_\omega$ in (3+1) dimensions

The AdS$_\omega$ algebra in (3+1)

The **(3+1)D AdS$_\omega$ Lie algebra**:

$$
\begin{aligned}
&[J_a, J_b] = \epsilon_{abc} J_c \,, && [J_a, P_b] = \epsilon_{abc} P_c \,, && [J_a, K_b] = \epsilon_{abc} K_c \,, \\
&[K_a, P_0] = P_a \,, && [K_a, P_b] = \delta_{ab} P_0 \,, && [K_a, K_b] = -\epsilon_{abc} J_c \,, \\
&[P_0, P_a] = \omega K_a \,, && [P_a, P_b] = -\omega \epsilon_{abc} J_c \,, && [P_0, J_a] = 0 \,.
\end{aligned}
$$

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\end{aligned}
$$

Explicilty, $\mathbf{AdS}_\omega^{3+1}$ comprises the three following Lorentzian spacetimes:

- $\omega > 0, \Lambda < 0$: AdS spacetime $\mathbf{AdS}^{3+1} \equiv \mathrm{SO}(3,2)/\mathrm{SO}(3,1)$.
- $\omega < 0, \Lambda > 0$: dS spacetime $\mathbf{dS}^{3+1} \equiv \mathrm{SO}(4,1)/\mathrm{SO}(3,1)$.
- $\omega = \Lambda = 0$: Minkowski spacetime $\mathbf{M}^{3+1} \equiv \mathrm{ISO}(3,1)/\mathrm{SO}(3,1)$.

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Casimir operators:

$$\mathcal{C} = P_0^2 - \mathbf{P}^2 + \omega \left(\mathbf{J}^2 - \mathbf{K}^2\right)$$

$$\mathcal{W} = W_0^2 - \mathbf{W}^2 + \omega\,(\mathbf{J}\cdot\mathbf{K})^2$$

$$W_0 = \mathbf{J}\cdot\mathbf{P} \qquad W_a = -J_a P_0 + \epsilon_{abc} K_b P_c$$

A Drinfel'd double structure for $\mathfrak{so}(5)$

Classical Lie algebra $\mathfrak{c}_2$ generated by $\{h_a, e_{\pm a}\}$ $(a = 1, 2)$:

$$
\begin{aligned}
&[h_1, e_{\pm 1}] = \pm e_{\pm 1}\,, \qquad && [h_1, e_{\pm 2}] = \mp e_{\pm 2}\,, \qquad && [e_{+1}, e_{-1}] = h_1\,,\\
&[h_2, e_{\pm 1}] = \mp e_{\pm 1}\,, && [h_2, e_{\pm 2}] = \pm 2 e_{\pm 2}\,, && [e_{+2}, e_{-2}] = h_2\,,\\
&[h_1, h_2] = 0\,, && [e_{-1}, e_{+2}] = 0\,, && [e_{+1}, e_{-2}] = 0\,.
\end{aligned}
$$

$$
\begin{aligned}
&[e_{+1}, e_{+2}] := e_{+3}\,, \qquad && [e_{-2}, e_{-1}] := e_{-3}\,,\\
&[e_{+1}, e_{+3}] := e_{+4}\,, && [e_{-3}, e_{-1}] := e_{-4}\,.
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- The generators $\{h_a, e_{\pm b}\}$ $(a = 1, 2;\, b = 1, \ldots, 4)$ span $\mathfrak{so}(5)$

$$
\begin{aligned}
e_0 &= -\tfrac{1}{\sqrt{2}}(J_{04} - \mathrm{i}J_{13})\,, & f_0 &= \tfrac{1}{\sqrt{2}}(J_{04} + \mathrm{i}J_{13})\,,\\
e_1 &= \tfrac{1}{\sqrt{2}}(J_{23} + \mathrm{i}J_{12})\,, & f_1 &= -\tfrac{1}{\sqrt{2}}(J_{23} - \mathrm{i}J_{12})\,,\\
e_2 &= \tfrac{1}{2}\left(J_{01} - J_{34} - \mathrm{i}(J_{03} + J_{14})\right)\,, & f_2 &= -\tfrac{1}{2}\left(J_{01} - J_{34} + \mathrm{i}(J_{03} + J_{14})\right)\,,\\
e_3 &= \tfrac{1}{\sqrt{2}}(J_{24} + \mathrm{i}J_{02})\,, & f_3 &= -\tfrac{1}{\sqrt{2}}(J_{24} - \mathrm{i}J_{02})\,,\\
e_4 &= \tfrac{1}{2}\left(J_{01} + J_{34} + \mathrm{i}(J_{03} - J_{14})\right)\,, & f_4 &= -\tfrac{1}{2}\left(J_{01} + J_{34} - \mathrm{i}(J_{03} - J_{14})\right)\,.
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e_2 &= \tfrac{1}{2}\left(J_{01} - J_{34} - \mathrm{i}(J_{03} + J_{14})\right)\,, & f_2 &= -\tfrac{1}{2}\left(J_{01} - J_{34} + \mathrm{i}(J_{03} + J_{14})\right)\,,\\
e_3 &= \tfrac{1}{\sqrt{2}}(J_{24} + \mathrm{i} J_{02})\,, & f_3 &= -\tfrac{1}{\sqrt{2}}(J_{24} - \mathrm{i} J_{02})\,,\\
e_4 &= \tfrac{1}{2}\left(J_{01} + J_{34} + \mathrm{i}(J_{03} - J_{14})\right)\,, & f_4 &= -\tfrac{1}{2}\left(J_{01} + J_{34} - \mathrm{i}(J_{03} - J_{14})\right)\,.
\end{aligned}
$$

- Let us consider $f_b \equiv e_{-b}$ $(b = 1, \ldots, 4)$ and

$$
e_0 := \frac{1}{\sqrt{2}}\left((1+\mathrm{i})h_1 + \mathrm{i}h_2\right)\,, \qquad f_0 := \frac{1}{\sqrt{2}}\left((1-\mathrm{i})h_1 - \mathrm{i}h_2\right)\,.
$$

A Drinfel'd double structure for $\mathfrak{so}(5)$

We take **the two Borel subalgebras as the DD subalgebras:**

$$X_i \equiv e_{+i} \qquad x^i \equiv f_i \equiv e_{-i}\,, \qquad i = 0, \ldots, 4\,.$$

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$$X_i \equiv e_{+i} \qquad x^i \equiv f_i \equiv e_{-i}\,, \qquad i = 0, \ldots, 4\,.$$

Therefore, $\mathfrak{so}(5)$ is endowed with the following DD structure:

- Canonical pairing

$$\langle e_i, e_j \rangle = 0\,, \qquad \langle f_i, f_j \rangle = 0\,, \qquad \langle f_i, e_j \rangle = \delta_{ij}\,, \qquad \forall i, j\,.$$

- Casimir element

$$C = \frac{1}{2} \sum_i (x^i\, X_i + X_i\, x^i) = \frac{1}{2} \sum_{i=0}^{4} (f_i e_i + e_i f_i)\,.$$

- Canonical DD classical r-matrix

$$r = \sum_i x^i \otimes X_i = \sum_{i=0}^{4} f_i \otimes e_i\,, \quad r_{\rm skew} = \frac{1}{2} \sum_i x^i \wedge X_i = \frac{1}{2} \sum_{i=0}^{4} f_i \wedge e_i\,.$$

The DD structure for AdS$_\omega$

- Change of basis

$$
\begin{array}{llll}
P_1 = \mathrm{i}\sqrt{\omega}\, J_{01}\,, & P_2 = \mathrm{i}\sqrt{\omega}\, J_{02}\,, & P_3 = \mathrm{i}\sqrt{\omega}\, J_{03}\,, & P_0 = -\sqrt{\omega}\, J_{04}\,, \\
K_1 = \mathrm{i}J_{14}\,, & K_2 = \mathrm{i}J_{24}\,, & K_3 = \mathrm{i}J_{34}\,, & \\
J_1 = J_{23}\,, & J_2 = -J_{13}\,, & J_3 = J_{12}\,. &
\end{array}
$$

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$$K_1 = \mathrm{i}J_{14}\,, \quad K_2 = \mathrm{i}J_{24}\,, \quad K_3 = \mathrm{i}J_{34}\,,$$
$$J_1 = J_{23}\,, \quad J_2 = -J_{13}\,, \quad J_3 = J_{12}\,.$$

- Pairing and Casimir operator:

$$\langle P_0, P_0\rangle_\omega = -\omega, \ \langle P_a, P_b\rangle_\omega = \omega\,\delta_{ab}, \ \langle K_a, K_b\rangle_\omega = \delta_{ab}, \ \langle J_a, J_b\rangle_\omega = -\delta_{ab}\,,$$

$$C_\omega = \omega\, C = \frac{1}{2}\Big(\sum_{a=1}^{3} P_a^2 - P_0^2 + \omega \sum_{a=1}^{3}\Big(K_a^2 - J_a^2\Big)\Big)$$

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$$C_\omega = \omega\, C = \frac{1}{2}\left(\sum_{a=1}^{3} P_a^2 - P_0^2 + \omega \sum_{a=1}^{3}\left(K_a^2 - J_a^2\right)\right)$$

The DD classical r-matrices in (2+1) and (3+1)

$$r_\omega \equiv \sqrt{\omega}\, r_J = z(\underbrace{K_1\wedge P_1 + K_2\wedge P_2 + K_3\wedge P_3 + \sqrt{\omega}\, J_3\wedge J_1}_{\kappa\text{-AdS}_\omega} + \underbrace{P_0\wedge J_2}_{\text{twist}})$$

$$r_{2+1} = z\underbrace{\left(K_1\wedge P_1 + K_2\wedge P_2\right)}_{\kappa\text{-AdS}_\omega} + \underbrace{\theta J\wedge P_0}_{\text{twist}}\,.$$

A DD quantum AdS$_\omega$ deformation

Cocommutator map in (3+1)

$$\delta(P_0) = 0\,, \qquad \delta(J_2) = 0\,,$$
$$\delta(J_1) = z\left(P_0 \wedge J_3 + \sqrt{\omega} J_1 \wedge J_2\right), \qquad \delta(J_3) = z\left(J_1 \wedge P_0 + \sqrt{\omega} J_3 \wedge J_2\right),$$
$$\delta(P_1) = z\Big((P_1 - P_3) \wedge P_0 + \omega\big(J_2 \wedge (K_1 - K_3) + J_3 \wedge K_2\big) + \sqrt{\omega} J_1 \wedge P_2\Big),$$
$$\delta(P_2) = z\Big(P_2 \wedge P_0 + \omega\big(J_1 \wedge K_3 + J_2 \wedge K_2 + K_1 \wedge J_3\big) + \sqrt{\omega}\big(P_1 \wedge J_1 + P_3 \wedge J_3\big)\Big),$$
$$\delta(P_3) = z\Big((P_1 + P_3) \wedge P_0 + \omega\big(J_2 \wedge (K_1 + K_3) + K_2 \wedge J_1\big) + \sqrt{\omega} J_3 \wedge P_2\Big),$$
$$\delta(K_1) = z\Big((K_1 - K_3) \wedge P_0 + (P_1 - P_3) \wedge J_2 + P_2 \wedge J_3 + \sqrt{\omega} J_1 \wedge K_2\Big),$$
$$\delta(K_2) = z\Big(K_2 \wedge P_0 + J_3 \wedge P_1 + P_2 \wedge J_2 + P_3 \wedge J_1 + \sqrt{\omega}\big(K_1 \wedge J_1 + K_3 \wedge J_3\big)\Big),$$
$$\delta(K_3) = z\Big((K_1 + K_3) \wedge P_0 + (P_1 + P_3) \wedge J_2 + J_1 \wedge P_2 + \sqrt{\omega} J_3 \wedge K_2\Big).$$

Note the strong effect of ω in the addition law for momenta.
The rotation subalgebra is also influenced by the twist.

First–order noncommutative spacetime

First-order Poisson–Lie brackets defined by the 4-dimensional spacetime PL subalgebra:

$$\begin{aligned} \{x^1, x^0\} &= z\,(x^1 + x^3) \\ \{x^2, x^0\} &= z\,x^2, \\ \{x^3, x^0\} &= z\,(x^3 - x^1)\,, \\ \{x^a, x^b\} &= 0, \qquad a, b = 1, 2, 3\,. \end{aligned}$$

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\{x^1, x^0\} &= z\left(x^1 + x^3\right) \\
\{x^2, x^0\} &= z\, x^2, \\
\{x^3, x^0\} &= z\left(x^3 - x^1\right), \\
\{x^a, x^b\} &= 0, \qquad a, b = 1, 2, 3\,.
\end{aligned}$$

This is **nonisomorphic to (3+1) κ–Minkowski spacetime**.

The x^2 **coordinate is distinguished**.

Restoring space isotropy

Space isotropy can be manifestly recovered in this DD quantum deformation by considering the following automorphism of the AdS$_\omega$ algebra: [22]

$$
\begin{aligned}
\widetilde{Y}_1 &= \frac{1}{\sqrt{6}}\,Y_1 + \frac{1}{\sqrt{3}}\,Y_2 + \frac{1}{\sqrt{2}}\,Y_3, &\qquad Y_1 &= \frac{1}{\sqrt{6}}\left(\widetilde{Y}_1 + \widetilde{Y}_2 - 2\widetilde{Y}_3\right),\\
\widetilde{Y}_2 &= \frac{1}{\sqrt{6}}\,Y_1 + \frac{1}{\sqrt{3}}\,Y_2 - \frac{1}{\sqrt{2}}\,Y_3, &\qquad Y_2 &= \frac{1}{\sqrt{3}}\left(\widetilde{Y}_1 + \widetilde{Y}_2 + \widetilde{Y}_3\right),\\
\widetilde{Y}_3 &= -\frac{2}{\sqrt{6}}\,Y_1 + \frac{1}{\sqrt{3}}\,Y_2, &\qquad Y_3 &= \frac{1}{\sqrt{2}}\left(\widetilde{Y}_1 - \widetilde{Y}_2\right),\\
&\text{for}\quad \mathbf{Y} \in \{\mathbf{P}, \mathbf{K}, \mathbf{J}\}, &\qquad \widetilde{P}_0 &= P_0.
\end{aligned}
$$

[22] A.B., F.J. Herranz, P. Naranjo, Phys. Lett. B 746 (2015) 37

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$$\widetilde{Y}_1 = \frac{1}{\sqrt{6}}\, Y_1 + \frac{1}{\sqrt{3}}\, Y_2 + \frac{1}{\sqrt{2}}\, Y_3, \qquad Y_1 = \frac{1}{\sqrt{6}}\left(\widetilde{Y}_1 + \widetilde{Y}_2 - 2\widetilde{Y}_3\right),$$

$$\widetilde{Y}_2 = \frac{1}{\sqrt{6}}\, Y_1 + \frac{1}{\sqrt{3}}\, Y_2 - \frac{1}{\sqrt{2}}\, Y_3, \qquad Y_2 = \frac{1}{\sqrt{3}}\left(\widetilde{Y}_1 + \widetilde{Y}_2 + \widetilde{Y}_3\right),$$

$$\widetilde{Y}_3 = -\frac{2}{\sqrt{6}}\, Y_1 + \frac{1}{\sqrt{3}}\, Y_2, \qquad Y_3 = \frac{1}{\sqrt{2}}\left(\widetilde{Y}_1 - \widetilde{Y}_2\right),$$

$$\text{for} \quad \mathbf{Y} \in \{\mathbf{P}, \mathbf{K}, \mathbf{J}\}, \qquad \widetilde{P}_0 = P_0 .$$

In this way, the classical r-matrix is transformed into

$$\tilde{r}_\omega = z\left(\tilde{K}_1 \wedge \tilde{P}_1 + \tilde{K}_2 \wedge \tilde{P}_2 + \tilde{K}_3 \wedge \tilde{P}_3 + \frac{1}{\sqrt{3}}\,\tilde{P}_0 \wedge (\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3) + \frac{\sqrt{\omega}}{\sqrt{3}}\,(\tilde{J}_1 \wedge \tilde{J}_2 + \tilde{J}_2 \wedge \tilde{J}_3 + \tilde{J}_3 \wedge \tilde{J}_1)\right)$$

[22] A.B., F.J. Herranz, P. Naranjo, Phys. Lett. B 746 (2015) 37

Restoring space isotropy

The **first-order noncommutative spacetime** spanned by the dual coordinates of the spacetime subalgebra reads

$$\{x^a, x^0\} = z\left(x^a + \frac{1}{\sqrt{3}}\left(x^{a+2} - x^{a+1}\right)\right)$$
$$\{x^a, x^b\} = 0 \qquad a, b = 1, 2, 3\,.$$

Full quantum twisted AdS$_\omega$ algebra

Instead of considering

$$r = z\left(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{\omega} J_3 \wedge J_1\right) + P_0 \wedge J_2,$$

we take the equivalent AdS$_\omega$ deformation generated by

$$r_{z,\vartheta} = z\left(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{\omega} J_1 \wedge J_2\right) + \vartheta J_3 \wedge P_0.$$

[23] A.B., F. Musso, J. Phys. A: Math. Theor 46 (2013) 195203

[24] A.B., F.J. Herranz, F. Musso, P. Naranjo, preprint (2015)

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$$r_{z,\vartheta} = z\left(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3 + \sqrt{\omega} J_1 \wedge J_2\right) + \vartheta J_3 \wedge P_0.$$

The **Poisson analogue of the corresponding all-orders quantum algebra can be explicitly computed** by following the **dual Poisson–Lie group approach based in the quantum duality principle and** presented in [23].

We end up with the following **coproduct in a 'bicrossproduct' basis**: [24]

$$\begin{aligned}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \qquad \Delta(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \\
\Delta(J_1) &= J_1 \otimes e^{z\sqrt{\omega} J_3} + \cos(\vartheta P_0) \otimes J_1 + \sin(\vartheta P_0) \otimes J_2, \\
\Delta(J_2) &= J_2 \otimes e^{z\sqrt{\omega} J_3} + \cos(\vartheta P_0) \otimes J_2 - \sin(\vartheta P_0) \otimes J_1,
\end{aligned}$$

[23] A.B., F. Musso, J. Phys. A: Math. Theor 46 (2013) 195203

[24] A.B., F.J. Herranz, F. Musso, P. Naranjo, preprint (2015)

Nonlinear composition of momenta

$$
\begin{aligned}
\Delta(P_1) &= P_1 \otimes \cosh(z\sqrt{\omega}J_3)\cos(\vartheta\sqrt{\omega}J_3) + e^{-zP_0}\cos(\vartheta P_0)\otimes P_1 \\
&\quad + P_2 \otimes \sinh(z\sqrt{\omega}J_3)\sin(\vartheta\sqrt{\omega}J_3) + e^{-zP_0}\sin(\vartheta P_0)\otimes P_2 \\
&\quad - \sqrt{\omega}K_2 \otimes \sinh(z\sqrt{\omega}J_3)\cos(\vartheta\sqrt{\omega}J_3) + \sqrt{\omega}K_1 \otimes \cosh(z\sqrt{\omega}J_3)\sin(\vartheta\sqrt{\omega}J_3) \\
&\quad - z\sqrt{\omega}\left[(P_3\otimes J_1 - \sqrt{\omega}K_3\otimes J_2)C_\vartheta(P_0,J_3) + (P_3\otimes J_2 + \sqrt{\omega}K_3\otimes J_1)S_\vartheta(P_0,J_3)\right] \\
&\quad + \frac{z^2\omega}{2}\left[2(\sqrt{\omega}K_1 - P_2)\otimes J_1J_2e^{-z\sqrt{\omega}J_3} - (\sqrt{\omega}K_2 + P_1)\otimes(J_1^2 - J_2^2)e^{-z\sqrt{\omega}J_3}\right]\widetilde{C}_\vartheta(P_0,J_3) \\
&\quad - \frac{z^2\omega}{2}\left[2(\sqrt{\omega}K_2 + P_1)\otimes J_1J_2e^{-z\sqrt{\omega}J_3} + (\sqrt{\omega}K_1 - P_2)\otimes(J_1^2 - J_2^2)e^{-z\sqrt{\omega}J_3}\right]\widetilde{S}_\vartheta(P_0,J_3), \\
\Delta(P_2) &= P_2 \otimes \cosh(z\sqrt{\omega}J_3)\cos(\vartheta\sqrt{\omega}J_3) + e^{-zP_0}\cos(\vartheta P_0)\otimes P_2 \\
&\quad - P_1 \otimes \sinh(z\sqrt{\omega}J_3)\sin(\vartheta\sqrt{\omega}J_3) - e^{-zP_0}\sin(\vartheta P_0)\otimes P_1 \\
&\quad + \sqrt{\omega}K_1 \otimes \sinh(z\sqrt{\omega}J_3)\cos(\vartheta\sqrt{\omega}J_3) + \sqrt{\omega}K_2 \otimes \cosh(z\sqrt{\omega}J_3)\sin(\vartheta\sqrt{\omega}J_3) \\
&\quad - z\sqrt{\omega}\left[(P_3\otimes J_2 + \sqrt{\omega}K_3\otimes J_1)C_\vartheta(P_0,J_3) - (P_3\otimes J_1 - \sqrt{\omega}K_3\otimes J_2)S_\vartheta(P_0,J_3)\right] \\
&\quad - \frac{z^2\omega}{2}\left[2(\sqrt{\omega}K_2 + P_1)\otimes J_1J_2e^{-z\sqrt{\omega}J_3} + (\sqrt{\omega}K_1 - P_2)\otimes(J_1^2 - J_2^2)e^{-z\sqrt{\omega}J_3}\right]\widetilde{C}_\vartheta(P_0,J_3) \\
&\quad - \frac{z^2\omega}{2}\left[2(\sqrt{\omega}K_1 - P_2)\otimes J_1J_2e^{-z\sqrt{\omega}J_3} - (\sqrt{\omega}K_2 + P_1)\otimes(J_1^2 - J_2^2)e^{-z\sqrt{\omega}J_3}\right]\widetilde{S}_\vartheta(P_0,J_3), \\
\Delta(P_3) &= e^{-zP_0}\otimes P_3 + P_3\otimes\cos(\vartheta\sqrt{\omega}J_3) + \sqrt{\omega}K_3\otimes\sin(\vartheta\sqrt{\omega}J_3) \\
&\quad + z\sqrt{\omega}\left[(\sqrt{\omega}K_2 + P_1)\otimes J_1e^{-z\sqrt{\omega}J_3} - (\sqrt{\omega}K_1 - P_2)\otimes J_2e^{-z\sqrt{\omega}J_3}\right]C_\vartheta(P_0,J_3) \\
&\quad + z\sqrt{\omega}\left[(\sqrt{\omega}K_2 + P_1)\otimes J_2e^{-z\sqrt{\omega}J_3} + (\sqrt{\omega}K_1 - P_2)\otimes J_1e^{-z\sqrt{\omega}J_3}\right]S_\vartheta(P_0,J_3),
\end{aligned}
$$

Quantum commutation rules

$$\{J_1, J_2\} = \frac{e^{2z\sqrt{\omega}J_3} - 1}{2z\sqrt{\omega}} - \frac{z\sqrt{\omega}}{2}\left(J_1^2 + J_2^2\right), \qquad \{J_1, J_3\} = -J_2, \qquad \{J_2, J_3\} = J_1,$$

$$\{J_1, P_1\} = z\sqrt{\omega}J_1 P_2, \qquad \{J_1, P_2\} = P_3 - z\sqrt{\omega}J_1 P_1, \qquad \{J_1, P_3\} = -P_2,$$

$$\{J_2, P_1\} = -P_3 + z\sqrt{\omega}J_2 P_2, \qquad \{J_2, P_2\} = -z\sqrt{\omega}J_2 P_1, \qquad \{J_2, P_3\} = P_1,$$

$$\{J_3, P_1\} = P_2, \qquad \{J_3, P_2\} = -P_1, \qquad \{J_3, P_3\} = 0,$$

$$\{J_1, K_1\} = z\sqrt{\omega}J_1 K_2, \qquad \{J_1, K_2\} = K_3 - z\sqrt{\omega}J_1 K_1, \qquad \{J_1, K_3\} = -K_2,$$

$$\{J_2, K_1\} = -K_3 + z\sqrt{\omega}J_2 K_2, \qquad \{J_2, K_2\} = -z\sqrt{\omega}J_2 K_1, \qquad \{J_2, K_3\} = K_1,$$

$$\{J_3, K_1\} = K_2, \qquad \{J_3, K_2\} = -K_1, \qquad \{J_3, K_3\} = 0,$$

$$\{K_a, P_0\} = P_a, \qquad \{P_0, P_a\} = \omega K_a, \qquad \{P_0, J_a\} = 0,$$

Quantum commutation rules

$$
\begin{aligned}
\{K_1, P_1\} &= \frac{1}{2z}\left(\cosh(2z\sqrt{\omega}J_3) - e^{-2zP_0}\right) + \frac{z^3\omega^2}{4}e^{-2z\sqrt{\omega}J_3}\left(J_1^2 + J_2^2\right)^2 + \frac{z}{2}\left(P_2^2 + P_3^2 - P_1^2\right) \\
&\quad + \frac{z\omega}{2}\left[K_2^2 + K_3^2 - K_1^2 + J_1^2\left(1 - e^{-2z\sqrt{\omega}J_3}\right) + J_2^2\left(1 + e^{-2z\sqrt{\omega}J_3}\right)\right], \\
\{K_2, P_2\} &= \frac{1}{2z}\left(\cosh(2z\sqrt{\omega}J_3) - e^{-2zP_0}\right) + \frac{z^3\omega^2}{4}e^{-2z\sqrt{\omega}J_3}\left(J_1^2 + J_2^2\right)^2 - \frac{z}{2}\left(P_1^2 + P_3^2 - P_2^2\right) \\
&\quad + \frac{z\omega}{2}\left[K_1^2 + K_3^2 - K_2^2 + J_1^2\left(1 + e^{-2z\sqrt{\omega}J_3}\right) + J_2^2\left(1 - e^{-2z\sqrt{\omega}J_3}\right)\right], \\
\{K_3, P_3\} &= \frac{1 - e^{-2zP_0}}{2z} + \frac{z}{2}\left[(P_1 + \sqrt{\omega}K_2)^2 + (P_2 - \sqrt{\omega}K_1)^2 - P_3^2 - \omega K_3^2\right] \\
&\quad + z\omega e^{-2z\sqrt{\omega}J_3}\left(J_1^2 + J_2^2\right), \\
\{P_1, K_2\} &= z\left(P_1P_2 + \omega K_1K_2 - \sqrt{\omega}P_3K_3 + \omega J_1J_2e^{-2z\sqrt{\omega}J_3}\right), \\
\{P_2, K_1\} &= z\left(P_1P_2 + \omega K_1K_2 + \sqrt{\omega}P_3K_3 + \omega J_1J_2e^{-2z\sqrt{\omega}J_3}\right), \\
\{P_1, K_3\} &= \frac{1}{2}\sqrt{\omega}J_1\left(1 - e^{-2z\sqrt{\omega}J_3}\left[1 - z^2\omega\left(J_1^2 + J_2^2\right)\right]\right) + z\left(P_1P_3 + \omega K_1K_3 + \sqrt{\omega}K_2P_3\right), \\
\{P_3, K_1\} &= \frac{1}{2}\sqrt{\omega}J_1\left(1 - e^{-2z\sqrt{\omega}J_3}\left[1 - z^2\omega\left(J_1^2 + J_2^2\right)\right]\right) + z\left(P_1P_3 + \omega K_1K_3 - \sqrt{\omega}P_2K_3\right), \\
\{P_2, K_3\} &= \frac{1}{2}\sqrt{\omega}J_2\left(1 - e^{-2z\sqrt{\omega}J_3}\left[1 - z^2\omega\left(J_1^2 + J_2^2\right)\right]\right) + z\left(P_2P_3 + \omega K_2K_3 - \sqrt{\omega}K_1P_3\right), \\
\{P_3, K_2\} &= \frac{1}{2}\sqrt{\omega}J_2\left(1 - e^{-2z\sqrt{\omega}J_3}\left[1 - z^2\omega\left(J_1^2 + J_2^2\right)\right]\right) + z\left(P_2P_3 + \omega K_2K_3 + \sqrt{\omega}P_1K_3\right),
\end{aligned}
$$

Quantum commutation rules

$$\{K_1, K_2\} = -\frac{\sinh(2z\sqrt{\omega}J_3)}{2z\sqrt{\omega}} - \frac{z\sqrt{\omega}}{2}\left(J_1^2 + J_2^2 + 2K_3^2\right) - \frac{z^3\omega^{3/2}}{4}\, e^{-2z\sqrt{\omega}J_3}\left(J_1^2 + J_2^2\right)^2$$

$$\{K_1, K_3\} = \frac{1}{2}J_2\left(1 + e^{-2z\sqrt{\omega}J_3}\left[1 + z^2\omega\left(J_1^2 + J_2^2\right)\right]\right) + z\sqrt{\omega}K_2K_3$$

$$\{K_2, K_3\} = -\frac{1}{2}J_1\left(1 + e^{-2z\sqrt{\omega}J_3}\left[1 + z^2\omega\left(J_1^2 + J_2^2\right)\right]\right) - z\sqrt{\omega}K_1K_3$$

$$\{P_1, P_2\} = -\omega\,\frac{\sinh(2z\sqrt{\omega}J_3)}{2z\sqrt{\omega}} - \frac{z\sqrt{\omega}}{2}\left(2P_3^2 + \omega(J_1^2 + J_2^2)\right) - \frac{z^3\omega^{5/2}}{4}e^{-2z\sqrt{\omega}J_3}\left(J_1^2 + J_2^2\right)^2$$

$$\{P_1, P_3\} = \frac{1}{2}\omega J_2\left(1 + e^{-2z\sqrt{\omega}J_3}\left[1 + z^2\omega\left(J_1^2 + J_2^2\right)\right]\right) + z\sqrt{\omega}P_2P_3$$

$$\{P_2, P_3\} = -\frac{1}{2}\omega J_1\left(1 + e^{-2z\sqrt{\omega}J_3}\left[1 + z^2\omega\left(J_1^2 + J_2^2\right)\right]\right) - z\sqrt{\omega}P_1P_3$$

Quantum casimir

The Poisson–deformed counterpart of the **second-order Casimir** reads

$$
\begin{aligned}
\mathcal{C} \;=\; & \frac{2}{z^2}\left[\cosh(zP_0)\cosh(z\sqrt{\omega}J_3)-1\right]+\omega\cosh(zP_0)(J_1^2+J_2^2)e^{-z\sqrt{\omega}J_3} \\
& -e^{zP_0}\left(\mathbf{P}^2+\omega\mathbf{K}^2\right)\left[\cosh(z\sqrt{\omega}J_3)+\frac{z^2\omega}{2}(J_1^2+J_2^2)e^{-z\sqrt{\omega}J_3}\right] \\
& +2\omega e^{zP_0}\left[\frac{\sinh(z\sqrt{\omega}J_3)}{\sqrt{\omega}}R_3+z\left(J_1R_1+J_2R_2+\frac{z\sqrt{\omega}}{2}(J_1^2+J_2^2)R_3\right)e^{-z\sqrt{\omega}J_3}\right],
\end{aligned}
$$

where $R_a=\epsilon_{abc}K_bP_c$.

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The Poisson–deformed counterpart of the **second-order Casimir** reads

$$\begin{aligned}\mathcal{C} \;=\; & \frac{2}{z^2}\left[\cosh(zP_0)\cosh(z\sqrt{\omega}J_3)-1\right]+\omega\cosh(zP_0)(J_1^2+J_2^2)e^{-z\sqrt{\omega}J_3}\\ & -e^{zP_0}\left(\mathbf{P}^2+\omega\mathbf{K}^2\right)\left[\cosh(z\sqrt{\omega}J_3)+\frac{z^2\omega}{2}(J_1^2+J_2^2)e^{-z\sqrt{\omega}J_3}\right]\\ & +2\omega e^{zP_0}\left[\frac{\sinh(z\sqrt{\omega}J_3)}{\sqrt{\omega}}R_3+z\left(J_1R_1+J_2R_2+\frac{z\sqrt{\omega}}{2}(J_1^2+J_2^2)R_3\right)e^{-z\sqrt{\omega}J_3}\right],\end{aligned}$$

where $R_a=\epsilon_{abc}K_bP_c$.

- In the $z\to 0$ limit, we obtain

$$\mathcal{C}=P_0^2-\mathbf{P}^2+\omega\left(\mathbf{J}^2-\mathbf{K}^2\right).$$

- In the $\omega\to 0$ limit, we obtain the κ-Poincaré quantum Casimir in the bicrossproduct basis:

$$\mathcal{C}=\frac{2}{z^2}\left[\cosh(zP_0)-1\right]-e^{zP_0}\mathbf{P}^2=\frac{4}{z^2}\sinh^2(zP_0/2)-e^{zP_0}\mathbf{P}^2$$

The twisted κ–Poincaré algebra in (3+1)

When $\omega \to 0$ we get a **twisted κ-Poincaré algebra** [25] [26] [27] generated by

$$r_{z,\vartheta} = z\,(K_1 \wedge P_1 + K_2 \wedge P_2 + K_3 \wedge P_3) + \vartheta J_3 \wedge P_0.$$

$$\begin{aligned}
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \qquad \Delta(J_3) = J_3 \otimes 1 + 1 \otimes J_3,\\
\Delta(J_1) &= J_1 \otimes 1 + \cos(\vartheta P_0) \otimes J_1 + \sin(\vartheta P_0) \otimes J_2,\\
\Delta(J_2) &= J_2 \otimes 1 + \cos(\vartheta P_0) \otimes J_2 - \sin(\vartheta P_0) \otimes J_1,\\
\Delta(P_1) &= P_1 \otimes 1 + e^{-zP_0} \cos(\vartheta P_0) \otimes P_1 + e^{-zP_0} \sin(\vartheta P_0) \otimes P_2,\\
\Delta(P_2) &= P_2 \otimes 1 + e^{-zP_0} \cos(\vartheta P_0) \otimes P_2 - e^{-zP_0} \sin(\vartheta P_0) \otimes P_1,\\
\Delta(P_3) &= P_3 \otimes 1 + e^{-zP_0} \otimes P_3,\\
\Delta(K_1) &= K_1 \otimes 1 + e^{-zP_0} \cos(\vartheta P_0) \otimes K_1 + e^{-zP_0} \sin(\vartheta P_0) \otimes K_2\\
&\quad + zP_2 \otimes J_3 - \vartheta P_1 \otimes J_3 - z\,(P_3 \cos(\vartheta P_0) \otimes J_2 - P_3 \sin(\vartheta P_0) \otimes J_1)\,,\\
\Delta(K_2) &= K_2 \otimes 1 + e^{-zP_0} \cos(\vartheta P_0) \otimes K_2 - e^{-zP_0} \sin(\vartheta P_0) \otimes K_1\\
&\quad - zP_1 \otimes J_3 - \vartheta P_2 \otimes J_3 + z\,(P_3 \cos(\vartheta P_0) \otimes J_1 + P_3 \sin(\vartheta P_0) \otimes J_2)\,,\\
\Delta(K_3) &= K_3 \otimes 1 + e^{-zP_0} \otimes K_3 - \vartheta P_3 \otimes J_3\\
&\quad + z\,(P_1 \cos(\vartheta P_0) \otimes J_2 - P_2 \cos(\vartheta P_0) \otimes J_1)\\
&\quad - z\,(P_1 \sin(\vartheta P_0) \otimes J_1 + P_2 \sin(\vartheta P_0) \otimes J_2)\,.
\end{aligned}$$

[25] J. Lukierski and V. Lyakhovsky, *Math. Phys. Contemp. Math.* **391** (2005) 281

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The twisted κ–Poincaré algebra

Deformed commutation rules are given by

$$
\begin{aligned}
&\{J_a, J_b\} = \epsilon_{abc} J_c, && \{J_a, P_b\} = \epsilon_{abc} P_c, && \{J_a, K_b\} = \epsilon_{abc} K_c,\\
&\{K_a, P_0\} = P_a, && \{K_a, K_b\} = -\epsilon_{abc} J_c, && \{P_0, J_a\} = 0,\\
&\{P_0, P_a\} = 0, && \{P_a, P_b\} = 0,
\end{aligned}
$$

$$
\{K_a, P_b\} = \delta_{ab}\left(\frac{1}{2z}\left(1 - e^{-2zP_0}\right) + \frac{z}{2}\mathbf{P}^2\right) - zP_aP_b,
$$

[28] S. Majid, H. Ruegg, *Phys. Lett. B* **334** (1994) 348

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$$\{K_a, P_b\} = \delta_{ab}\left(\frac{1}{2z}\left(1 - e^{-2zP_0}\right) + \frac{z}{2}\mathbf{P}^2\right) - zP_aP_b,$$

The deformed quadratic Casimir reduces to

$$\mathcal{C} = \frac{2}{z^2}\left[\cosh(zP_0) - 1\right] - e^{zP_0}\mathbf{P}^2 = \frac{4}{z^2}\sinh^2(zP_0/2) - e^{zP_0}\mathbf{P}^2.$$

All these expressions correspond to the (twisted) κ-Poincaré algebra **in the bicrossproduct basis.** [28]

[28] S. Majid, H. Ruegg, *Phys. Lett. B* **334** (1994) 348

6. Conclusions

Conclusions

- **Quantum gravity models with cosmological constant** should be considered in order to describe the interplay between quantum effects and cosmology. [29] [30] [31]

[29] A.B., F.J. Herranz, N.R. Bruno, arXiv:hep-th/0401244 (2004).

[30] A. Marciano, G. Amelino-Camelia, N.R. Bruno, G. Gubitosi, G. Mandanici, A. Melchiorri, J. Cosmol. Astropart. Phys. B 06 (2010) 030

[31] G. Amelino-Camelia G, Living Rev. Rel. 16 (2013), 5

[32] J. Kowalski-Glikman, Phys. Lett. B 547 (2002) 291

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[34] A.B., F.J. Herranz, P. Naranjo, Phys. Lett. B 746 (2015) 37

Conclusions

- **Quantum gravity models with cosmological constant** should be considered in order to describe the interplay between quantum effects and cosmology. [29] [30] [31]
- **Quantum groups with cosmological constant** incorporate many **new features** with respect to the flat (Poincaré) deformations:

  The cosmological constant would modify in an essential way both the associated **dispersion relations** and **curved momentum spaces**. [32] [33]

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- The role of twists seems to be outstanding in the DD setting. The (A)dS **κ-deformation** has to be **enlarged by a twist in order to be consistent** with a DD structure.

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- This DD construction can be **fully extended to (3+1) dimensions**. [34]

---

[29] A.B., F.J. Herranz, N.R. Bruno, arXiv:hep-th/0401244 (2004).

[30] A. Marciano, G. Amelino-Camelia, N.R. Bruno, G. Gubitosi, G. Mandanici, A. Melchiorri, J. Cosmol. Astropart. Phys. B 06 (2010) 030

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For detailed proofs and bibliography see
arXiv:1502.07518
arXiv:1408.3689
arXiv:1403.4773
arXiv:1402.2884
arXiv:1303.3080

THANKS FOR YOUR ATTENTION