Fermion production in magnetic fields on de Sitter Universe

Cosmin Crucean and Mihaela-Andreea Băloi

West University of Timisoara
Faculty of Physics

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Outline

Introduction

The line element of the de Siter expanding universe is:

$$ds^2 = dt^2 - e^{2\omega t} d\vec{x}^2, \tag{1}$$

where $\omega > 0$ is the expansion factor. On curved spacetimes the fields with spin $s = 1/2$ are defined fixing the local frames and the corresponding coframes with the tetrad fields $e_{\hat{\mu}}(x)$ and $\hat{e}^{\hat{\mu}}(x)$, which satisfy the orthonormalization relations [1]:

$$e_{\hat{\mu}} \cdot e_{\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}} \quad e^{\hat{\mu}} \cdot e^{\hat{\nu}} = \eta^{\hat{\mu}\hat{\nu}} \quad e^{\hat{\mu}} \cdot e_{\hat{\nu}} = \delta^{\hat{\mu}}_{\hat{\nu}}, \tag{2}$$

where $\hat{\mu}, \hat{\nu} = 0, 1, 2, 3$ and $\eta_{\hat{\mu}\hat{\nu}} = diag.(1, -1, -1, -1)$ is the Minkowski metric. For the line element (1) we can choose the Cartesian gauge with nonvanishing tetrad components [1]:

$$e^0_{\hat{0}} = e^{-\omega t} \qquad e^i_{\hat{j}} = \delta^i_{\hat{j}} e^{-\omega t} \tag{3}$$

Introduction

The positive/negative frequency solutions of the Dirac equation, in momentum-helicity basis on de Sitter geometry are [1]:

$$U_{\vec{p},\lambda}(t,\vec{x}\,) = \frac{\sqrt{\pi p/\omega}}{(2\pi)^{3/2}} \left(\begin{array}{c} \frac{1}{2} e^{\pi k/2} H^{(1)}_{\nu_-}(\frac{p}{\omega} e^{-\omega t}) \xi_\lambda(\vec{p}\,) \\ \lambda e^{-\pi k/2} H^{(1)}_{\nu_+}(\frac{p}{\omega} e^{-\omega t}) \xi_\lambda(\vec{p}\,) \end{array} \right) e^{i\vec{p}\cdot\vec{x}-2\omega t}$$

$$V_{\vec{p},\lambda}(t,\vec{x}\,) = \frac{\sqrt{\pi p/\omega}}{(2\pi)^{3/2}} \left(\begin{array}{c} -\lambda\, e^{-\pi k/2} H^{(2)}_{\nu_-}(\frac{p}{\omega} e^{-\omega t})\, \eta_\lambda(\vec{p}\,) \\ \frac{1}{2}\, e^{\pi k/2} H^{(2)}_{\nu_+}(\frac{p}{\omega} e^{-\omega t})\, \eta_\lambda(\vec{p}\,) \end{array} \right) e^{-i\vec{p}\cdot\vec{x}-2\omega t}, \qquad (4)$$

where $H^{(1,2)}_{\nu}(z)$ are the Hankel functions of the first/second kind, $k = \frac{m}{\omega}$ and $\nu_\pm = \frac{1}{2} \pm ik$. The unit normalized helicity spinors are given by:

$$\xi_{\frac{1}{2}}(\vec{p}\,) = \sqrt{\frac{p_3 + p}{2p}} \left(\begin{array}{c} 1 \\ \frac{p_1 + ip_2}{p_3 + p} \end{array} \right), \quad \xi_{-\frac{1}{2}}(\vec{p}\,) = \sqrt{\frac{p_3 + p}{2p}} \left(\begin{array}{c} \frac{-p_1 + ip_2}{p_3 + p} \\ 1 \end{array} \right), \qquad (5)$$

while $\eta_\sigma(\vec{p}\,) = i\sigma_2[\xi_\sigma(\vec{p}\,)]^*$.

[1] Ion I. Cotăescu, *Phys. Rev. D* **65** (2002).

Introduction

These spinors satisfy the following relation [1]:

$$\vec{\sigma}\vec{p}\,\xi_\lambda(\vec{p}\,) = 2p\lambda\xi_\lambda(\vec{p}\,) \tag{6}$$

with $\lambda = \pm 1/2$ and where $\vec{\sigma}$ are the Pauli matrices and $p = \mid \vec{p} \mid$ is the modulus of the momentum vector.

The vector potential which defines the dipolar magnetic field on de Sitter spacetime is obtained considering the conformal invariance of Maxwell equations i.e. $A^\mu_{dS} = \Omega^{-1} A^\mu_M$, where $\Omega = (\omega t_c)^2$ and $t_c = -\frac{e^{-\omega t}}{\omega}$ is the conformal time [2]:

$$\widehat{\vec{A}}(x) = \frac{\vec{m} \times \vec{x}}{|\vec{x}|^3} e^{-\omega t} \qquad\qquad A^{\widehat{0}}(x) = 0, \tag{7}$$

where $\vec{m}$ is the dipole magnetic moment. In (7) the hat indices indicate that we refer to the components in the local Minkowski frames $A^{\widehat{\mu}} = e^{\widehat{\mu}}_{\nu} A^\nu$.

[2] Ion I. Cotăescu, C. Crucean, *Progress of Theor. Phys.* **124** (2010).

The transition amplitude

The expression of the transition amplitude in the first order of perturbation theory is [3]:

$$\mathcal{A}_{e^- e^+} = -ie \int d^4x \sqrt{-g}\, \bar{U}_{\vec{p},\lambda}(x) \gamma_{\widehat{\mu}} A^{\widehat{\mu}}(x) V_{\vec{p}\,',\lambda'}(x). \tag{8}$$

where e is the unit charge of the field and $\sqrt{-g} = e^{3\omega t}$.
In our case $A^{\widehat{0}}(x) = 0$ and therefore the transition amplitude takes the following form:

$$\mathcal{A}_{e^- e^+} = -ie \int d^4x \sqrt{-g}\, \bar{U}_{\vec{p},\lambda}(x) \widehat{\vec{\gamma}} \widehat{\vec{A}}(x) V_{\vec{p}\,',\lambda'}(x). \tag{9}$$

After is replaced the solution of the Dirac equation (4) and the expression (7) of vector potential in (9), we obtain two types of integrals.

[3] Ion I. Cotăescu, C. Crucean, *Phys. Rev. D* **87** (2013).

The transition amplitude

The spatial integral can be solved as in Minkowski theory and has the following result:

$$\int d^3x \frac{\vec{x}}{|\vec{x}|^3}\, e^{-i(\vec{p}+\vec{p}\,')\vec{x}} = -\frac{4\pi i(\vec{p}+\vec{p}\,')}{|\vec{p}+\vec{p}\,'|^2}. \tag{10}$$

The temporal integral is solved using the new variable of integration $z = e^{-\omega t}/\omega$, obtaining integrals of the type:

$$\int_0^\infty dzz \left[-sgn(\lambda\lambda')e^{-\pi k} H^{(2)}_{\nu_-}(pz) H^{(2)}_{\nu_-}(p'z) + e^{\pi k} H^{(2)}_{\nu_+}(pz) H^{(2)}_{\nu_+}(p'z) \right]. \tag{11}$$

These integrals can be solved using the following relation between Hankel and Bessel K functions [7]:

$$H^{(1,2)}_{\nu}(z) = \mp \left(\frac{2i}{\pi}\right) e^{\mp i\pi\nu/2} K_\nu(\mp iz). \tag{12}$$

The transition amplitude

For computing the transition amplitude we use the following integral [7]:

$$\int_0^\infty dz z^{-\lambda} K_\mu(az) K_\nu(bz) = \frac{2^{-2-\lambda} a^{-\nu+\lambda-1} b^\nu}{\Gamma(1-\lambda)} \Gamma\left(\frac{1-\lambda+\mu+\nu}{2}\right)$$
$$\times \Gamma\left(\frac{1-\lambda-\mu+\nu}{2}\right) \Gamma\left(\frac{1-\lambda+\mu-\nu}{2}\right) \Gamma\left(\frac{1-\lambda-\mu-\nu}{2}\right)$$
$$\times {}_2F_1\left(\frac{1-\lambda+\mu+\nu}{2}, \frac{1-\lambda-\mu+\nu}{2}; 1-\lambda; 1-\frac{b^2}{a^2}\right),$$
$$Re(a+b) > 0\,, Re(\lambda) < 1 - |Re(\mu)| - |Re(\nu)|. \tag{13}$$

and the identity between the Gauss hypergeometric functions [7]:

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a,b;a+b-c+1;1-z) \\ &+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\ &\times {}_2F_1(c-a,c-b;c-a-b+1;1-z). \end{aligned} \tag{14}$$

The transition amplitude

The final result of the transition amplitude reads:

$$\begin{aligned}\mathcal{A}_{e^-e^+} &= -\frac{ie}{4\pi^3|\vec{p}+\vec{p}\,'|^2}\Bigg\{\frac{\theta(p-p\,')}{p}\left[f_k^*\left(\frac{p\,'}{p}\right)-sgn(\lambda\lambda\,')f_k\left(\frac{p\,'}{p}\right)\right]\\ &+\frac{\theta(p\,'-p)}{p\,'}\left[f_k^*\left(\frac{p}{p\,'}\right)-sgn(\lambda\lambda\,')f_k\left(\frac{p}{p\,'}\right)\right]\Bigg\}\\ &\times\xi_\lambda^+(\vec{p})[\vec{\sigma}\cdot(\vec{m}\times(\vec{p}+\vec{p}\,'))]\eta_{\lambda\,'}(\vec{p}\,'), \end{aligned} \tag{15}$$

where the functions $f_k\left(\frac{p}{p\,'}\right)$ have the following expressions:

$$\begin{aligned} f_k\left(\frac{p\,'}{p}\right) &= \left(\frac{p\,'}{p}\right)^{1-ik}\Gamma\left(\frac{3}{2}-ik\right)\Gamma\left(\frac{1}{2}+ik\right)\\ &\times {}_2F_1\left(1,\frac{3}{2}-ik;2;1-\left(\frac{p\,'}{p}\right)^2\right)\\ &= \frac{\pi}{\cosh(\pi k)\left(1-\left(\frac{p\,'}{p}\right)^2\right)}\left(\left(\frac{p\,'}{p}\right)^{ik}-\left(\frac{p\,'}{p}\right)^{1-ik}\right). \end{aligned} \tag{16}$$

Probability of transition

The probability of transition has the following expression:

$$
\begin{aligned}
\mathcal{P}_{e^-e^+} &= \frac{1}{2}\sum_{\lambda\lambda'}|\mathcal{A}_{e^-e^+}|^2 = \frac{1}{2}\sum_{\lambda\lambda'}\frac{e^2}{16\pi^6}\frac{1}{|\vec{p}+\vec{p}\,'|^4} \\
&\times\left\{\frac{\theta(p-p')}{p^2}\left[2\left|f_k\left(\frac{p'}{p}\right)\right|^2 - sgn(\lambda\lambda')\left(f_k^2\left(\frac{p'}{p}\right)+f_k^{*2}\left(\frac{p'}{p}\right)\right)\right]\right. \\
&\left.+\frac{\theta(p'-p)}{p'^2}\left[2\left|f_k\left(\frac{p}{p'}\right)\right|^2 - sgn(\lambda\lambda')\left(f_k^2\left(\frac{p}{p'}\right)+f_k^{*2}\left(\frac{p}{p'}\right)\right)\right]\right\} \\
&\times|\xi^+_\lambda(\vec{p})[\vec{\sigma}\cdot(\vec{m}\times(\vec{p}+\vec{p}\,'))]\eta_{\lambda'}(\vec{p}\,')|^2. \qquad (17)
\end{aligned}
$$

Helicity bispinors summation

Let $\{\vec{e_i}\}$ be an orthogonal frame, $i = 1, 2, 3$.
Then the spherical coordinates of the electron and positron momenta are $\vec{p} = (p, \alpha, \beta)$ and $\vec{p}\,' = (p', \gamma, \varphi)$ with $\alpha, \gamma \in (0, \pi)$ and $\beta, \gamma \in (0, 2\pi)$.
We also fix the magnetic moment on the $\vec{e}_3$ direction such that $\vec{m} = m\,\vec{e}_3$.
This choice allows us to express the vectorial product in the form:

$$\vec{\sigma} \cdot (\vec{m} \times (\vec{p} + \vec{p}\,')) = m((p_1 + p_1')\sigma_2 - (p_2 + p_2')\sigma_1), \tag{18}$$

from which we observe that the momenta $\vec{p}$, $\vec{p}\,'$ have components only in the plane $(1, 2)$. Knowing that in the spherical coordinates the components of momenta are:

$$\begin{aligned} p_1 &= p \sin\alpha \cos\beta \ ; \qquad & p_1' &= p' \sin\gamma \cos\varphi \\ p_2 &= p \sin\alpha \sin\beta \ ; \qquad & p_2' &= p' \sin\gamma \sin\varphi, \end{aligned} \tag{19}$$

and using the Pauli matrices $\sigma_1\,, \sigma_2$, we can explicitly compute the summation in the probability.

Helicity bispinors summation

Further we take $\beta = \pi$ and $\varphi = 0$, then $\theta_{pp'} = \alpha + \gamma$.

1. The case when helicity is conserved $\lambda = -\lambda'$

$$\mathcal{P}_{e^- e^+} \sim 2m^2 \frac{(p\,\sin\alpha - p'\sin\gamma)^2}{(p^2 + p'^2 + 2pp'\cos(\alpha+\gamma))^2} \sin^2\left(\frac{\alpha+\gamma}{2}\right) \tag{20}$$

For $\theta_{pp'} = \pi$ the probability is maximum. The fermion pair is emitted perpendicular to the direction of the dipolar magnetic field and the momenta of the fermions are parallel and opposite as orientation.

For $\theta_{pp'} = 0$ the probability is vanishing.

2. The case when helicity is not conserved $\lambda = \lambda'$

$$\mathcal{P}_{e^- e^+} \sim 2m^2 \frac{(p\,\sin\alpha - p'\sin\gamma)^2}{(p^2 + p'^2 + 2pp'\cos(\alpha+\gamma))^2} \cos^2\left(\frac{\alpha+\gamma}{2}\right). \tag{21}$$

For $\theta_{pp'} = 0$ the probability is maximum. The fermion pair is emitted perpendicular to the direction of the dipolar magnetic field and the momenta of the fermions are parallel and have the same orientation.

For $\theta_{pp'} = \pi$ the probability is vanishing.

Graphical results

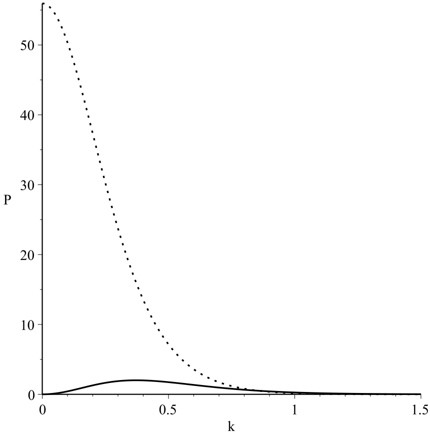

Figure: Probability as function of parameter k, for $p/p' = 0.4$ and $p/p' = 0.1$. The dashed line represents the case of helicity conservation and the solid line the case when helicity is not conserved.

Graphical results

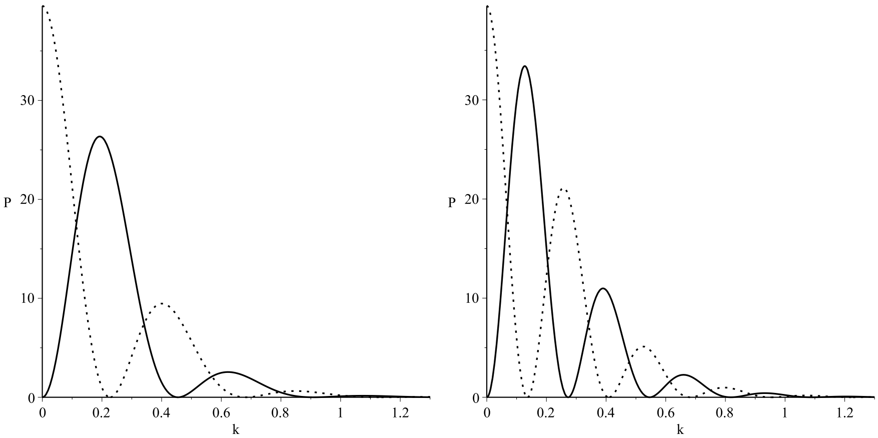

Figure: Probability as function of parameter k, for $p/p' = 0.001$ and $p/p' = 0.00001$. The dashed line represents the case of helicity conservation and the solid line the case when helicity is not conserved.

Limit cases

1. The case when $k = \frac{m}{\omega} \to \infty$; Minkowski limit

The functions $f_k\left(\frac{p'}{p}\right)$ that define our amplitude (15) vanish in this limit $\mathcal{A}_{e^-e^+}|_{k=\infty} = \mathcal{A}_{e^-e^+}(flat) = 0$, since it depends on the factor $\cosh^{-1}(\pi k)$ which is highly convergent for large k.

2. The case when $k = \frac{m}{\omega} \to 0$; Null mass limit

Setting $k = 0$ in the functions $f_k\left(\frac{p'}{p}\right)$ that define the amplitude we obtain:

$$f_0\left(\frac{p'}{p}\right) = \frac{\pi}{1 + \frac{p'}{p}}. \tag{22}$$

The transition probability in the null mass case has the expression:

$$\mathcal{P}_{e^-e^+}|_{(k=0)} = \sum_{\lambda} \frac{e^2}{2\pi^4|\vec{p} + \vec{p}\,'|^4} \frac{1}{(p + p')^2} \left|\xi^+_{\lambda}(\vec{p})[\vec{\sigma}\cdot(\vec{m}\times(\vec{p}+\vec{p}\,'))]\eta_{-\lambda}(\vec{p}\,')\right|^2. \tag{23}$$

Conclusion

1. The process *vacuum* $\rightarrow e^- + e^+$ in the presence of an external magnetic field was studied using perturbative methods.

2. The most probable transitions are those in which the fermion pair is emitted perpendicular to the direction of the magnetic field.

3. In the helicity conservation case the probability is maximum when $\theta_{pp'} = \pi$. In this case the fermion pair could separate.

4. In the helicity non-conservation case the probability is maximum when $\theta_{pp'} = 0$. In this case the fermion pair annihilates into the vacuum.

5. It was shown that this process is significant only in the strong gravitational fields of the early Universe.

6. For $\omega = 0$ it was recovered the Minkowski limit where this process is forbidden by the energy-momentum conservation laws.

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