Symmetry reductions in loop quantum gravity

based on classical gauge fixings

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Plan of the talk

1. Approaches to symmetry reductions
2. General strategy
3. Bianchi I: Details on classical derivation
4. Bianchi I: Details on quantum theory
5. Spherical symmetry (sketch)
6. Conclusion

Outline

Proposals for a symmetry reduced quantum theory in LQG

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- Mini / midi-superspace quantisation
  - LQC [Bojowald '99-; Ashtekar, Bojowald, Lewandowski '03; ...]
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  - Weave states [Ashtekar, Rovelli, Smolin '92; Bombelli '00]
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- Code symmetry as $f(p,q)=0$, impose $\widehat{f(p,q)}\,|\Psi\rangle_{\mathrm{sym}}=0 \leftarrow$ this talk
  - Bianchi I models [NB '14]
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Outline

Recipe for canonical loop quantisation

Hamiltonian connection formulation of a gravitational theory

Metric $\rightarrow$ **connection** with gauge symmetry under a local gauge group (e.g. SU(2))

+ certain mathematical properties (avoid with new vacua? [Dittrich, Geiller '14; Bahr, Dittrich, Geiller '15])

- compact gauge group
- real variables
- canonical Poisson brackets
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$\Downarrow$

Methods of
Algebraic quantum field theory
(C^*-algebras, GNS-construction, ...)

Kinematically quantised gravitational theory

[Rovelli, Smolin, Ashtekar, Isham, Lewandowski, Marolf, Mourao, Thiemann, Sahlmann, ...]

- Regularise Hamiltonian constraint [Thiemann '96; ...]
- Compute quantum constraint algebra [Thiemann '96; ...; Varadarajan, Laddha et al. '11-]

General strategy for the symmetry reduction

1. **Suitable classical starting point**
    - Gauge fix spatial diffeomorphisms adapted to the symmetry reduction
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   - Find phase space functions $f_i(p, q) = 0$ in the symmetric subspace
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   - Find subspace of quantum reduced states $\left|\Psi\right\rangle_{\text{sym}}$
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5. **Relate observables** $\hat{\mathcal{O}}_{\text{sym}}$ to mini- / midisuperspace parameters
   - Map observables
   - Study dynamics

Outline

Classical preparations I: Phase space

Gauge fixing to obtain suitable coordinates

1. Start with ADM phase space $\{q_{ab}(\sigma), P^{cd}(\sigma')\} = \delta^{(3)}(\sigma, \sigma')\delta^c_{(a}\delta^d_{b)}$
2. Impose **diagonal metric** gauge $q_{a\neq b} = 0 \quad \Leftrightarrow \quad q = \mathrm{diag}(q_{xx}, q_{yy}, q_{zz})$

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4. Coordinatise the reduced phase space via $q_{xx}, q_{yy}, q_{zz}, P^{xx}, P^{yy}, P^{zz}$
5. Solve $C_a = 0$ for $P^{a\neq b} \quad \Rightarrow \quad P^{a\neq b}(q_{aa}, P^{bb}) \quad \Rightarrow \quad$ insert in Hamiltonian

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Choose connection type variables

1. Define $e_a e_a = q_{aa}$, $e_a e^a = 1$, without summation, and $E^a = \sqrt{\det q}\, e^a$
2. Define $K_a = K_{ab} e^b$ with K_{ab} being the extrinsic curvature constructed form P^{ab}
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   $\Rightarrow$ Abelian gauge theory (Poisson bracket of Maxwell theory)

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At this stage, only Hamiltonian constraint and reduced spatial diffeomorphisms left.

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Result:

Direct consequences of a Bianchi I reduction can be imposed as spatial diffeomorphisms and a Gauß law on the (quantised) reduced phase space (as operator equations).

Classical preparations III: Summary

Phase space: (full GR admitting diagonal metric gauge)

1. $K_a(\sigma)$, $E^b(\sigma)$ are $3+3$ canonical variables per spatial point σ
2. Remaining constraints are
   1. reduced spatial diffeomorphisms (preserving the diagonal gauge)
   2. Hamiltonian constraint

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Strategy:

1. Quantise full phase space via LQG techniques
2. Impose symmetry reduction by imposing $\tilde{C}_a = 0 = G$ at the quantum level

Outline

Quantisation I: Full theory in diagonal gauge

Standard LQG type quantisation

1. Compute holonomies $h_\gamma^\lambda(K) := \exp\left(i\lambda \int_\gamma K_a \, ds^a\right)$ and fluxes $E(S) = \int_S E^a \, d^2 s_a$
   γ path, S surface, $\lambda \in \mathbb{Z}$ for U(1), or $\lambda \in \mathbb{R}$ for $\mathbb{R}_{\text{Bohr}}$ see e.g. [Corichi, Krasnov '97] for U(1)

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2. Define positive linear Ashtekar-Lewandowski functional on holonomy-flux algebra
3. Representation follows from the GNS construction: Hilbertspace $= L^2(\bar{\mathcal{A}}, d\mu_{\text{AL}})$
   $\bar{\mathcal{A}}$ = generalised U(1) or $\mathbb{R}_{\text{Bohr}}$ connections

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Remarks

- For $\mathbb{R}_{\text{Bohr}}$: $\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} dx\, f(x) = \int_{\mathbb{R}_{\text{Bohr}}} d\mu_{\text{H}}\, f(x)$ provides **normalised** and translation invariant Haar measure $\Rightarrow$ per edge: $\mathcal{H} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{H}})$

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2. Define positive linear Ashtekar-Lewandowski functional on holonomy-flux algebra
3. Representation follows from the GNS construction: Hilbertspace $= L^2(\bar{\mathcal{A}}, d\mu_{\text{AL}})$
   $\bar{\mathcal{A}} =$ generalised U(1) or $\mathbb{R}_{\text{Bohr}}$ connections

Remarks

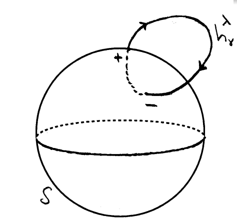
- For $\mathbb{R}_{\text{Bohr}}$: $\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} dx\, f(x) = \int_{\mathbb{R}_{\text{Bohr}}} d\mu_{\text{H}}\, f(x)$ provides **normalised** and translation invariant Haar measure $\Rightarrow$ per edge: $\mathcal{H} = L^2(\mathbb{R}_{\text{Bohr}}, d\mu_{\text{H}})$
- Choosing $\lambda \in \mathbb{Z}$ over $\lambda \in \mathbb{R}$ (i.e. compactifying $\int_\gamma K_a ds^a$) has no justification at this stage (also not later)

Quantisation II: Area operator

Area operator for Abelian theory

- $A(S) = |E(S)| = |\int_S E^a \, d^2 s_a|$ is analogous to (absolute value of) electric flux
- Important difference to non-Abelian, e.g. SU(2), area op. $\int_S \sqrt{|E^i E_i|}$:
  - \- Absolute value is **outside** of the integral
  - \- $E(S)$ does not detect closed contractible loops for closed S

    While one can also define "non-Abelian like" area operator here, the Abelian one will turn out to be most useful.



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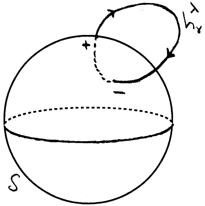
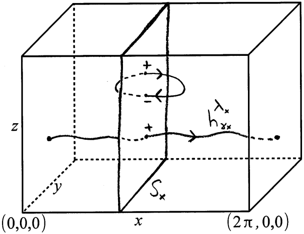
    While one can also define "non-Abelian like" area operator here, the Abelian one will turn out to be most useful.
  - - Non-trivial topology: $A(S)$ can detect Wilson loops even for closed S





$x \sim x + 2\pi, \quad y \sim y + 2\pi, \quad z \sim z + 2\pi$

$\hat{A}(S)$ measures intersection number N_{int} × rep. label: $\hat{A}(S) \left| h_\gamma^\lambda \right\rangle = |N_{\text{int}} \lambda| \left| h_\gamma^\lambda \right\rangle$

Quantisation III: Imposing the symmetry reduction

Reduction constraints are very familiar from full theory

1. All spatial diffeomorphisms: $\tilde{C}_a[N^a] = \int_\Sigma d^3\sigma \, E^a \mathcal{L}_{\vec{N}} K_a = 0$
2. Abelian Gauß law: $G[\omega] = \int_\Sigma d^3\sigma \, \omega \, \partial_a E^a = 0$

$\Rightarrow$ **spatially diffeomorphism invariant** and **gauge invariant** charge (spin) networks!

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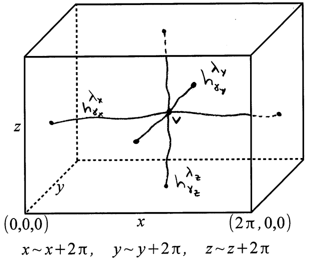
Simplest choice of quantum state

Consider spin network made from 3 Wilson loops wrapping around $\mathbb{T}^1_x, \mathbb{T}^1_y, \mathbb{T}^1_z$, meeting in a single vertex v. [c.f. Husain '91, '05]

Mapping to Bianchi I LQC states of [Ashtekar, Wilson-Ewing '09] $|\lambda_x, \lambda_y, \lambda_z\rangle \mapsto |p_1, p_2, p_3\rangle$



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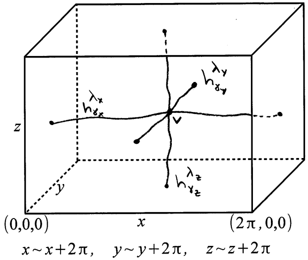

Observables w.r.t. the reduction constraints

1. Area of closed surfaces $\rightarrow$ 3 non-trivial areas $A(\mathbb{T}^2_x), A(\mathbb{T}^2_y), A(\mathbb{T}^2_z)$
2. Diff-equiv. classes of Wilson loops $\rightarrow$ 3 non-trivial closed loops along $\mathbb{T}^1_x, \mathbb{T}^1_y, \mathbb{T}^1_z$

Quantisation IV: Dynamics

Hamiltonian constraint / true Hamiltonian (via deparametrisation)

Take original Hamiltonian:

- Evaluate at $q_{a \neq b} = 0$ because of gauge fixing
- Discard $P^{a \neq b}$, $\partial_a e_b$, and $\partial_a K_b$ terms because of reduction constraints

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- Substitute e_a either by fluxes or Thiemann's trick $e_a = 2\{K_a, V\}$
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- $\mathbb{R}_{\text{Bohr}}$ allows arbitrarily small $\lambda \in \mathbb{R}$ for better approximation.
  "improved" LQC choice: $1/\lambda_x = \sqrt{|E^y E^z / E^x|}$ = size of universe in x-direction
  $\Rightarrow$ "new" LQC dynamics [Ashtekar, Pawlowski, Singh '06; Ashtekar, Wilson-Ewing '09]

Outline

Classical preparations for reduction to spherical symmetry

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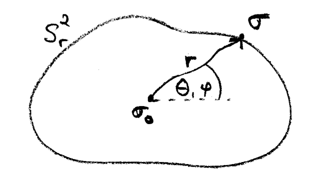

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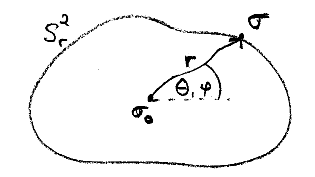

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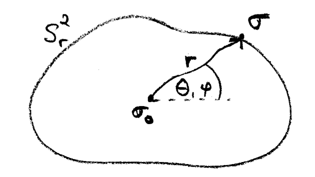

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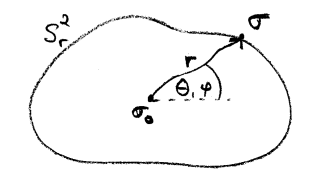

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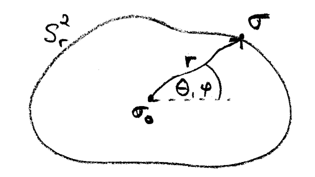

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Find conditions compatible with spherical symmetry

- $P^{rA} = 0 \quad \Leftrightarrow \quad$ generator of all spatial diffeomorphisms preserving all S^2_r
  (Follows form non-existence of non-zero spherically symmetric vector field on S^2)

Impose spherical symmetry as invariance under S^2_r-preserving diffeomorphisms.
(These are **active** diffeomorphisms with respect to the (r, θ, ϕ) coordinate system)

Quantisation and reduction to spherical symmetry

Perform standard LQG-type quantisation (roughly similar to lattice field theory)

1. SU(2) gauge theory with holonomies restricted to lie in an S_r^2
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1. Areas of the S^2_r $\rightarrow$ $4\pi R(r)^2 := \int_{S^2_r} d^2x \sqrt{\det q_{AB}}$
2. Averaged trace of momenta $\rightarrow$ $P_R(r) := \frac{2}{R(r)} \int_{S^2_r} d^2x \, P^{AB} q_{AB}$

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What about dynamics?

More challenging than for Bianchi I, ongoing work with A. Zipfel. First steps

- Map states in classically reduced $\rightarrow$ quantum reduced theory
- Compute quantum algebra $[\hat{R}(r), \hat{P}_R(r')]$ from full theory

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Conclusion: Proposed reduction programme successful

1. **Suitable classical starting point**
   - Bianchi I: ADM in diagonal metric gauge
   - Sph. sym.: ADM in radial gauge
2. **Identification of constraints** imposed by symmetry reduction
   - Bianchi I: all spatial diffeomorphisms and Abelian Gauß law
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5. **Relate observables** w.r.t. reduction constraints to mini- / midisuperspace
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