Results and applications of canonical effective methods in quantum theories

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Quantum gravity and general relativity

Effective theory well-developed for perturbative approaches on a given space-time: Low-energy effective action.

New ingredients for background-independent quantum gravity:

→ Canonical formulation in some approaches.

→ Meaning of “low energy”? Vacuum problem.

→ Quantum space-time structure:
Form of covariance to be derived.
Terms to be included in effective action?

Canonical effective theory

→ Useful not only for canonical quantum gravity:

- Non-associative algebra of observables.
- No vacuum required.

  But then more freedom in parameters,
  role of symmetries weaker.

→ Loop quantum gravity: Surprises in Planck regime.

Space-time structure can change dramatically:
signature change.

New picture of evaporating black holes:

Unexpected Cauchy horizons.

Canonical effective theory: Starting point

Ehrenfest equations

$$\frac{\mathrm{d}\langle\hat{x}\rangle}{\mathrm{d}t} = \frac{\langle\hat{p}\rangle}{m}$$

$$\frac{\mathrm{d}\langle\hat{p}\rangle}{\mathrm{d}t} = -\langle V'(\hat{x})\rangle = -V'(\langle\hat{x}\rangle) - \frac{1}{2}V'''(\langle\hat{x}\rangle)(\Delta x)^2 + \cdots$$

Fluctuation dynamical:

$$\frac{\mathrm{d}(\Delta x)^2}{\mathrm{d}t} = \frac{2}{m}C_{xp}$$

proportional to covariance.

Infinitely many coupled ordinary differential equations for expectation values and moments

$$\langle(\hat{x}-\langle\hat{x}\rangle)^a(\hat{p}-\langle\hat{p}\rangle)^b\rangle_{\mathrm{symm}}$$

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Effective phase space

Parameterize state by expectation values $\langle\hat{q}\rangle$ and $\langle\hat{p}\rangle$ of basic operators and moments

$$G^{a,n} = \langle(\hat{q}-\langle\hat{q}\rangle)^{n-a}(\hat{p}-\langle\hat{p}\rangle)^a\rangle_{\rm symm}$$

Commutator of operators determines Poisson bracket of moments:

$$\{\langle\hat{A}\rangle,\langle\hat{B}\rangle\} = \frac{\langle[\hat{A},\hat{B}]\rangle}{i\hbar}$$

Hamiltonian evolution for $\langle\hat{q}\rangle$, $\langle\hat{p}\rangle$ and $G^{a,n}$ generated by

$$\begin{aligned}\langle\hat{H}\rangle(\langle\hat{q}\rangle,\langle\hat{p}\rangle,G^{a,n}) &= \langle H(\langle\hat{q}\rangle+(\hat{q}-\langle\hat{q}\rangle),\langle\hat{p}\rangle+(\hat{p}-\langle\hat{p}\rangle))\rangle \\ &= H(\langle\hat{q}\rangle,\langle\hat{p}\rangle)+\sum_{n=2}^{\infty}\sum_{a=0}^{n}\frac{1}{a!(n-a)!}\frac{\partial^n H(\langle\hat{q}\rangle,\langle\hat{p}\rangle)}{\partial\langle\hat{q}\rangle^{n-a}\partial\langle\hat{p}\rangle^a}G^{a,n}\end{aligned}$$

Equations of motion

Anharmonic oscillator $V(q) = \frac{1}{2}m\omega^2 q^2 + U(q)$:

$$
\begin{aligned}
\dot{q} &= \frac{p}{m} \\
\dot{p} &= -m\omega^2 q - U'(q) - \sum_n \frac{1}{n!}\left(\frac{\hbar}{m\omega}\right)^{n/2} U^{(n+1)}(q)\tilde{G}^{0,n} \\
\dot{\tilde{G}}^{a,n} &= -a\omega\tilde{G}^{a-1,n} + (n-a)\omega\tilde{G}^{a+1,n} - a\frac{U''(q)}{m\omega}\tilde{G}^{a-1,n} \\
&\quad + \frac{\sqrt{\hbar}aU'''(q)}{2(m\omega)^{\frac{3}{2}}}\tilde{G}^{a-1,n-1}\tilde{G}^{0,2} + \frac{\hbar aU''''(q)}{3!(m\omega)^2}\tilde{G}^{a-1,n-1}\tilde{G}^{0,3} \\
&\quad - \frac{a}{2}\left(\frac{\sqrt{\hbar}U'''(q)}{(m\omega)^{\frac{3}{2}}}\tilde{G}^{a-1,n+1} + \frac{\hbar U''''(q)}{3(m\omega)^2}\tilde{G}^{a-1,n+2}\right) + \cdots
\end{aligned}
$$

∞ly many coupled equations for ∞ly many variables.

Low-energy effective action

To second adiabatic order, first order in $\hbar$:

$$\left(m+\frac{\hbar U'''(q)^2}{32m^2\omega^5\left(1+\frac{U''(q)}{m\omega^2}\right)^{5/2}}\right)\ddot{q}$$
$$+\frac{\hbar\dot{q}^2\left(4m\omega^2U'''(q)U''''(q)\left(1+\frac{U''(q)}{m\omega^2}\right)-5U'''(q)^3\right)}{128m^3\omega^7\left(1+\frac{U''(q)}{m\omega^2}\right)^{7/2}}$$
$$+m\omega^2q+U'(q)+\frac{\hbar U'''(q)}{4m\omega\left(1+\frac{U''(q)}{m\omega^2}\right)^{1/2}}=0\,.$$

as it results from

$$\Gamma_{\rm eff}[q(t)] = \int {\rm d}t\left(\frac{1}{2}\left(m+\frac{\hbar U'''(q)^2}{2^5m^2\left(\omega^2+m^{-1}U''(q)\right)^{5/2}}\right)\dot{q}^2 -\frac{1}{2}m\omega^2q^2-U(q)-\frac{\hbar\omega}{2}\left(1+\frac{U''(q)}{m\omega^2}\right)^{1/2}\right)$$

Vacuum

Harmonic oscillator: $\hat{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{q}^2$ implies effective Hamiltonian

$$\langle\hat{H}\rangle = \frac{1}{2m}\langle\hat{p}\rangle^2 + \frac{1}{2}m\omega^2\langle\hat{q}\rangle^2 + \frac{1}{2m}G^{2,0} + \frac{1}{2}m\omega^2 G^{0,2}$$

Moments solved for stationary states saturating uncertainty relation: effective potential

$$V_{\rm eff}(\langle\hat{q}\rangle) = \frac{1}{2}m\omega^2\langle\hat{q}\rangle^2 + \frac{1}{2}\hbar\omega\,.$$

Analog in quantum field theory: Coleman–Weinberg potential.

Effective potentials

Example: $V(\phi_1,\phi_2)=\lambda_1\phi_1^4+\lambda_2\phi_1^2\phi_2^2+\lambda_3\phi_3^4$, $m_1=m_2=m$.

$$
\begin{aligned}
V_{\rm eff}(\phi_1,\phi_2) &= V(\phi_1,\phi_2)\\
&+\frac{\hbar}{2\sqrt{m}}\left(\frac{2g^2}{f_1-f_2}+f_1\right)\sqrt{\frac{1}{f_1+g^2/(f_1-f_2)}\cdot\left(1-\frac{g^2/(f_2-f_1)}{\sqrt{f_1f_2-g^2}}\right)}\\
&+\frac{\hbar}{2\sqrt{m}}\left(\frac{2g^2}{f_2-f_1}+f_2\right)\sqrt{\frac{1}{f_2+g^2/(f_2-f_1)}\cdot\left(1-\frac{g^2/(f_1-f_2)}{\sqrt{f_1f_2-g^2}}\right)}
\end{aligned}
$$

with $f_1(\phi_1,\phi_2):=12\lambda_1\phi_1^2+2\lambda_2\phi_2^2$, $f_2(\phi_1,\phi_2):=12\lambda_3\phi_2^2+2\lambda_2\phi_1^2$
and $g(\phi_1,\phi_2):=4\lambda_2\phi_1\phi_2$.

Computational implementation in progress:
Large set of coupled linear and quadratic equations for moments at higher $\hbar$-orders.

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Algebraic states

Canonical effective equations rewrite

$$\frac{\mathrm{d}\langle\hat{O}\rangle}{\mathrm{d}t} = \frac{\langle[\hat{O},\hat{H}]\rangle}{i\hbar}$$

in terms of moments.

Need only expectation-value functional on operator algebra, no wave functions or Hilbert space.

→ Includes mixed states.

→ Link to algebraic quantum field theory.

Useful in context of problem of time:
Avoid physical Hilbert space.

→ Algebra may be non-associative.

Non-associative quantum mechanics

Why?

- → Magnetic monopoles.
  Canonical momentum $\vec{p} = m\vec{v} + q\vec{A}$ implies
  (twisted) Poisson brackets $\{mv_i, mv_j\} = q\epsilon_{ijk}B^k$.
  Jacobiator
  $$\epsilon^{ijk}\{\{v_i, v_j\}, v_k\} \propto \partial_l B^l$$
- → Chiral anomalies.
- → Double-field theory.

How?

- → Non-associative $*$-product. [Mylonas, Schupp, Szabo]
- → States on algebra. (Canonical effective theory.)
  [with Brahma, Büyükçam, Strobl]

Formalism

→ Existence of positive functionals on algebra?
(No GNS construction.)

→ Relations?

Alternative algebra with totally antisymmetric associator

$$[O_i, O_j, O_k] := (O_i O_j) O_k - O_i (O_j O_k)$$

does not seem possible:

- Commutators of composite observables more complicated.
- Axioms of Günaydin, Piron, Ruegg (1978) not available.

→ Eigenvalues and eigenstates?

Quantum space-time

Metric and matter perturbations on a background:

- → General covariance, gauge transformations of modes.
- → May fix the gauge. Field theory on a background.

General relativity: Gauge transformations and dynamics generated by the same phase-space functions (constraints).

Space-time structure: Hypersurface deformations as gauge.

- → Quantum gravity: Dynamics and gauge transformations for modes potentially quantum corrected.
- → Quantum field theory on a background: Quantize only the dynamics of modes, after gauge fixing or other method to restrict classical space-time transformations.

Can lead to very different outcomes.

Loop modification

Higher-curvature corrections in Friedmann equation:

$$\frac{8\pi G}{3}\rho = \mathcal{H}^2\left(1+O(\ell^2\mathcal{H}^2)\right)+O(\ell^2\dot{\mathcal{H}}^2)+\cdots$$

Loop quantum cosmology:

$$\frac{8\pi G}{3}\rho = \frac{\sin(\ell\bar{\mathcal{H}})^2}{\ell^2} = \mathcal{H}^2\left(1+O(\ell^2\mathcal{H}^2)\right)$$

Effective Friedmann equation ($\rho_{\rm QG}=3/(8\pi G\ell^2)$)

$$\mathcal{H}^2 = \frac{8\pi G}{3}\rho\left(1-\frac{\rho}{\rho_{\rm QG}}\right)+\cdots$$

Part of covariant theory? Evolution of inhomogeneity?

Anomaly problem

General relativity: System with several classical constraints, structure functions.

$$\{C_I, C_J\} = f_{IJ}^K C_K$$

with phase-space functions f_{IJ}^K.

Should be turned into operators $\hat{C}_I$ and $\hat{f}_{IJ}^K$ such that

$$[\hat{C}_I, \hat{C}_J] = i\hbar \hat{f}_{IJ}^K \hat{C}_K$$

in this ordering.

Just one problem: $\{f_{IJ}^K, C_L\} \neq 0$ in general relativity.

Constraint operators cannot be symmetric. [Komar]

Effective constraints

States annihilated by $\hat{C}_I$:

$$C_{I,\mathrm{pol}} = \langle \widehat{\mathrm{pol}} \hat{C}_I \rangle \approx 0$$

for all polynomials in a set of basic operators. May be complex.

Effective constraint algebra first class if $[\hat{C}_I, \hat{C}_J]$ first class:

$$\{\langle \widehat{\mathrm{pol}}_A \hat{C}_I \rangle, \langle \widehat{\mathrm{pol}}_B \hat{C}_J \rangle\} = \frac{\langle [\widehat{\mathrm{pol}}_A \hat{C}_I, \widehat{\mathrm{pol}}_B \hat{C}_J] \rangle}{i\hbar}$$

Classical structure functions $f_{IJ}^K(x_i)$ fully determine structure functions of $C_{I,1}$ (with $\widehat{\mathrm{pol}} = 1$):

$$\{C_{I,1}, C_{J,1}\} = \langle \hat{f}_{IJ}^K \hat{C}_K \rangle = f_{IJ}^K(\langle \hat{x}_i \rangle) C_{K,1} + \sum_j \frac{\partial f_{IJ}^K(\langle \hat{x}_i \rangle)}{\partial \langle \hat{x}_j \rangle} C_{K,x_j} + \cdots$$

Holonomies and space-time

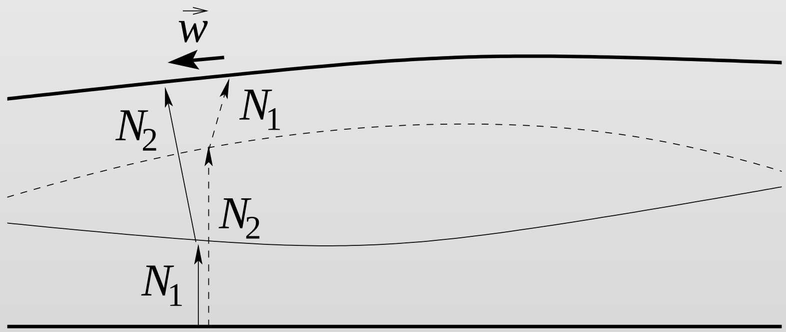
[Reyes; Barrau, Cailleteau, Grain, Mielczarek]

$\mathcal{H}^2 \longrightarrow f(\mathcal{H})$ anomaly-free if constraint algebra deformed to

$$\{H[N_1], H[N_2]\} = D[\beta h^{ab}(N_1\nabla_b N_2 - N_2\nabla_b N_1)]$$

with

$$\beta(\mathcal{H}) = \frac{1}{2}\frac{\mathrm{d}^2 f(\mathcal{H})}{\mathrm{d}\mathcal{H}^2}$$



Example: $\beta(\mathcal{H}) = \cos(2\ell\mathcal{H})$ for $f(\mathcal{H}) = \ell^{-2}\sin^2(\ell\mathcal{H})$.

→ Well-defined and consistent canonical effective theory.

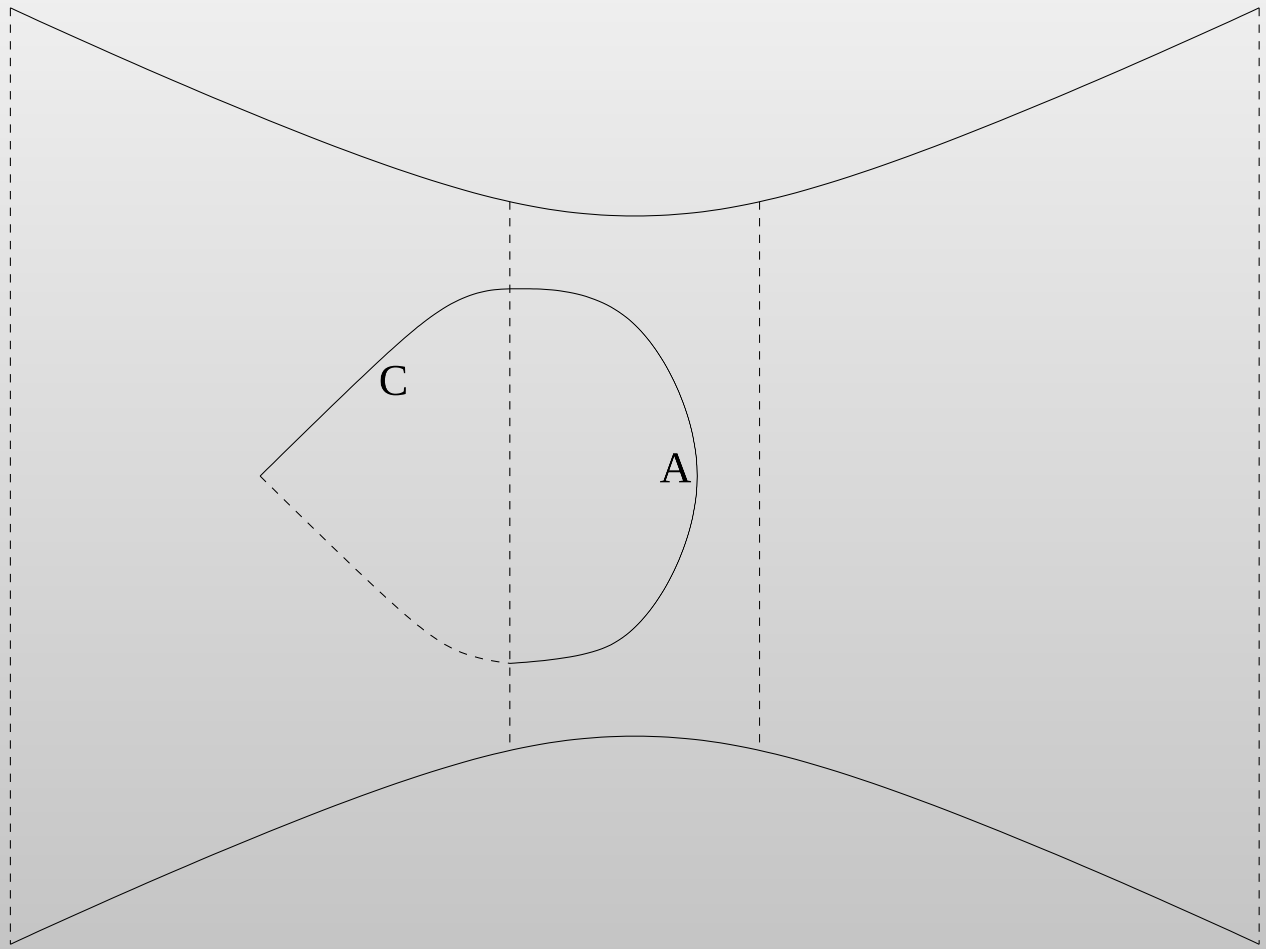
But no classical or Riemannian space-time
(unless field redefinition).

→ Signature change: $\beta(\mathcal{H}) < 0$ around maximum of $f(\mathcal{H})$.

Tricomi problem and cosmic boom

[with J. Mielczarek]

$-\frac{\partial^2 u}{\partial t^2} + \beta(\mathcal{H})\Delta u = 0$: Characteristic C connected to arc A.



Need future data: no deterministic evolution. Poles generic.

Information loss

Non-singular black-hole model:

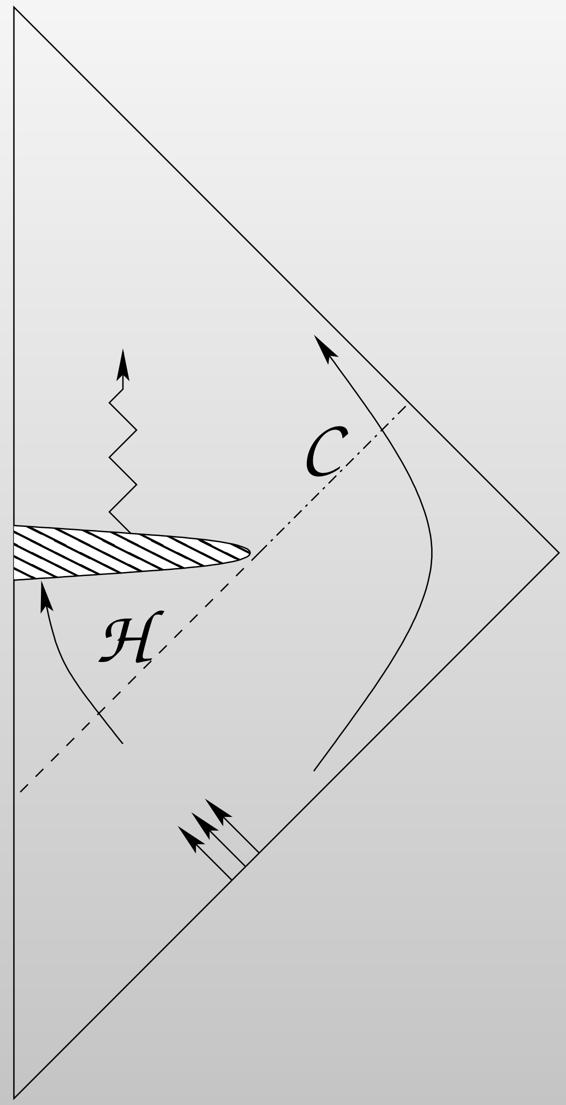
Evolve through classical singularity by quantum evolution of homogeneous interior.

No event horizon. [Ashtekar, MB 2006]

Anomaly-free space-time structure (spherically symmetric models):

- → High-curvature region Euclidean.
- → Arbitrary boundary values affect future space-time.
- → Event horizon $\mathcal{H}$ and Cauchy horizon $\mathcal{C}$.

No-heir theorem?



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Conclusions

Systematic effective framework exists for loop quantum gravity (and other fields), but much remains to be worked out.

Highlighted importance of anomaly problem.
Some surprising results, inequivalent to background treatment.

Signature change:

→ Instabilities in evolution picture.

  Solutions sensitive to initial data and precise equations of motion.

  Quantization ambiguities dangerous.

→ Global problems:

  - Cauchy horizons.
  - Isolated poles in generic solutions.
    Cosmological perturbation theory applicable?