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QUANTUM-DEFORMED COVARIANT PHASE SPACES AS HOPF ALGEBROIDS

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(in collaboration with M. Woronowicz and Z .Skoda; arXiv:1507.02612[hep-th])

1. QUANTUM SPACES, HOPF ALGEBRAS AND HEISENBERG ALGEBRA

Let us consider symmetries in three basic physical theories

Classical mechanics $\xrightarrow{\hbar \neq 0}$ QM +QFT without QG effects $\xrightarrow[\text{deformation}]{\text{quantum}}$ QM+QFT with QG effects

↖ ↗ ↗

classical Lie groups F(G) — Quantum Lie group $F(G_q)$

↕ dual (Fourier) — ↕ dual (Hopf)

classical Lie algebras $U(\widehat{g})$ — Quantum Lie algebras $U_q(\widehat{g})$

↑ ↑ ↑

usually matrix realizations — usually differential realizations — realizations on non-commutative Hopf algebra modules

Hopf algebras describe quantum (algebraic) generalization of notion of groups (noncommutative group parameters) and quantum generalization of Lie algebras.

$$H = (A, m, \Delta, \hat{\gamma}, \epsilon) \supset \text{bialgebra}\,(A, m, \Delta, \epsilon) \supset \text{algebra}(A, m)$$

$\Uparrow$

Hopf algebra

Basic property: coalgebra defined by coproducts $\Delta : A \to A \otimes A$ which describe homomorphisms of algebra A

Coproducts for classical Lie algebra generators $\widehat{g}$ primitive $(\Delta^{(0)}(\widehat{g}) = \widehat{g} \otimes 1 + 1 \otimes \widehat{g})$, for quantum Lie algebras noncocommutative $(\Delta^T(\widehat{g}) \neq \Delta(\widehat{g}))$

Coproducts for classical and quantum groups are chosen in Woronowicz – Faddeev quantum deformation scheme the same in classical and quantum cases

$$\Delta(G_{ij}) = G_{ik} \otimes G_{kj} \;\Leftrightarrow\; \Delta(G) = G \dot{\otimes} G \qquad \left(\begin{array}{l} G_{ij} \text{ – matrix reali-} \\ \text{zation of groups} \end{array} \right)$$

Quantum space algebra V $\leftrightarrow$ noncommutative vector space as generators of V

In standard physics – commutative representation spaces for classical Lie groups and Lie algebras (vector spaces, classical functions)

In quantum-deformed case – noncommutative Hopf algebra modules with the Hopf-algebraic action $h \triangleright x \in V$ $(h \in A, x \in V)$ with property

$$h \triangleright (x \cdot y) = (h_{(1)} \triangleright x) \cdot (h_{(2)} \triangleright y)$$

Hopf algebra modules – important class of quantum spaces described by noncommutative algebra V with defined Hopf-algebraic action (Hopf-algebraic symmetry properties buildt-in).

For physics (in theories with QG effects) important deformations of D=4 space-time symmetries, i.e. pairs of dual deformed Poincaré – Hopf algebras $H = \mathcal{U}_q(\widehat{p}_{3,1})$, $\widetilde{H} = G_q(\mathcal{P}_{3,1})$ (q – deformation parameter, not necessarily Drinfeld–Jimbo!)

Most popular examples: "canonical" $\theta_{\mu\nu}$-deformation and κ-deformation (in applications, e.g. in DSR theories, one postulates $\kappa \sim m_p$)

Problem: How to describe in this framework QM phase space (Heisenberg algebra) and its quantum deformations?

Standard (nondeformed) relativistic Heisenberg algebra with generators $(\widehat{x}_\mu, \widehat{p}_\nu)$:

$$[\widehat{x}_\mu, \widehat{x}_\nu] = [\widehat{p}_\mu, \widehat{p}_\nu] = 0 \qquad [\widehat{x}_\mu, \widehat{p}_\nu] = i\hbar\, g_{\mu\nu} \cdot 1$$

$\hbar$ is c-number; $g = \mathrm{diag}(1, 1, 1, -1)$. $\hbar$ as universal constant, a number, is required by postulates of QM – the same value for all QM states (Hilbert space vectors).

Canonical phase space contains a pair of Abelian Hopf algebras with

$$H_x: \ \Delta(\widehat{x}_\mu) = \widehat{x}_\mu \otimes 1 + 1 \otimes \widehat{x}_\mu \qquad H_p: \ \Delta(\widehat{p}_\mu) = \widehat{p}_\mu \otimes 1 + 1 \otimes \widehat{p}_\mu$$

Unfortunately it is easy to check that

$$[\Delta(\widehat{x}_\mu), \Delta(p_\nu)] = 2i\hbar\, g_{\mu\nu} 1 \otimes 1 \quad \leftarrow \text{lack of homomorphism!}$$

i.e. Heisenberg algebra is not a Hopf algebra.

Two ways out:

i) We replace $\hbar \cdot 1$ by a central charge $\hbar \cdot \widehat{c}$ with coproduct $\Delta \widehat{c} = \widehat{c} \otimes 1 + 1 \otimes \widehat{c}$. But it leads to various values of $\hbar$ for different quantum states, e.g. $\hbar \to n\hbar$ for n-particle states (for n=2 $\Delta(\hbar \cdot \widehat{c})|1> \otimes |2> = 2\hbar |1> \otimes |2>$) − unphysical!

ii) Homomorphism of coalgebra is cured if we can postulate e.g.

$$\Delta(\widehat{x}_\mu) = 1 \otimes \widehat{x} \qquad \Delta(\widehat{p}_\mu) = \widehat{p}_\mu \otimes 1 + 1 \otimes \widehat{p}_\mu$$

Mathematical justification of introducing such coproduct: replace bialgebras by bialgebroids.

2. FROM HOPF ALGEBRAS TO HOPF ALGEBROIDS

Algebraic part of Hopf algebroid has a particular structure:
total algebra A and base subalgebra $B \subset A$ $(h \in A, b \in B)$.
The algebroid structure is characterized by two important maps:

source algebra homomorphic map $B \to A$	:	$b \cdot h = s(b)h$	$s(b) \in h$
target antialgebra antihomomorphic map $B \to A^T$	:	$h \cdot b = t(b)h$	$t(b) \in h$

It appears that quantum phase space has the structure of Hopf bialgebroid

$$\mathcal{H} = (\underbrace{A, m}_{\text{algebra}}; \underbrace{B, s, t}_{\text{base algebra}}; \underset{\substack{\uparrow \\ \text{coproduct}}}{\Delta}, \underset{\substack{\uparrow \\ \text{antipode}}}{\widehat{\gamma}}, \epsilon)$$

Hopf algebra ≡ bialgebroid with antipode $\widehat{\gamma}$

Assignements in relativistic physics:

Heisenberg algebra: $A = (\widehat{x}_\mu, \widehat{p}_\nu)$ $\quad B = (\widehat{x}_\mu)$ (spinless dynamics)

Heisenberg double of Poincaré algebra: $A = (\widehat{x}_\mu, \widehat{\Lambda}_{\mu\rho}, \widehat{p}_\nu, \widehat{m}_{\nu\lambda})$

$B = (\widehat{x}_\mu, \widehat{\Lambda}_{\mu\rho})$ (spin $\neq 0$)

Important feature: source and target maps commute

$$[s(b), t(b')] = 0 \qquad \forall b, b' \in B$$

and introduce the algebra A as (B,B)-module (left–right B-module)

$$b \cdot h \cdot b' = s(b)t(b')h$$

The new tensor product $A \underset{\mathrm{B}}{\otimes} A$ over noncommutative ring B introduced by Takeuchi (1977) has a (B,B) bimodule structure and can be obtained from standard tensor products if we factorize standard tensor product $A \otimes A$ by left ideal (we choose right bialgebroid)

$$I_L = s(b) \otimes 1 - 1 \otimes t(b) \qquad b \in B$$

i.e. $A \underset{\mathrm{B}}{\otimes} A$ defined by the equivalence classes of $\{h \otimes h'\}_B$ in $A \otimes A$ using the rule

$$h \underset{\mathrm{B}}{\otimes} h' \simeq \{h \otimes h'\}_B \;\text{ iff }\; I_L \circ (h \otimes h') = s(b)h \otimes h' - h \otimes t(b)h' = 0 \quad \forall b \in B$$

The ideal generates nonuniqueness in terms of standard tensor product $A \otimes A$ consistent with the homomorphism property of the map $A \to A \otimes A$. Such freedom defines for bialgebroid the choice of coproduct gauges.

Antipode $\widehat{\gamma}$ is an antialgebra map $\widehat{\gamma} \triangleright h_1 h_2 = (\widehat{\gamma} \triangleright h_2)(\widehat{\gamma} \triangleright h_1)$ with the condition

$$\widehat{\gamma} \triangleright s(b) = t(b)$$

One can choose natural inclusion for the source map: $s(b) = b$. In such a case one gets $t(b) = \widehat{\gamma} \triangleright b$. Further we can put consistently

$$\varepsilon(h) = 1 \quad h \in B \qquad\qquad \varepsilon(h) = 0 \quad h \notin B \quad h \in A$$

Besides we have coproduct / antipode consistency relations

$$m(\widehat{\gamma} \otimes 1)\Delta_\Gamma = s\varepsilon \qquad m(1 \otimes \widehat{\gamma})\Delta = t\varepsilon\widehat{\gamma}$$

where in general case one should introduce a particular projection $\Delta \to \Delta_\Gamma$ of coproducts within the gauge freedom described by ideal I_L.

Remark: For the pair of algebras (A,B) one can introduce a left and right bialgebroid $\mathcal{H}^L, \mathcal{H}^R$. Left bialgebroids $\mathcal{H}^L$ require the presence of right ideal $I_R = t(b) \otimes 1 - 1 \otimes s(b)$. $\mathcal{H}^R$ requires opposite antialgebra multiplication rule ($m^R(a \otimes b)$ $= ba$) and left ideal $I_L = s(b) \otimes 1 - 1 \otimes t(b)$ effectively obtained from I_R by the replacement $s \leftrightarrow t$. From calculational reasons we choose $\mathcal{H}^R$.

3. COVARIANT QUANTUM PHASE SPACES AS HEISENBERG DOUBLES

Standard Heisenberg algebra can be obtained as Hopf-algebraic semi-direct product $H_p \rtimes H_x$ of two dual Abelian Hopf algebras of momenta and positions

$$H_p \equiv H \qquad H_x \equiv \widetilde{H}$$

Heisenberg double is the generalization of such construction to any dual pair $(H, \widetilde{H})$ of Hopf algebras (H – generalized momenta, $\widetilde{H}$ – generalized coordinates)

$$H = (A, m, \Delta, s, \varepsilon) \qquad \widetilde{H} = (\widetilde{A}, \widetilde{m}, \widetilde{\Delta}, \widetilde{s}, \widetilde{\varepsilon})$$

Duality requires existence of bilinear nongenerate map:

$$A \otimes \widetilde{A} \to C : \quad < a, \widetilde{a} > \in C \quad a \in A \quad \widetilde{a} \in \widetilde{A}$$

Duality provides links which are known from Hopf-algebraic framework

multiplication in H $\leftrightarrow$ comultiplication in $\widetilde{H}$
comultiplication in $\widetilde{H}$ $\leftrightarrow$ multiplication in H

i.e. having H ($\widetilde{H}$) one can calculate by duality relations $\widetilde{H}$ (H).

In the construction $\mathcal{H} \rtimes \widetilde{\mathcal{H}}$ the algebra $\widetilde{A} \in \widetilde{\mathcal{H}}$ is a $\mathcal{H}$-module with the action of H

$$a \triangleright \widetilde{a} = \widetilde{a}_{(1)} < a, \widetilde{a}_{(2)} > \quad \leftarrow$$

consistent with Hopf-algebraic action in $\widetilde{A}$

If we introduce Heisenberg double algebra $\mathcal{A}$

$$\mathcal{A} = A \oplus \widetilde{A}$$

one can derive the cross-multiplication rules using the formula

$$a \cdot \widetilde{a} \equiv (a \otimes 1)(1 \otimes \widetilde{a}) = \widetilde{a}_{(1)} < a_{(1)}, \widetilde{a}_{(2)} > a_{(2)}$$

which completes the multiplication in $\mathcal{A}$.

Important theorem (Lu, 1996): Heisenberg double algebra is endowed with Hopf algebroid structure.

Physical application: Important class of quantum-deformed generalized covariant phase spaces are defined as Heisenberg doubles of quantum Poincaré–Hopf algebras (in D=4 ten generalized momenta) and quantum Poincaré groups (in D=4 ten generalized coordinates). Advantage of such definition: buildt-in quantum covariance, rigorous mathematical Hopf algebroid framework.

4. EXAMPLE: κ-DEFORMED D=4 COVARIANT QUANTUM PHASE SPACES

i) Generalized κ-deformed quantum phase space $\mathcal{H}^{(10+10)}$

We shall consider quantum κ-Poincaré algebra in bicrossproduct (Majid–Ruegg) basis. Bicrossproduct ≡ consistent crossproduct structures of algebra and coalgebra with right action $U(so(1;3)) \rtimes T^4$ and left coaction $U(so(1;3)) \blacktriangleright\!\!\!< T^4$

$$H = U(so(1;3)) \underset{\kappa}{\triangleright\!\!\blacktriangleleft} T^4 \qquad \overset{\text{duality}}{\longleftrightarrow} \qquad \widetilde{H} = \widetilde{T}^4_\kappa \underset{\kappa}{\triangleright\!\!\blacktriangleleft} \mathcal{L}^6,$$

where $\mathcal{L}^6$ is the algebra of functions of Lorentz parameters $\widehat{\lambda}_{\mu\nu}$

T^4	$\overset{\text{duality}}{\longleftrightarrow}$	$\widetilde{T}^4_\kappa$	$U(so(3,1))$	$\overset{\text{duality}}{\longleftrightarrow}$	$f(\widehat{\lambda}^\nu_\mu)$ $\updownarrow$
momenta $\widehat{p}_\mu$		Minkowski space coordinates $\widehat{x}^\mu$	nondeformed Lorentz algebra $\widehat{m}_{\mu\nu}$		Abelian Lorentz group parameters

The duality of H and $\widetilde{H}$ determined by duality relations of generators

$$< \widehat{x}^\mu, \widehat{p}_\nu >= \delta^\mu{}_\nu \qquad < \widehat{\lambda}^\mu{}_\nu, \widehat{m}_{\rho\tau} >= i(\delta^\mu{}_\rho g_{\nu\tau} - \delta^\mu{}_\tau g_{\nu\rho})$$

We consider the Heisenberg double $H \underset{\kappa}{\rtimes} \widetilde{H}$ defining generalized phase space $\mathcal{H}^{(10,10)}$ which contains translational $(\widehat{x}_\mu)$ and spin $(\widehat{\lambda}^\mu{}_\nu)$ degrees of freedom

Such nondeformed phase spaces were used earlier to introduce the dynamics on generalized coordinate space given by Poincaré group or its cosets (Lurcat 1968; Souriau 1970; Balachandran, Stern 1980–85, Bette, Zakrzewski 1997).

Hopf algebra H: κ-deformed Poincaré algebra in Majid–Ruegg basis (1994)

algebra sector:

$$
\begin{aligned}
[\hat{m}_{\mu\nu}, \hat{m}_{\lambda\sigma}] &= i\,(g_{\mu\sigma}\hat{m}_{\nu\lambda} + g_{\nu\lambda}\hat{m}_{\mu\sigma} - g_{\mu\lambda}\hat{m}_{\nu\sigma} - g_{\nu\sigma}\hat{m}_{\mu\lambda}) \\
[\hat{m}_{ij}, \hat{p}_{\mu}] &= -i\,(g_{i\mu}\hat{p}_j - g_{j\mu}\hat{p}_i) \\
[\hat{m}_{i0}, \hat{p}_0] &= i\hat{p}_i \qquad , \qquad [\hat{p}_\mu, \hat{p}_\nu] = 0 \\
[\hat{m}_{i0}, \hat{p}_j] &= i\delta_{ij}\left(\kappa \sinh\left(\frac{\hat{p}_0}{\kappa}\right) e^{-\frac{\hat{p}_0}{\kappa}} + \frac{1}{2\kappa}\vec{\hat{p}}^{\,2}\right) - \frac{i}{\kappa}\hat{p}_i\hat{p}_j
\end{aligned}
$$

coalgebra sector:

$$
\begin{aligned}
\Delta(\hat{m}_{ij}) &= \hat{m}_{ij}\otimes I + I\otimes\hat{m}_{ij} \\
\Delta(\hat{m}_{k0}) &= \hat{m}_{k0}\otimes e^{-\frac{\hat{p}_0}{\kappa}} + I\otimes\hat{m}_{k0} + \frac{1}{\kappa}\hat{m}_{kl}\otimes\hat{p}_l \\
\Delta(\hat{p}_0) &= \hat{p}_0\otimes I + I\otimes\hat{p}_0 \\
\Delta(\hat{p}_k) &= \hat{p}_k\otimes e^{-\frac{\hat{p}_0}{\kappa}} + I\otimes\hat{p}_k
\end{aligned}
$$

counits and antipodes:

$$
\begin{aligned}
S(\hat{m}_{ij}) &= -\hat{m}_{ij} \quad , \quad S(\hat{m}_{i0}) = -\hat{m}_{i0} + \frac{3i}{2\kappa}\hat{p}_i \\
S(\hat{p}_i) &= -e^{\frac{\hat{p}_0}{\kappa}}\hat{p}_i, \quad S(\hat{p}_0) = -\hat{p}_0 \\
\epsilon(\hat{p}_\mu) &= \epsilon(\hat{m}_{\mu\nu}) = 0.
\end{aligned}
$$

By Hopf–algebraic duality one gets the κ-Poincaré quantum group $\widetilde{H}$

algebra sector:

$$[\widehat{x}^\mu, \widehat{x}^\nu] = \frac{i}{\kappa}(\delta^\mu_0 \widehat{x}^\nu - \delta^\nu_0 \widehat{x}^\mu), \qquad [\widehat{\Lambda}^\mu_\nu, \widehat{\Lambda}^\alpha_\beta] = 0$$

$$[\widehat{\Lambda}^\mu_\nu, \widehat{x}^\lambda] = -\frac{i}{\kappa}\Big((\widehat{\Lambda}^\mu_0 - \delta^\mu_0)\widehat{\Lambda}^\lambda_\nu + (\widehat{\Lambda}^0_\nu - \delta^0_\nu) g^{\mu\lambda}\Big)$$

coalgebra sector:

$$\Delta(\widehat{x}^\mu) = \widehat{\Lambda}^\mu{}_\alpha \otimes \widehat{x}^\alpha + \widehat{x}^\mu \otimes I$$

$$\Delta(\widehat{\Lambda}^\mu{}_\nu) = \widehat{\Lambda}^\mu{}_\alpha \otimes \widehat{\Lambda}^\alpha{}_\nu$$

antipodes and counits:

$$S(\widehat{\Lambda}^\mu{}_\nu) = \widehat{\Lambda}_\nu{}^\mu \qquad S(\widehat{x}^\mu) = -\widehat{\Lambda}_\nu{}^\mu \widehat{x}^\nu$$

$$\epsilon(\widehat{x}^\mu) = 0 \qquad \epsilon(\widehat{\Lambda}^\mu{}_\nu) = \delta^\mu{}_\nu$$

One can further calculate using Heisenberg double formulae the cross relations for generalized κ-deformed phase space algebra $\mathcal{H}^{(10,10)}$ with the basis $(\widehat{p}_\mu, \widehat{m}_{\mu\nu}; \widehat{x}^\mu, \widehat{\Lambda}^\mu{}_\nu)$ which supplement the algebra sectors of H and $\widehat{H}$:

cross relations:

$$[\widehat{p}_k, \widehat{x}_l] = -i\delta_{kl} \qquad [\widehat{p}_0, \widehat{x}_0] = \imath\hbar$$

$$[\widehat{p}_k, \widehat{x}_0] = -\frac{i}{\kappa}\widehat{p}_k \qquad [\widehat{p}_0, \widehat{x}_l] = 0$$

$$[\widehat{m}_{\alpha\beta}, \widehat{\Lambda}^\mu_\nu] = i\Big(\delta^\mu_\beta \widehat{\Lambda}_{\alpha\nu} - \delta^\mu_\alpha \widehat{\Lambda}_{\beta\nu}\Big), \qquad [\widehat{p}_\mu, \widehat{\Lambda}^\alpha_\beta] = 0$$

$$[\widehat{m}_{\alpha\beta}, \widehat{x}^\mu] = i\Big(\delta^\mu_\beta \widehat{x}_\alpha - \delta^\mu_\alpha \widehat{x}_\beta\Big) + \frac{i}{\kappa}\Big(\delta^0_\beta \widehat{m}_\alpha{}^\mu - \delta^0_\alpha \widehat{m}_\beta{}^\mu\Big)$$

In bialgebroid framework $A = \mathcal{H}^{(10,10)}$ and $B = U_\kappa(\widehat{g})$ $(\widehat{g} = (\widehat{p}_\mu, \widehat{m}_{\mu\nu}))$.

ii) Standard covariant κ-deformed phase space $\mathcal{H}^{(4,4)}$

a) In $\mathcal{H}^{(10,10)}$ one can put consistently $\widehat{\Lambda}^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu}$, i.e remove spin degrees of freedom from quantum Poincaré group. One gets

$$\mathcal{H}^{(10,4)} = H \rtimes \widetilde{T}^4_\kappa = (so(3,1) \rtimes T^4) \rtimes T^4_\kappa \Leftarrow \quad \kappa\text{-deformed DSR algebra}$$

From the above structure one gets κ-Poincaré covariance of κ-deformed Minkowski space (Majid, Ruegg 1994).

b) Subsequently one can remove Lorentz sector so(3,1) as well from H.
One gets standard κ-deformed quantum phase space $\mathcal{H}^{(4,4)} = (\widehat{x}_\mu, \widehat{p}_\mu)$

$$\begin{aligned}
[\widehat{x}^\mu, \widehat{x}^\nu] &= \tfrac{i}{\kappa}(\delta^\mu_0 \widehat{x}^\nu - \delta^\nu_0 \widehat{x}^\mu) \\
[p_\mu, p_\nu] &= 0 \\
[\widehat{p}_k, \widehat{x}_l] &= -i\delta_{kl} \qquad\qquad [\widehat{p}_0, \widehat{x}_0] = i \\
[\widehat{p}_k, \widehat{x}_0] &= -\tfrac{i}{\kappa}\widehat{p}_k \qquad\qquad [\widehat{p}_0, \widehat{x}_l] = 0.
\end{aligned}$$

These algebra relations can be lifted to coalgebraic sector only in Hopf algebroid framework, but not as Hopf–algebra.

5. TARGET MAPS, ANTIPODES AND COPRODUCT GAUGES FOR $\mathcal{H}^{(4,4)}$

We shall present explicite formulae for Hopf algebroid $\mathcal{H}^{(4,4)}$

a) Target map and the ideal defining tensor product $\mathcal{H}^{(4,4)} \underset{\widetilde{T}^4}{\otimes} \mathcal{H}^{(4,4)}$

We shall consider the canonical choice $s(\widehat{x}_\mu) = \widehat{x}_\mu$. In such case $t(\widehat{x}_\mu)$ can be calculated from

$$[\widehat{x}_\mu, t(\widehat{x}_\nu)] = 0$$

One gets the unique formulae

$$t(\widehat{x}_i) = \widehat{x}_i \, e^{\frac{\widehat{p}_0}{\kappa}} \qquad\qquad t(\widehat{x}_0) = \widehat{x}_0 - \frac{1}{\kappa}\widehat{p}_i \widehat{x}_i$$

One can check that t is a homomorphic map in $\widetilde{T}^4_\kappa$

$$[t(\widehat{x}_0), t(\widehat{x}_i)] = \frac{i}{\kappa} t(\widehat{x}_i) \qquad\qquad [t(\widehat{x}_i), t(\widehat{x}_j)] = 0$$

One gets the following basis of $I_L = s(b) \otimes 1 - 1 \otimes t(b) \quad (b \in \widetilde{T}^4_\kappa)$

$$I_i \equiv I_L(\widehat{x}_i) = \widehat{x}_i \otimes 1 - 1 \otimes x_i \, e^{-\frac{\widehat{p}_0}{\kappa}} \qquad\qquad I_0 \equiv I_L(\widehat{x}_0) = \widehat{x}_0 \otimes 1 - 1 \otimes (\widehat{x}_0 - \frac{1}{\kappa}\widehat{x}_i \widehat{p}_i)$$

b) Antipodes and counits

Using the canonical choice $s(b) = b$ one gets from $\widehat{\gamma} \triangleright s(b) = t(b)$ that

$$\widehat{\gamma}(\widehat{x}_\mu) = t(\widehat{x}_\mu) \qquad \widehat{\gamma}(\widehat{p}_i) = -e^{-\frac{\widehat{p}_0}{\kappa}}\widehat{p}_i \qquad \widehat{\gamma}(\widehat{p}_0) = -\widehat{p}_0$$

One gets the antipodes of $\widehat{x}_\mu$ (we use notation $\widehat{\gamma} \triangleright h \equiv \widehat{\gamma}(h)$)

$$\widehat{\gamma}(\widehat{x}_i) = e^{-\frac{\widehat{p}_0}{\kappa}}\widehat{x}_i = t(\widehat{x}_i)$$

$$\widehat{\gamma}(\widehat{x}_0) = \widehat{x}_0 - \tfrac{1}{\kappa}\widehat{p}_i\,\widehat{x}_i = t(\widehat{x}_0)$$

Using the formulae for $t(\widehat{x}_\mu)$ and $\widehat{\gamma} \triangleright (h \cdot h') = \widehat{\gamma}(h') \cdot \widehat{\gamma}(h)$ one obtains

$$\gamma^2(\widehat{x}_i) = \widehat{x}_i \qquad \widehat{\gamma}^2(\widehat{x}_0) = \widehat{x}_0 - \frac{3i}{\kappa} \qquad \widehat{\gamma}^2(\widehat{p}_\mu) = \widehat{p}_\mu$$

Further $\epsilon(\widehat{x}_\mu) = \widehat{x}_\mu$, $\epsilon(1) = 1$, $\epsilon(\widehat{p}_\mu) = 0$ i.e. the generators of base algebra B behave as unity element in Hopf–algebraic case.

One can check the consistency relations

$$\varepsilon(s(\widehat{x}_\mu)) = \varepsilon(t(\widehat{x}_\mu)) = \widehat{x}_\mu \qquad \varepsilon(h\,s(\widehat{x}_\mu)) = \varepsilon(h\,t(\widehat{x}_\mu)) \qquad \text{for all } h \in \mathcal{H}^{(4,4)}$$

c) Coproduct gauges.

In bialgebroid scheme the choice of coproducts is not unique – we introduce generating class of coproducts homomorphic to the phase space algebra

$$\tilde{\Delta}(\hat{x}_\mu) = \Delta(\hat{x}_\mu) + \Lambda_\mu(\hat{x},\hat{p}) = \hat{x}_\nu \otimes f^\nu{}_\mu(\hat{p}) \qquad \Lambda_\mu \text{ - basic coproduct gauge}$$

where $\Delta(\hat{x}_\mu) = 1 \otimes \hat{x}_\mu$ and $f^\nu{}_\mu$ should be calculated. If $(\tilde{\Delta}(\hat{x}_\mu), \Delta(\hat{p}_\mu))$ satisfy the $\mathcal{H}^{(4,4)}$ algebraic relations one calculates $f^\nu{}_\mu$ in unique way

$$\tilde{\Delta}(\hat{x}_i) = \hat{x}_i \otimes e^{\frac{\hat{p}_0}{\kappa}} \longrightarrow \Lambda_i(\hat{x},\hat{p}) = \hat{x}_i \otimes e^{\frac{\hat{p}_0}{\kappa}} - 1 \otimes \hat{x}_i$$

$$\tilde{\Delta}(\hat{x}_0) = \hat{x}_o \otimes 1 + \frac{1}{\kappa}\hat{x}_i \otimes e^{\frac{\hat{p}_0}{\kappa}}\hat{p}_i \longrightarrow$$

$$\longrightarrow \Lambda_0(\hat{x},\hat{p}) = \hat{x}_0 \otimes 1 + \frac{1}{\kappa}\hat{x}_i \otimes e^{\frac{\hat{p}_0}{\kappa}}\hat{x}_i - 1 \otimes \hat{x}_o$$

One can check that Λ_μ satisfies κ-Minkowski space algebra

$$[\Lambda_\mu, \Lambda_\nu] = C^{(\kappa)\,\rho}_{\mu\nu}\Lambda_\rho \qquad C^{(\kappa)\,\rho}_{\mu\nu} = \frac{1}{\kappa}(\delta^0_\mu \delta^\rho_\nu - \delta^0_\nu \delta^\rho_\mu)$$

and

$$[\Delta(\hat{x}_\mu), \Lambda_\nu] = 0$$

If $\kappa \to \infty$ (canonical Heisenberg algebra) we get $\Lambda_\mu(\hat{x}_\mu) = 1 \otimes \hat{x}_\mu - \hat{x}_\mu \otimes 1$ and $\tilde{\Delta}(\hat{x}_\mu) = \hat{x}_\mu \otimes 1$ i.e. $f^\nu{}_\mu = \delta^\mu{}_\nu$.

The basic coproduct gauge can be generalized in four steps:

α) One can simply replace $\Lambda_\mu \longrightarrow \alpha\Lambda_\mu$ (α - arbitrary constant), one gets

$$\widetilde{\Delta}_{(\alpha)}(\widehat{x}_i) = (1-\alpha)(1\otimes\widehat{x}_i) + \alpha\widehat{x}_i \otimes e^{\frac{\widehat{p}_0}{\kappa}}$$

$$\widetilde{\Delta}_{(\alpha)}(\widehat{x}_0) = (1-\alpha)(1\otimes\widehat{x}_0) + \alpha(\widehat{x}_0\otimes 1 + \tfrac{1}{\kappa}\widehat{x}_i \otimes e^{\frac{\widehat{p}_0}{\kappa}}\widehat{p}_i)$$

Coproducts $\widetilde{\Delta}_\alpha(\widehat{x}_\mu)$ are the homomorphic maps, i.e.

$$[\widetilde{\Delta}_{(\alpha)}(\widehat{x}_\mu), \widetilde{\Delta}_{(\alpha)}(\widehat{x}_\nu)] = c_{\mu\nu}{}^{\rho}\widetilde{\Delta}_{(\alpha)}(\widehat{x}_\nu)$$

β) One can further extend the coproducts by introducing as basis the monomials

$$\Lambda_\mu \longrightarrow \Lambda_\mu^{(k)} = A_\mu^{\nu_1\ldots\nu_k}\Lambda_{\nu_1\ldots}\ldots\Lambda_{\nu_k} \qquad (k=2,3\ldots)$$

We get (using short–hand notation)

$$\left[\Lambda^{(k)}, \Lambda^{(l)}\right] \subset \Lambda^{(k+l-1)}$$

$$\left[\widetilde{\Delta}(\widehat{x}_\mu), \Lambda^{(l)}\right] = 0$$

Follows that the homomorphism of coproducts for $k \geqslant 2$ remains valid only modulo the coproduct gauge, i.e. the $\mathcal{H}^{(4,4)}$ algebra is valid in the equivalence class defined by the coproduct gauge freedom which is spanned by 2-tensors $\Lambda_\mu^{(k)}$ for all $k \geqslant 2$.

γ) The maximal class of coproduct gauges for $\Delta(\widehat{x}_\mu)$ is spanned by the following basis $(k \geqslant 1, l \geqslant 0, m \geqslant 0)$

$$\Lambda_\mu \longrightarrow \Lambda_\mu^{k,l,m} \equiv A_\mu^{\nu_1 \ldots \nu_k;\rho_1 \ldots \rho_l;\sigma_1 \ldots \sigma_m} \Lambda_{\nu_1} \ldots \Lambda_{\nu_k} \Delta(\widehat{x}_{\rho_1}) \ldots \Delta(x_{\rho_l}) \cdot$$
$$\cdot \Delta(\widehat{p}_{\sigma_1}) \ldots \Delta(\widehat{p}_{\sigma_m})$$

The homomorphism is satisfied again modulo coproduct gauge, i.e.

$$[\Delta(\widehat{x}_\mu) + \Lambda_\mu^{k,l,m}, \Delta(\widehat{x}_\nu) + \Lambda_\nu^{k',l',m'}] = c_{\mu\nu}{}^\rho (\Delta(\widehat{x}_\rho) + \Sigma \lambda_{k'',l'',m''} \Lambda_\rho^{k'',l'',m''})$$

δ) Finally one shows that one can add the derived above general coproduct gauge to original Hopf–algebraic fourmomenta coproducts, i.e.

$$\widetilde{\Delta}(\widehat{p}_\mu) = \Delta(\widehat{p}_\mu) + \sum_{k,l,m}^{\infty} \lambda_{klm} \Lambda_\mu^{k,l,m} \qquad \lambda_{klm} \text{ - constants}$$

and check that algebraically $(\widetilde{\Delta}(\widehat{x}_\mu), \widetilde{\Delta}(\widehat{p}_\mu))$ will describe the homomorphism of $\mathcal{H}^{(4,4)}$ algebra modulo the most general coproduct gauge.

Remark: coproduct gauge freedom describes the equivalence classes which can be as well described as generated by the left ideal $I_L = \widehat{y}_\mu \otimes 1 - 1 \otimes t(\widehat{y}_\mu)$, characterizing right bialgebroid, with the generators $\widehat{y}_\mu = \widehat{x}_\nu (f^{-1})^\nu{}_\mu$ of the base algebra B.

6. DISCUSSION

i) Important question: physical interpretation of the freedom in coproducts for Hopf algebroids which describe quantum phase spaces.

Basic remark: coproduct gauges are not unphysical as gauge degrees of freedom in standard gauge theories – they describe model - dependent ways of composing two-particle coordinates and momenta which provide the same quantum phase space algebra for global coordinates and momenta for two-particle system.

We have conceptual analogy:

gauge theories:		**quantum phase spaces:**
gauge-invariant quantities	$\longleftrightarrow$	**quantum phase space algebra**
gauge degrees of freedom	$\longleftrightarrow$	**different ways of composing 2-particle coordinates and momenta by the homomorphic coproduct formulae (described by coproduct gauges)**

Simple example: free 2-particle nonrelativistic phase space:

The momenta have assigned the primitive coproduct, e.g. for total momentum (i=1,2,3)

$$\widehat{p}_i^{(1+2)} = \widehat{p}_i^{(1)} + \widehat{p}_i^{(2)} \qquad \widehat{p}_i^{(1)} = \widehat{p}_i \otimes 1 \qquad \widehat{p}_i^{(2)} = 1 \otimes \widehat{p}_i$$

The coordinates $\widehat{x}_i$ (generators of base algebra) have nonunique bialgebroid. Let us choose

$$\Delta(x_i) = \alpha(\widehat{x}_i \otimes 1) + (1-\alpha)(1 \otimes \widehat{x}_i)$$

Such formula has physical interpretation in nonrelativistic quantum phase space $(x_i^{(a)}, p_i^{(a)};$ $i=1,2,3,\ a=1,2)$, because if we put $\alpha = \frac{m_1}{m_1+m_2}$ we obtain correctly the formula describing nonrelativistic center of mass coordinates

$$\widehat{x}_i^{(1+2)} = \frac{m_1}{m_1+m_2}\widehat{x}_i^{(1)} + \frac{m_2}{m_1+m_2}\widehat{x}_i^{(2)}$$

choice of parameter α physical - depends on masses of considered particles

Such formula can be extended to N-particle system.

Unfortunately the center–of–mass coordinate $\widehat{x}^{(1+2)}$ for relativistic system (Pryce 1948; Newton, Wigner 1968) depends in nonpolynomial way on p_μ which can not be fitted to the bialgebroid coproduct formula providing relativistic center–of–mass coordinate (with fourmomenta no problem)

ii) The choice of algebra basis for Hopf bialgebroid

We used Majid–Ruegg basis for κ–Poincaré with bicrossproduct structure of κ–deformed Poincaré algebra, very convenient for the general derivation of κ–covariance properties, and provides $\mathcal{H}^{(4,4)}$ as centrally extended Lie algebra.

Bicrossproduct structure remains valid if we change the fourmomenta (basis in T^4) in arbitrarily nonlinear way

$$\widehat{p}_\mu \longrightarrow \widehat{p}'_\mu = F_\mu(\widehat{p})$$

In particular one can chose $F_\mu(p)$ in a way leading back to the classical algebra basis of κ-deformed Poincaré algebra– then the Lie–algebraic structure of κ-deformed quantum phase space is lost. Only if $F_\mu(\widehat{p})$ is linear ($F_\mu(\widehat{p}) = \alpha_{\mu\nu}\, p^\nu$) the resulting quantum phase spaces are described by 8-dimensional centrally extended Lie algebras, as in Majid-Ruegg basis.

iii) Not every Hopf algebroid (every quantum phase space) by Hopf-algebroid duality leads to the dual Hopf algebroid. Duality property is valid only for the subclass of so–called Frobenius bialgebroids. In general case by duality one obtains from bialgebroids object called cobialgebroid. The self-dual Hopf algebroids describing self-dual quantum phase spaces are Frobenius bialgebroids with antipode; self-duality would correspond to Born reciprocity in phase space.

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