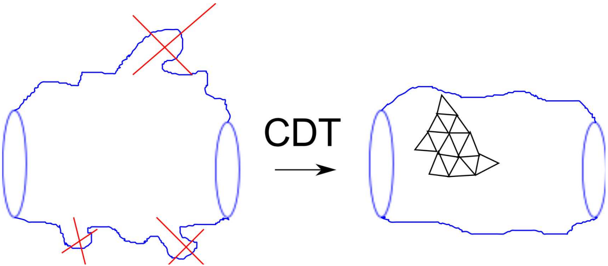
Phenomenolgy of Causal Dynamical Triangulations

Jakub Mielczarek

Jagiellonian University, Cracow
LPSC, Grenoble

9 September, 2015, Wrocław

Causal Dynamical Triangulations



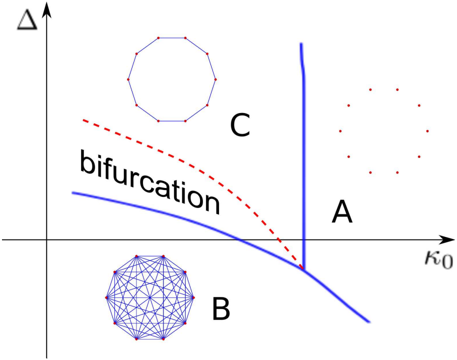
$$\langle \text{out}|\text{in}\rangle = \int Dg_{\mu\nu} e^{\frac{i}{\hbar}(S_{HE}[g_{\mu\nu}]+S_{\Lambda}[g_{\mu\nu}])} \to \text{Wick rotation} \to$$

$$Z = \int Dg_{\mu\nu} e^{-\frac{1}{\hbar}(S^E_{HE}[g_{\mu\nu}]+S^E_{\Lambda}[g_{\mu\nu}])} \to \text{Discretization} \to$$

$$Z_a = \sum_T \frac{1}{C_T} e^{-\frac{1}{\hbar}(S^E_{HE}[T]+S^E_{\Lambda}[T])}$$

Two characteristic features

- Non-trivial phase structure



- The phases are characterized by different scalings $\rightarrow$ different Hausdorff and spectral dimensions. The spectral dimension is running as a function of diffusion time $\rightarrow$ Dimensional reduction in phase C.

Spectral dimension

Heat kernel equation:

$$\frac{\partial}{\partial \sigma} K(x,y;\sigma) = \Delta_x K(x,y;\sigma),$$

where σ is a diffusion time. Eigenproblem:

$$\Delta_x \phi_n(x) = \lambda_n \phi_n(x).$$

The heat kernel can be decomposed as follows:

$$K(x,y;\sigma) = \sum\!\!\!\!\!\!\!\int_n e^{\sigma \lambda_n} \phi_n(x) \phi_n^*(y).$$

By using the expression for the trace of the heat kernel $P(\sigma) := \mathrm{tr} K$ one can introduce

$$d_S \equiv -2 \frac{\partial \log P(\sigma)}{\partial \log \sigma} = -2\sigma \frac{\sum\!\!\!\!\!\!\int_n \lambda_n e^{\lambda_n \sigma}}{\sum\!\!\!\!\!\!\int_n e^{\lambda_n \sigma}}.$$

Spectral dimension in CDT

In the phase C of CDT the spectral dimension can be parametrized as follows

$$d_S(\sigma) = a - \frac{b}{c+\sigma},$$

where σ is the diffusion time. At the representative point in phase C $(\kappa_0 = 2.2,\ \Delta = 0.6)$ the values $a = 4.02$, $b = 119$ and $c = 54$ [1] and $a = 4.06$, $b = 135$ and $c = 67$ [2] have been obtained from numerical simulations. In both cases, four dimensional space-time ($d_S \approx 4$) is correctly recovered in the IR limit ($\sigma \to \infty$). However, in the UV limit ($\sigma \to 0$) dimensional reduction to $d_S \approx 2$ is predicted.

The dimensional reduction to $d_S \approx 2$ in the UV limit is, however, not a generic prediction of CDT! Other values of $d_{\mathrm{UV}} := d_S(\sigma \to 0)$ can be found outside of the neighborhood of the canonical point $(\kappa_0 = 2.2,\ \Delta = 0.6)$!

[1] J. Ambjorn, J. Jurkiewicz and R. Loll, Phys. Rev. Lett. **95** (2005) 171301.

[2] D. N. Coumbe and J. Jurkiewicz, JHEP **1503** (2015) 151.

Dispersion relation

The spectral dimension is related to the eigenvalues of the Laplace operator. Once again:

$$d_S(\sigma) \equiv -2\frac{\partial \log P(\sigma)}{\partial \log \sigma},$$

where in the continuous limit

$$P(\sigma) = \int d\mu e^{\sigma \triangle_p}.$$

The $\triangle_p$ is a 4D momentum space Laplace operator. In the $\mathbb{R}^4$ space we have $\triangle_p = -E^2 - p^2$, where $p := \sqrt{\vec{p}\cdot\vec{p}}$. Wick rotating back to the Lorentzian case $E \to iE$, the momentum-space wave (Klein-Gordon) equation becomes

$$\triangle_p \phi = 0 \to (E^2 - p^2)\phi = 0 \Rightarrow E^2 = p^2 \text{ (dispersion relation)}$$

Quantum space-time reverse engineering

Deformed dispersion relations → non-trivial diffusion time dependence of spectral dimensions[3].

We want to do the opposite: The task is to recover the form of Δ_p from the CDT spectral dimension.

At the moment we will focus on the case with $d_{\rm UV} \approx 2$.

After fixing the IR value d_S to be precisely equal to the topological space-time dimension $d = 4$ and parametrizing the departure from $d_S = 2$ in UV by ϵ we obtain

$$d_S(\sigma) = 4 - \frac{2-\epsilon}{1+\sigma/c},$$

such that $d_{\rm UV} = 2 + \epsilon$. The parameter c can be related to the energy scale of the dimensional reduction: $E_* := \frac{1}{\sqrt{c}}$.

[3] D. Benedetti, Phys. Rev. Lett. **102** (2009) 111303; L. Modesto, Class. Quant. Grav. **26** (2009) 242002

Following Ref. [4], we simplify the problem by assuming that the departure from classical case is parametrized by the single variable function $\Omega(p)$:

$$\triangle_p = -E^2 - \Omega(p)^2.$$

The corresponding dispersion relation will be $E = \Omega(p)$.

Furthermore, we enforce that the momentum space is flat as in the classical case. Therefore, invariant (with respect to the isometries of a given space-time) measure in the momentum space is $d\mu = \frac{dEd^3\vec{p}}{(2\pi)^4}$.

Taking the two above assumptions into account we obtain:

$$P(\sigma) = \frac{1}{\sqrt{4\pi\sigma}} \frac{4\pi}{(2\pi)^2} \int_0^\infty dp\ p^2 e^{-\sigma\Omega^2(p)}.$$

[4] JM, "From Causal Dynamical Triangulations To Astronomical Observations," arXiv:1503.08794 [gr-qc].

As it has been shown in [5], the relation $P(\Omega)$ can be converted into the form of the inverse Laplace transform:

$$p^3 = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT}\left(12\pi^{5/2}\frac{P(\sigma)}{\sqrt{\sigma}}\right)e^{\sigma\Omega^2(p)}d\sigma.$$

The form of function $P(\sigma)$ is predicted by CDT and is given by

$$P(\sigma) = \frac{1}{16\pi^2\sigma^{1+\epsilon/2}(\sigma+c)^{1-\epsilon/2}},$$

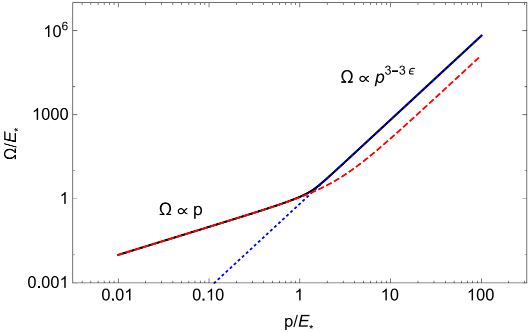
(Obtained by integrating $d_S(\sigma) \equiv -2\frac{\partial \log P(\sigma)}{\partial \log \sigma}$ with the CDT parametrization of $d_S(\sigma)$). Performing the Laplace transformation we obtain:

$$p^3 = \Omega^3 {}_1F_1\left(1-\epsilon/2, 5/2; -c\Omega^2\right),$$

where ${}_1F_1(a,b;z)$ is the confluent hypergeometric function.

[5] T. P. Sotiriou, M. Visser and S. Weinfurtner, Phys. Rev. D **84** (2011) 104018.

CDT-deformed dispersion relation



Asymptotic behaviours:

$$\Omega_{\mathrm{IR}}(p) \approx p + \frac{E_*}{15}(2-\epsilon)\left(\frac{p}{E_*}\right)^3,$$

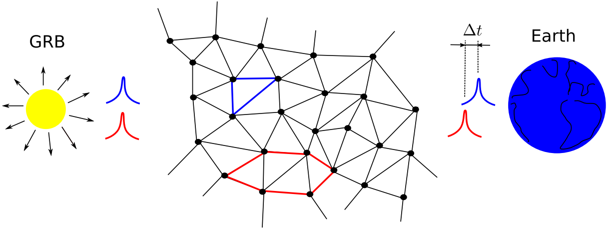
$$\Omega_{\mathrm{UV}}(p) \approx \frac{2}{3}E_*\left(\frac{p}{E_*}\right)^{3-3\epsilon} \propto p^{3-3\epsilon}.$$

For the particles having energy $E \ll E_*$ the IR limit approximation may be applied, leading to the quadratic deviation

$$v_{\rm gr} := \frac{\partial E}{\partial p} = \frac{\partial \Omega(p)}{\partial p} \approx 1 + \frac{3}{15}(2-\epsilon)\left(\frac{E}{E_*}\right)^2.$$

This expression predicts that the group velocity is increasing with the energy $\rightarrow$ superluminal motion.

The effect can be constrained with use of the signals from the high-energy astrophysical sources such as GRBs[6].



[6]G. Amelino-Camelia, *et al.*, Nature **393** (1998) 763

In our case

$$\Delta t \simeq -L\frac{3}{15}(2-\epsilon)\left(\frac{E}{E_*}\right)^2,$$

where L is the distance to a source. Although a huge value of the energy scale E_* is expected the multiplication by a sufficiently large distance L may bring the value of Δt close to the observational window.

In particular, the GRB 090510 remote by $L \approx 5,8$ Mpc may be used. Using the constraint from the *Fermi*-Large Area Telescope one can find that the energy scale of the dimensional reduction is

$$E_* > 6.7 \cdot 10^{10}\mathrm{GeV\ at\ (95\%CL)}.$$

This constraint is definitely much stronger than any obtained from the accelerator physics experiments (which are reaching energies of the order of 10^4 GeV).

Cosmology

Because of the quadratic nature of the IR variation of the group velocity, perhaps the more promising is an application of dimensional reduction to cosmology. Here, instead of the IR, the UV part of the dispersion relation will play a crucial role. The potential relevance of the UV scaling of the modified dispersion relation in cosmology has been suggested in Ref. [7]. It has been shown that dimensional reduction to $d_{\rm UV} = 2$ leads to the scale invariant vacuum fluctuations of the primordial perturbations in the UV domain. What has not been shown is how the shape of the spectrum changes when we try to relate it with the one that is observationally relevant

[7] G. Amelino-Camelia, M. Arzano, G. Gubitosi and J. Magueijo, Phys. Rev. D **87** (2013) 12, 123532.

A short reminder

Primordial perturbations are generated in the quantum process of particle creation from vacuum in the expanding universe.

Let us quantize the Fourier modes $u_{\mathbf{k}}(\tau)$. This follows the standard canonical procedure. Promoting this quantity to be an operator, one performs the decomposition

$$\hat{u}_{\mathbf{k}}(\tau) = f_k(\tau)\hat{b}_{\mathbf{k}} + f_k^*(\tau)\hat{b}^{\dagger}_{-\mathbf{k}},$$

where $f_k(\tau)$ is the so-called mode function which satisfies the same equation as $u_{\mathbf{k}}(\tau)$. The creation ($\hat{b}^{\dagger}_{\mathbf{k}}$) and annihilation ($\hat{b}_{\mathbf{k}}$) operators fulfill the commutation relation $[\hat{b}_{\mathbf{k}}, \hat{b}^{\dagger}_{\mathbf{q}}] = \delta^{(3)}(\mathbf{k} - \mathbf{q})$.

Canonical commutation relation between quantum field $\hat{\phi} = \frac{1}{z}\hat{u}$ and its conjugated momenta leads to the Wronskian condition

$$W(f_k, f_k') \equiv f_k \frac{df_k^*}{d\tau} - f_k^* \frac{df_k}{d\tau} = i.$$

Power spectrum

Two-point correlation function $\hat{\phi}(\mathbf{x}, \tau)$ field is given by

$$\langle 0|\hat{\phi}(\mathbf{x}, \tau)\hat{\phi}(\mathbf{y}, \tau)|0\rangle = \int_0^\infty \frac{dk}{k}\mathcal{P}_\phi(k, \tau)\frac{\sin kr}{kr},$$

where

$$\mathcal{P}_\phi(k, \eta) = \frac{k^3}{2\pi^2}\frac{|f_k(\tau)|^2}{z^2},$$

is the power spectrum, $|0\rangle$ is the vacuum state and $r = |\mathbf{x} - \mathbf{y}|$. This spectrum is the fundamental observable associated with production of primordial perturbations.

The mode functions $f_k(\tau)$ fulfill the same equation as $u_\mathbf{k}$:

$$\frac{d^2}{d\tau^2}f_k + \left(k^2 - \frac{z''}{z}\right)f_k = 0.$$

The k^2 is due to spatial part of the Laplace operator. The CDT modification of the dispersion relation can be, therefore, effectively introduced by the following replacement

$$k^2 \to a^2\Omega^2(k/a).$$

Consequently, our CDT-corrected equation of modes is

$$\frac{d^2}{d\tau^2} f_k + \left(a^2\Omega^2(k/a) - \frac{z''}{z} \right) f_k = 0.$$

Here, $k = pa$ (a is the scale factor) and the classical limit $a^2\Omega^2(k/a) \to a^2(k/a)^2 = k^2$ is recovered correctly for $p \to 0$. The value of z depends on which kind of perturbations is considered. For the gravitational waves $z = z_T := a$ and for the scalar perturbations $z = z_S := a\frac{\dot{\phi}}{H}$.

In the sub-Hubble limit $\left(a^2\Omega^2(k/a) \gg \frac{z''}{z}\right)$, the vacuum normalization of the f_k mode functions leads to

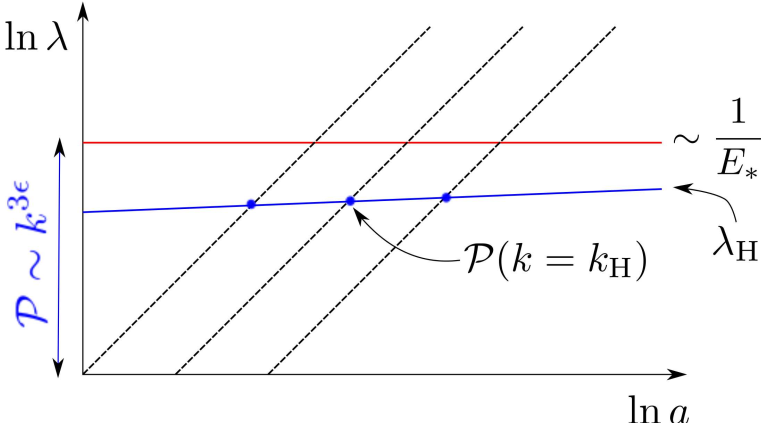
$$|f_k|^2 = \frac{1}{2a\Omega(k/a)}.$$

This is might be viewed as an instantaneous vacuum state, the adiabaticity condition is to be checked for a particular background dynamics.

Applying this to the definition of the scalar power spectrum we find that for $k/a \gg E_*$

$$\mathcal{P}_S(k) := \frac{k^3}{2\pi^2}\frac{|f_k|^2}{z_S^2} = \frac{1}{\pi(1+w)}\left(\frac{E_*}{E_{\rm Pl}}\right)^2\left(\frac{1}{E_*}\frac{k}{a}\right)^{3\epsilon},$$

where $n_S = 1 + 3\epsilon = 1 + 3(d_{\rm UV} - 2)$. Furthermore, we assume that the universe is filled with barotropic matter: $P = w\rho$, with $w = const$.



The value of the power spectrum accessible observationally is at the horizon-crossing. In our case, the horizon-crossing condition $a^2\Omega^2(k/a) = \frac{z''}{z}$ translates into $k \simeq aE_* \left(\frac{3}{2}\frac{H}{E_*}\right)^{\frac{1}{3(1-\epsilon)}} := k_H$ for $H > E_*$.

In this particular situation the resulting tensor-to-scalar ratio

$$r := \frac{\mathcal{P}_T(k_0)}{\mathcal{P}_S(k_0)} = 64\pi G \left(\frac{z_S}{z_T}\right)^2 = 24(1+w),$$

can be directly related to the CMB data. The best current observational bound at the pivot scale $k_0 = 0.05\ \mathrm{Mpc}^{-1}$ is $r < 0.12$ (BICEP2/*Keck Array* and *Planck*)[8], which implicates that $w < -0.995$ when the perturbations were formed.

The universe had to, therefore, be very close to the de Sitter phase when the perturbations were formed. Cosmic inflation is still needed!

But, this is in agreement with the CDT assumption, where positive cosmological constant is included from the very beginning.

[8] P. A. R. Ade *et al.* [BICEP2 and Planck Collaborations], Phys. Rev. Lett. **114** (2015) 101301

The spectral index of scalar perturbations at the horizon-crossing is

$$n_S - 1 \equiv \frac{d \ln \mathcal{P}_S(k = k_H)}{d \ln k} \approx \frac{3\epsilon r}{r + 48(\epsilon - 1)}.$$

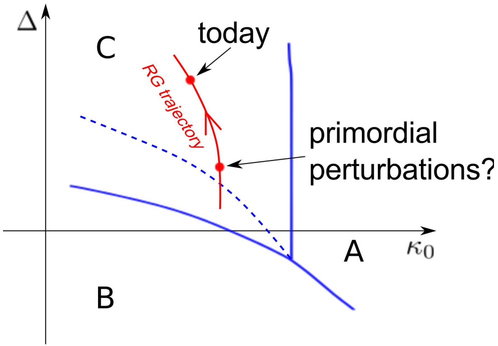
The scale invariance is still recovered for $\epsilon = 0$. However, by comparing the prediction with the *Planck* satellite measurement of the spectral index ($n_S = 0.9616 \pm 0.0094$) and the constraint on the tensor-to-scalar ratio we find that

$$\epsilon > 0.835.$$

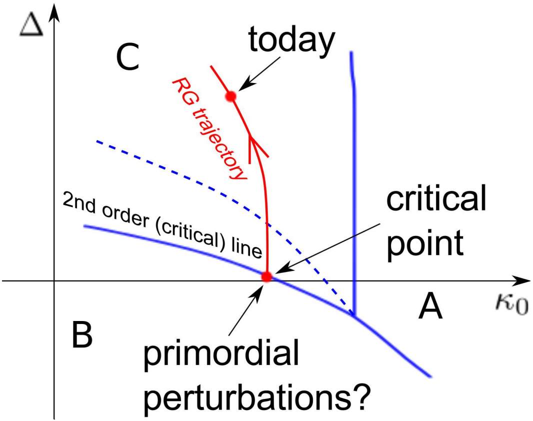
The value is in overt contradiction with the CDT-motivated assumption that $\epsilon \approx 0$. The scenario considered can be, therefore, ruled out based on the CMB data. Other scenarios, such as those with $H < E_*$ at the horizon-crossing have to be studied separately, taking into account the evolution of modes in the intermediate energy range.

Interpretation in terms of RG flow

Let us suppose that the phase C describes state of our Universe. Depending on the energy scale (energy density) the Universe is in another place at a RG trajectory.



Regaining a gives value of $d_{\rm UV}$ from the cosmological observations could tell as about conditions at which the primordial perturbations were formed.



Some evidence of this sort of RG lines: J. Ambjorn, A. Grlich, J. Jurkiewicz, A. Kreienbuehl and R. Loll, "Renormalization Group Flow in CDT," Class. Quant. Grav. **31** (2014) 165003. The issue is discussed in JM, "Big Bang as a critical point," arXiv:1404.0228 [gr-qc].

At the critical point correlation length $\xi \sim |T - T_C|^{-\nu}$ tends to infinity. Because of this, the correlation function of fluctuations at the critical point takes the power-law form

$$G_{\text{critical}}(r) \propto \frac{1}{r^{d-2+\eta}},$$

which is scale-free:

$$G_{\text{critical}}(r\lambda) = \lambda^{2-d-\eta} G_{\text{critical}}(r),$$

but is not scale-invariant (in the cosmological nomenclature). Some Authors tried to use the 2nd order transition to generate scale-invariant primordial perturbations anyway:

- O. Dreyer, "The world is discrete," arXiv:1307.6169 [gr-qc].
- J. Magueijo, L. Smolin and C. R. Contaldi, "Holography and the scale-invariance of density fluctuations," Class. Quant. Grav. **24** (2007) 3691

Inspiring ideas, but unclear assumptions at the same time.

Gravitational phase transitions in the early universe?

It is not clear whether gravitational phase transitions can generate cosmologically relevant spectrum of primordial perturbations. Numerous conceptual issues. E.g. what is the meaning of d at the transition point? If $d = 2$ then nearly (due to η) scale invariant spectrum can indeed be generated. Anyway, measurements of the critical exponents in CDT might tell us something.

In particular, by analyzing domains formation we can perform predictions regarding the number of gravitational defects. The concentration of domains can be constrained observationally.

The non-equilibrium process of the domains formations might be studied using the equilibrium characteristics such as correlation length and relaxation time (Żurek, Kibble):

$$\xi = \xi_0 \left|1 - T/T_C\right|^{-\nu}, \quad \tau = \tau_0 \left|1 - T/T_C\right|^{-\nu z}$$

Kibble-Żurek mechanism, domains and defects

Relaxation horizon

$$\xi_c = \int_0^t \frac{\xi(T)}{\tau(T)} dt \propto \xi_0 \frac{\tau_Q}{\tau_0} \left(1 - T/T_C\right)^{\nu(z-1)+1},$$

where τ_Q (quench time) characterizes a speed of the transition. At some point $\xi_c(T_F) = \xi(T_F)$, which gives us a domain size:

$$\xi \approx \xi_0 \left(\frac{\tau_Q}{\tau_0}\right)^{\frac{\nu}{\nu z+1}}.$$

Gravitational defects are formed at the domains boundary. For typical values $\xi \approx l_{\mathrm{Pl}}$. In case there is no inflation, the present size of the domains is

$$\xi_{\mathrm{today}} \approx \xi \frac{T_{\mathrm{Pl}}}{T_{\mathrm{CMB}}} \sim 1 \ \mathrm{mm}.$$

An average defect concentration $d \sim \frac{1}{\mathrm{mm}^3}$. Have you seen any? A similar problem like in case of the standard topological defects.

Conclusions

- Phenomenology of CDT can be built out based on non-trivial phase structure and diffusion time dependences of spectral dimensions. But, more possibilities in future.
- A particular scenario has been shown to be in disagreement with observations of the CMB. That is good! Some of the QG predictions are falsifiable.
- Other scenarios are to be studied: different d_{UV}, different Δ_p parameterizations, different vacua, curved momentum space, $E_* > H$, $\ldots \rightarrow$ more definitive statements. (partially done)
- A possibility of gravitational phase transitions in the early universe (*e.g.* geometrogenesis) $\rightarrow$ a new mechanism of generation of primordial perturbations?
- Critical behavior $\rightarrow$ non-equilibrium transitions $\rightarrow$ KZM $\rightarrow$ domains $\rightarrow$ gravitational defects
- I wish you a lot of disproved QG models/scenarios!