Information-theoretic approach to inhomogeneous cosmology

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a part of this talk based on a collaboration with

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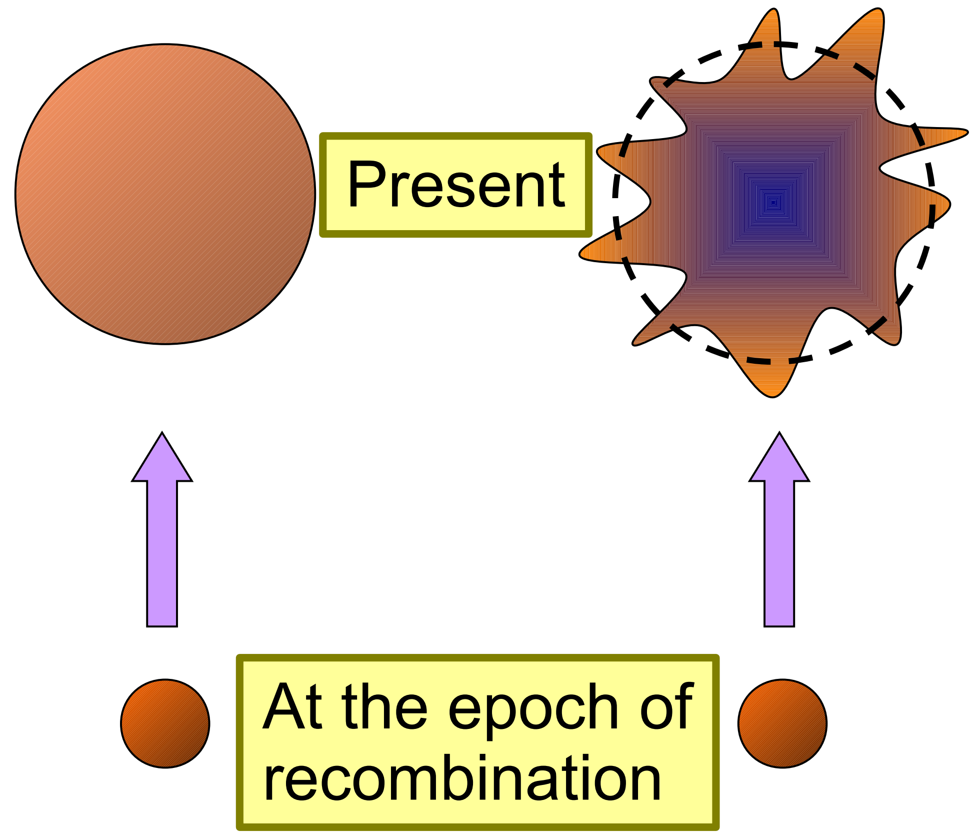
Plan of this talk:

1. Averaging problem in cosmology
2. Basics of averaging inhomogeneous universe
   with the definition of entropy and backreaction
3. Temporal behavior of the entropy
   with the estimation using linear perturbation theory
4. Summary and conclusion

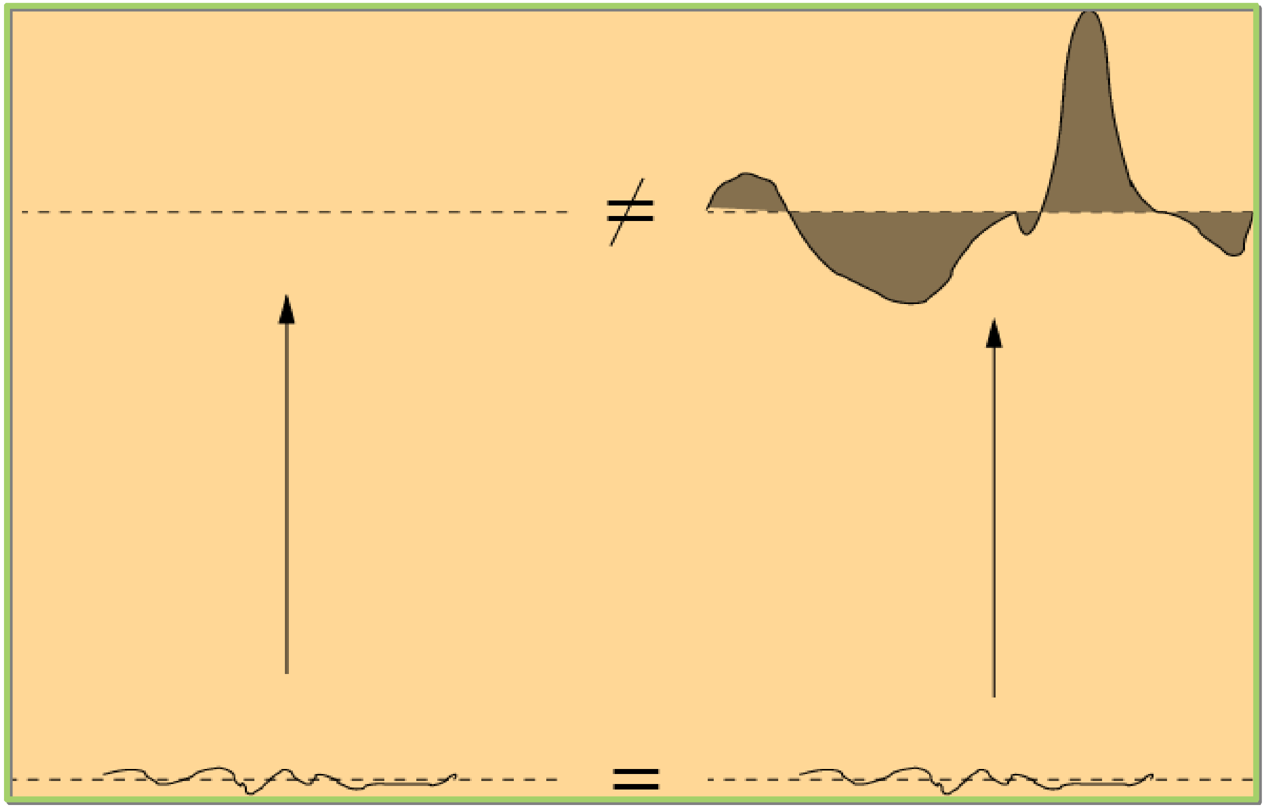
1. Averaging problem in cosmology

In standard cosmology, cosmic inhomogeneities are treated only **linearly**

But in actual universe, inhomogeneities grow **non-linearly** because of non-linearity of Einstein's equation



It is highly non-trivial whether we can regard a real inhomogeneous universe (averaged over some large scales) as a Friedmannian universe model



Time evolution and spatial average do not commute, in general

$$\frac{\partial}{\partial t}\langle A\rangle \neq \left\langle \frac{\partial A}{\partial t}\right\rangle$$

The presence of inhomogeneities may lead the average properties of a realistic universe to deviates from those of a Friedmannian one

↓

Inhomogeneities could be a alternative to dark energy for cosmic acceleration

Spatial average of an inhomogeneous universe and back-reaction (how the effective scale factor behaves)

Ellis (1984) – pointing out the non-commutativity for the first time

Futamase (1988, 1996), Russ et al (1997), Buchert (2000, 2001), Rasanen (2004), Kolb et al (2005), ...

(the conclusions are “method dependent”)

2. Basics of averaging inhomogeneous universe

(Buchert 2000, 2001)

(July 2008 in Lyon)

Comoving synchronous gauge

$$ds^2 = -dt^2 + g_{ij}\, dX^i dX^j,$$

$$u^\mu = (1, \mathbf{0})$$

for irrotational dust matter $T^{\mu\nu} = \rho u^\mu u^\nu$

Expansion rate $\theta := u^\mu{}_{;\mu} = \frac{1}{2} g^{ij} \partial_t g_{ij} = \frac{\partial_t(\sqrt{g})}{\sqrt{g}}$, $g := \det g_{ij}$

Continuity eq. $\partial_t \rho + \rho\theta = 0$

Raychaudhuri eq. $\partial_t \theta = -4\pi G\rho - \frac{1}{3}\theta^2 - 2\sigma^2$

Hamiltonian constr. $\frac{1}{2}{}^{(3)}R + \frac{1}{3}\theta^2 - \sigma^2 = 8\pi G\rho$

Spatial average over a compact domain D

$$\langle \cdot \rangle_D := \frac{1}{V_D} \int_D (\cdot) \sqrt{g}\, \mathrm{d}^3 X$$

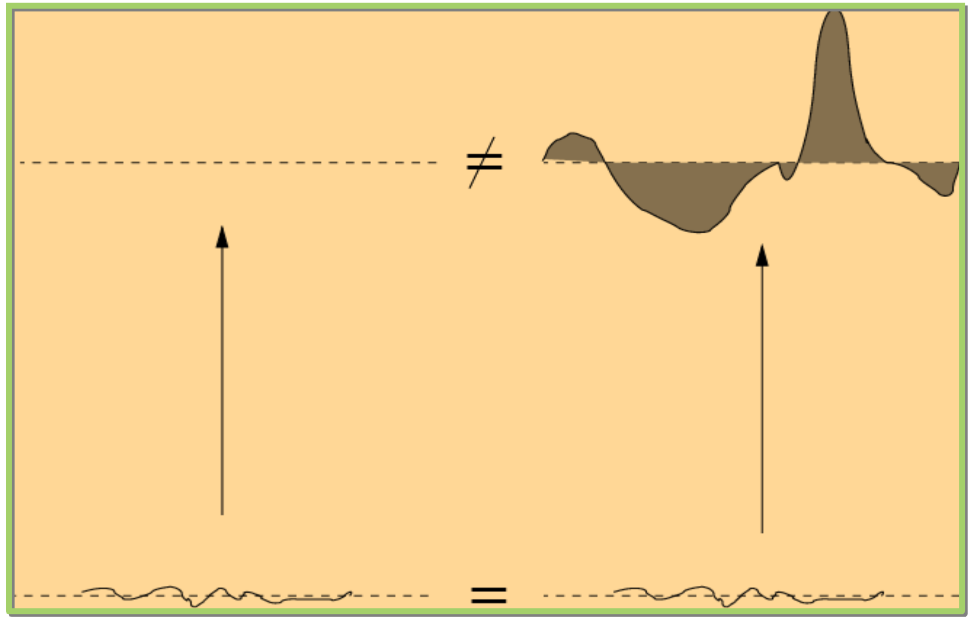
Commutation relation $\partial_t \langle A \rangle_D - \langle \partial_t A \rangle_D = \langle A\theta \rangle_D - \langle A \rangle_D \langle \theta \rangle_D$ between time evolution and spatial average

Commutation relation for the density field:

$$\partial_t \langle \rho \rangle_D - \langle \partial_t \rho \rangle_D = \langle \rho\theta \rangle_D - \langle \rho \rangle_D \langle \theta \rangle_D = \langle \delta\rho \delta\theta \rangle_D$$

$$\delta\rho := \rho - \langle \rho \rangle_D, \quad \delta\theta := \theta - \langle \theta \rangle_D$$

- Averaged Continuity eq. $\partial_t \langle \rho \rangle_D + \langle \rho \rangle_D \langle \theta \rangle_D = 0$



$$\frac{\partial}{\partial t}\langle \rho \rangle_D - \left\langle \frac{\partial \rho}{\partial t} \right\rangle_D = -\frac{1}{V_D}\frac{\mathrm{d}S}{\mathrm{d}t}$$

Evolution of averaged density field

Average after evolving the density field

Entropy production (per volume)

The answer for S is ...

Relative information entropy for cosmic matter distribution

$$S\{\rho \| \langle \rho \rangle\} := \int_D \rho \ln \frac{\rho}{\langle \rho \rangle} \sqrt{g}\, \mathrm{d}^3 X$$

$$S \geq 0$$

$$S = 0 \quad \text{iff} \quad \rho = \langle \rho \rangle$$

Hosoya, Buchert, Morita (2004)
Morita, Buchert, Hosoya, Li (2010)

cf. Relative information entropy (Kullback-Leibler divergence) in Information Theory

$$S_{\mathrm{KL}}\{p \| q\} := \sum p_i \ln \frac{p_i}{q_i}$$

$\{p_i\}$: actual prob. distribution

$\{q_i\}$: assumed prob. distribution

``*distance*'' between p and q

Commutation relation for the expansion rate:

$$\partial_t \langle \theta \rangle_D - \langle \partial_t \theta \rangle_D = \langle \theta^2 \rangle_D - \langle \theta \rangle_D^2$$

Raychaudhuri eq.

- Averaged Raychaudhuri eq. (or generalized Friedmann eq.)

$$3\frac{\ddot{a}_D}{a_D} + 4\pi G \langle \rho \rangle_D = Q_D \qquad a_D := V_D^{1/3}$$

`backreaction' term $$Q_D = \frac{2}{3}\left(\langle \theta^2 \rangle - \langle \theta \rangle^2\right) - 2\langle \sigma^2 \rangle$$

acceleration **deceleration**

3. Temporal behavior of the entropy

$$\frac{\mathrm{d}}{\mathrm{d}t} S = -\int_D \delta\rho \cdot \delta\theta \sqrt{g}\, \mathrm{d}^3 X$$

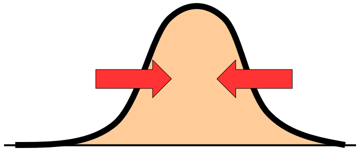
$$\delta\rho := \rho - \langle \rho \rangle, \quad \delta\theta := \theta - \langle \theta \rangle$$

Cluster (overdense): $\delta\rho > 0, \quad \delta\theta < 0$

Void (underdense): $\delta\rho < 0, \quad \delta\theta > 0$

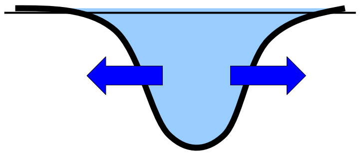
$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t} S > 0$

(but depends on the initial data)



collapsing overdense region

expanding void region

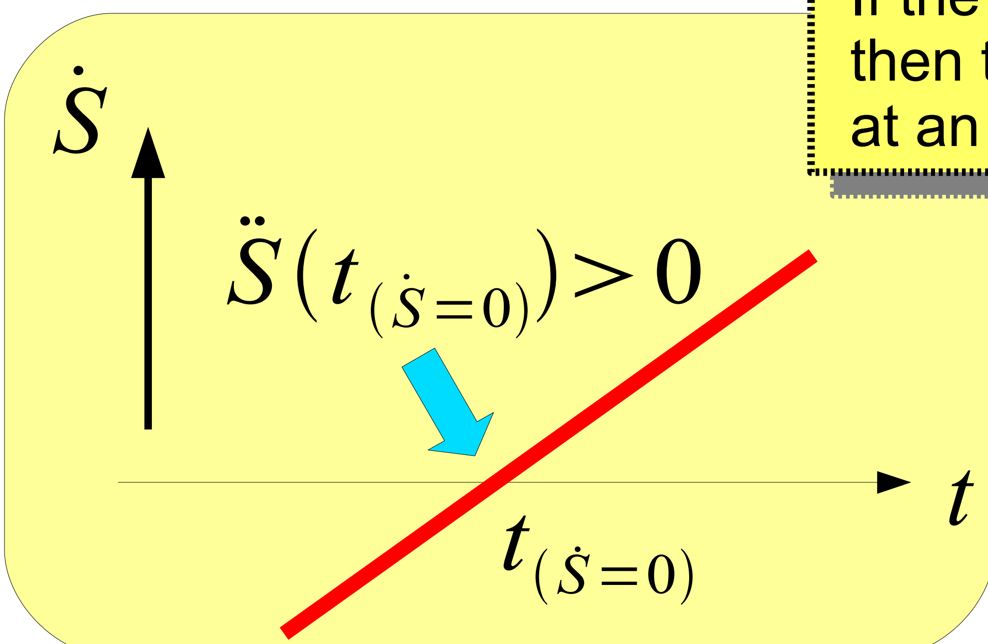


Time-convexity of the entropy

$$\frac{1}{V_D}\frac{\mathrm{d}^2}{\mathrm{d}t^2}S = 4\pi G(\Delta\rho)^2 + \frac{1}{3}\langle\rho(\delta\theta)^2\rangle + 2\langle\rho\sigma^2\rangle$$

$$+\langle\rho\rangle_D Q_D - \frac{2}{3}\langle\theta\rangle_D\frac{\dot{S}}{V_D}$$

$$(\Delta A)^2 := \langle(\delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$$



If the backreaction term is positive,
then the second derivative is positive
at an instant when the first derivative is zero

The first derivative always stays positive thereafter

- The time derivative of the relative information entropy is written as the correlation of the deviations of density and expansion from their mean values, which is intuitively positive

- The second time derivative is related to the backreaction; if the backreaction is positive (i.e. behaves as DE), the time derivative becomes positive after a sufficient time

The temporal increase of the entropy and DE-like backreaction are somehow linked

suggestive thing ...

Concrete evidence and rigorous proof for the temporal increase of the entropy have been given with:

Linear perturbations, LTB solution, behavior near singularity (Buchert, Hosoya, Li, Morita, in preparation)

Estimation of the backreaction with linear perturbations

Li & Schwarz (2007)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \left(\gamma_{ij} + h_{ij}\right) \mathrm{d}X^i \mathrm{d}X^j$$

Solution of scalar mode for a spatially flat FLRW background:

$$h_{ij} = \left(1 + \frac{20}{9}\Psi\right)\gamma_{ij} + 2t^{2/3}\Psi_{,ij} + 2t^{-1}\Phi_{,ij} \qquad (t_{\mathrm{in}} := 1)$$

$\Psi = \Psi(X)$, $\Phi = \Phi(X)$: small spatial functions determined by the initial data

The \`backreaction'

$$Q_D = \frac{2}{3}\left(\langle\theta^2\rangle - \langle\theta\rangle^2\right) - 2\langle\sigma^2\rangle = \frac{1}{4}\left\langle \dot{h}^2 - \dot{h}^i{}_j \dot{h}^j{}_i \right\rangle - \frac{1}{6}\langle\dot{h}\rangle^2 + O(h^3)$$

$$\approx \frac{4}{9}t^{-2/3}\left[\left\langle (\nabla^2\Psi)^2 - \Psi^{,i}{}_{,j}\Psi^{,j}{}_{,i} \right\rangle - \frac{2}{3}\langle\nabla^2\Psi\rangle^2\right]$$

Estimation of the relative entropy with linear perturbations

Straightforward calculations yield, to the leading order, (neglecting the decaying mode)

$$\frac{S}{V_D} \propto t^{-2/3}\left[\left\langle (\nabla^2\Psi)^2\right\rangle - \left\langle \nabla^2\Psi\right\rangle^2\right] \qquad (\geq 0)$$

$$\propto \frac{t^2}{8}\left\langle C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}\right\rangle + Q_D$$

$$\frac{\dot{S}}{V_D} \propto t^{-5/3}\left[\left\langle (\nabla^2\Psi)^2\right\rangle - \left\langle \nabla^2\Psi\right\rangle^2\right] \qquad (\geq 0)$$

$$\propto \frac{t}{8}\left\langle C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}\right\rangle + \frac{Q_D}{t}$$

$$\frac{\ddot{S}}{V_D} \propto t^{-8/3}\left[\left\langle (\nabla^2\Psi)^2\right\rangle - \left\langle \nabla^2\Psi\right\rangle^2\right] \qquad (\geq 0)$$

$$\propto \frac{1}{8}\left\langle C_{\mu\nu\lambda\rho}C^{\mu\nu\lambda\rho}\right\rangle + \frac{Q_D}{t^2}$$

Li, Buchert, Hosoya, Morita, Schwarz (2012)

4. Summary and conclusion

- Averaging inhomogeneous universe models has been considered in the context of dark energy cosmology

- Measure of inhomogeneity naturally arises from the averaging formulation, which is known as ***Kullback-Leibler relative entropy*** in information theory

- It has been shown that **the relative entropy is increasing in time** after a sufficient time; this temporal increase is somehow linked to the signature of the backreaction

- To the leading order in perturbation, the relative entropy is given as a sum of the Weyl curvature invariant and the backreaction