Infinitely many nonlocal conservation laws for vacuum anti-self-dual Einstein equations with nonzero cosmological constant

Artur Sergyeyev

Silesian University in Opava, Czech Republic

XXXV Max Born Symposium

Wrocław 2015

Integrable systems

- nonlinear
- tractable:
  - infinitely many explicit exact solutions
  - rich symmetry algebras
  - infinitely many conservation laws
- arise as compatibility conditions of overdetermined linear systems (Lax pairs)

Anti-self dual gravity in $4D$ with $\Lambda \neq 0$

The metric in this case can be locally written as

$$g = 2\left(u_{w\bar{w}}dwd\bar{w} + u_{w\bar{z}}dwd\bar{z} + u_{z\bar{w}}dzd\bar{w}\right)$$
$$+2\left(u_{z\bar{z}} + \frac{2}{\Lambda}\exp(\Lambda u)\right)dzd\bar{z} \tag{1}$$

Here $u = u(w, \bar{w}, z, \bar{z})$ is a real function on our four-manifold M (the *Przanowski function*), w and z are holomorphic coordinates on M and $\bar{w}$ and $\bar{z}$ denote their complex conjugates; $\Lambda \neq 0$ is the cosmological constant. As usual, the subscripts indicate partial derivatives, e.g. $u_w = \frac{\partial u}{\partial w}$ etc.

The Przanowski equation

The metric (1) is an anti-self-dual Einstein metric if and only if u satisfies the *Przanowski equation*

$$u_{z\bar{w}}u_{w\bar{z}} - u_{w\bar{w}}\left(u_{z\bar{z}} + \frac{2\exp(\Lambda u)}{\Lambda}\right) + u_w u_{\bar{w}} \exp(\Lambda u) = 0. \quad (2)$$

If $u_w = 0$ or $u_{\bar{w}} = 0$ we get $0 = 0$, and if $u_z = 0$ or $u_{\bar{z}} = 0$ we get

$$\frac{2u_{w\bar{w}}}{\Lambda} = u_w u_{\bar{w}},$$

whence

$$u = -\frac{2}{\Lambda}\ln(f_1(w,s) + f_2(\bar{w},s)),$$

where f_i are arbitrary (smooth) functions and $s = z$ if $u_{\bar{z}} = 0$ and $s = \bar{z}$ if $u_z = 0$.

Lax pair for the Przanowski equation: general structure

Hoegner (JMP 2012) has found a nonisospectral Lax pair for (2). Namely, (2) is equivalent to the commutativity condition

$$[l_1, l_2] = 0,$$

where the operators l_i have the following structure:

$$l_\alpha = l_\alpha^j \frac{\partial}{\partial z^j} + b_i \frac{\partial}{\partial \xi},$$

where z^j are local coordinates, and ξ is the so-called variable spectral parameter (essentially an additional independent variable such that $u_\xi = 0$).

Lax pair for the Przanowski equation: explicit form

$$
\begin{aligned}
l_1 &= \partial_w - \xi\frac{u_{w\bar{z}}}{Q}\partial_{\bar{w}} + \xi\frac{u_{w\bar{w}}}{Q}\partial_{\bar{z}} \\
&\quad + \left(\frac{\partial_w Q + \exp(\Lambda u) u_w u_{w\bar{w}}}{Q} - \frac{u_{w\bar{w}}}{u_{\bar{w}}}\right)\xi\partial_\xi, \\
l_2 &= \partial_z - \frac{\xi}{Q}\left(u_{z\bar{z}} + \frac{2}{\Lambda}\exp(\Lambda u)\right)\partial_{\bar{w}} + \xi\frac{u_{z\bar{w}}}{Q}\partial_{\bar{z}} \\
&\quad + \left(\frac{\partial_z Q + \exp(\Lambda u)(u_w u_{z\bar{w}} - u_{\bar{w}}\xi)}{Q} - \frac{u_{z\bar{w}}}{u_{\bar{w}}} + \frac{\xi}{u_w}\right)\xi\partial_\xi,
\end{aligned}
$$

where $Q = -u_w u_{\bar{w}} \exp(\Lambda u)$.

Modified Lax pair

Pass from ξ to $p = \xi \exp(\Lambda u) u_w / u_{\bar{w}}$. Then the l_α go into

$$\begin{aligned} L_1 &= \partial_w - \frac{p u_{w\bar{w}}}{u_{\bar{w}}^2}\partial_{\bar{z}} + \frac{p u_{w\bar{z}}}{u_{\bar{w}}^2}\partial_{\bar{w}} \\ &+ \frac{(u_{\bar{w}\bar{w}}u_{w\bar{z}} - u_{w\bar{w}}u_{\bar{w}\bar{z}} + \Lambda u_{\bar{w}}(u_{\bar{z}}u_{w\bar{w}} - u_{\bar{w}}u_{w\bar{z}}))p^2}{u_{\bar{w}}^3}\partial_p \\ L_2 &= \partial_z - \frac{p u_{\bar{w}z}}{u_{\bar{w}}^2}\partial_{\bar{z}} + \frac{(u_{\bar{w}z}u_{w\bar{z}} + \exp(\Lambda u)u_w u_{\bar{w}})p}{u_{\bar{w}}^2 u_{w\bar{w}}}\partial_{\bar{w}} \\ &- \frac{u_{\bar{w}z}(u_{w\bar{w}}u_{\bar{w}\bar{z}} - u_{\bar{w}\bar{w}}u_{w\bar{z}} - \Lambda u_{\bar{w}}(u_{\bar{z}}u_{w\bar{w}} - u_{\bar{w}}u_{w\bar{z}}))p^2}{u_{\bar{w}}^3 u_{w\bar{w}}}\partial_p \\ &+ \frac{(\exp(\Lambda u)u_{\bar{w}}(u_{\bar{w}}u_{w\bar{w}} + u_w u_{\bar{w}\bar{w}} - \Lambda u_w u_{\bar{w}}^2))p^2}{u_{\bar{w}}^3 u_{w\bar{w}}}\partial_p. \end{aligned} \tag{3}$$

Formal Taylor expansion

Substituting into the equations $L_\alpha \chi = 0$ a formal Taylor expansion $\chi = \sum_{i=0}^{\infty} \chi_i p^i$ shows that $\chi_0 = \chi_0^0(\bar{w}, \bar{z})$ is an arbitrary smooth function of $\bar{w}$ and $\bar{z}$, and χ_1 satisfies the equations

$$
\begin{aligned}
(\chi_1)_w &= \frac{u_{w\bar{w}}}{u_{\bar{w}}^2}(\chi_0^0)_{\bar{z}} - \frac{u_{w\bar{z}}}{u_{\bar{w}}^2}(\chi_0^0)_{\bar{w}}, \\
(\chi_1)_z &= \frac{u_{\bar{w}z}}{u_{\bar{w}}^2}(\chi_0^0)_{\bar{z}} - \frac{(u_{\bar{w}z}u_{w\bar{z}} + \exp(\Lambda u)u_w u_{\bar{w}})}{u_{\bar{w}}^2 u_{w\bar{w}}}(\chi_0^0)_{\bar{w}}.
\end{aligned}
$$

Expression for χ_1

We obtain

$$\chi_1 = -\frac{(\chi_0^0)_{\bar{z}}}{u_{\bar{w}}} - \omega_1(\chi_0^0)_{\bar{w}} + \chi_1^0,$$

where $\chi_1^0(\bar{w}, \bar{z})$ is an arbitrary smooth function of $\bar{w}$ and $\bar{z}$, and ω_1 satisfies

$$(\omega_1)_w = \frac{u_{w\bar{z}}}{u_{\bar{w}}^2}, \quad (\omega_1)_z = \frac{(u_{\bar{w}z}u_{w\bar{z}} + \exp(\Lambda u)u_w u_{\bar{w}})}{u_{\bar{w}}^2 u_{w\bar{w}}}. \tag{4}$$

The quantity ω_1 is a nonlocal variable: it is a potential for the local conservation law for (2)

$$\left(\frac{u_{w\bar{z}}}{u_{\bar{w}}^2}\right)_z = \left(\frac{(u_{\bar{w}z}u_{w\bar{z}} + \exp(\Lambda u)u_w u_{\bar{w}})}{u_{\bar{w}}^2 u_{w\bar{w}}}\right)_w \tag{5}$$

New form for L_α

$$
\begin{aligned}
L_1 &= \partial_w + p\left(\frac{1}{u_{\bar{w}}}\right)_w \partial_{\bar{z}} - p(\omega_1)_w \partial_{\bar{w}} \\
&\quad + p^2 \left(\frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}\bar{z}}}{2u_{\bar{w}}^2}\right)_w \partial_p \\
L_2 &= \partial_z + p\left(\frac{1}{u_{\bar{w}}}\right)_z \partial_{\bar{z}} - p(\omega_1)_z \partial_{\bar{w}} \\
&\quad + p^2 \left(\frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}\bar{z}}}{2u_{\bar{w}}^2}\right)_z \partial_p.
\end{aligned}
\tag{6}
$$

More on new form for L_α

Note that if we put

$$A = z\partial_z + w\partial_w + \frac{p}{u_{\bar{w}}}\partial_{\bar{z}} - p\omega_1\partial_{\bar{w}} + \frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}\bar{z}}}{2u_{\bar{w}}^2} p^2\partial_p$$

then we can write L_α as

$$L_1 = [\partial_w, A], \quad L_2 = [\partial_z, A].$$

Recursion relations for χ_k

A formal Taylor expansion $\chi = \sum_{i=0}^{\infty} \chi_i p^i$ now leads to the following recursion relations, where $k = 1, 2, 3, \ldots$:

$$
\begin{aligned}
(\chi_{k+1})_w &= -\left(\frac{1}{u_{\bar{w}}}\right)_w (\chi_k)_{\bar{z}} + (\omega_1)_w (\chi_k)_{\bar{w}} \\
&\quad - \left(\frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}\bar{z}}}{2u_{\bar{w}}^2}\right)_w k\chi_k, \\
(\chi_{k+1})_z &= -\left(\frac{1}{u_{\bar{w}}}\right)_z (\chi_k)_{\bar{z}} + (\omega_1)_z (\chi_k)_{\bar{w}} \\
&\quad - \left(\frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}\bar{z}}}{2u_{\bar{w}}^2}\right)_z k\chi_k.
\end{aligned} \tag{7}
$$

Nonlocal conservation laws

$$\begin{aligned}
&\left(-\left(\frac{1}{u_{\bar{w}}}\right)_w (\chi_k)_{\bar{z}} + (\omega_1)_w (\chi_k)_{\bar{w}}\right. \\
&\left. - \left(\frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}\bar{z}}}{2u_{\bar{w}}^2}\right)_w k\chi_k\right)_z \\
&= \left(-\left(\frac{1}{u_{\bar{w}}}\right)_z (\chi_k)_{\bar{z}} + (\omega_1)_z (\chi_k)_{\bar{w}}\right. \\
&\left. - \left(\frac{u_{\bar{w}}(2\Lambda u_{\bar{z}} + (\omega_1)_{\bar{w}}) - u_{\bar{w}\bar{z}}}{2u_{\bar{w}}^2}\right)_z k\chi_k\right)_w .
\end{aligned} \tag{8}$$

Here $k = 1, 2, 3, \ldots$.

Conclusions and outlook

- We have constructed an infinite hierarchy of nonlocal conservation laws for the Przanowski equation
- The potentials for these conservation laws can serve as nonlocal variables; using these it is possible to construct nonlocal symmetries for the Przanowski equation (work in progress)

Conclusions and outlook

- We have constructed an infinite hierarchy of nonlocal conservation laws for the Przanowski equation
- The potentials for these conservation laws can serve as nonlocal variables; using these it is possible to construct nonlocal symmetries for the Przanowski equation (work in progress)

Dziękuję za uwagę