A stability criterion for coherent states

[arXiv:1506.08613]

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joined work with T. Thiemann

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Motivation

Coherent states

- semiclassical limit
- quantum optics
- geometric quantization
- harmonic analysis

Why stable?-I

- semiclassical dynamics
- e.g. LQC

Why stable?-II

- constrained systems
- group averaging resembles evolution

Plan of the talk

Complexifier coherent states

[Hall '94],[Ashtekar, Lewandowski, Marolf, Mourão, Thiemann '96],[Thiemann, Winkler, Sahlmann '00]

Harmonic oscillator

- Complexification: $(q, p) \mapsto a = q - ip$
- Coherent states ψ_a: eigenstates of $\hat{a} = \hat{q} - i\hat{p}$

$\mathcal{M}$ Phase space, $\mathcal{H}$ Hilbert space, $C \in C^\infty(\mathcal{M})$ complexifier, $\mathcal{L}_C$ Lie derivative along Hamiltonian vector field of C

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Harmonic oscillator

- Complexification: $(q,p) \mapsto a = q - ip = e^{i\mathcal{L}_C} q$ with $C = p^2/2$
- Coherent states ψ_a: $\psi_a = \left[e^{-\hat{C}} \delta_{q'}(q)\right]_{q' \to a}$

$\delta_{q'}(q) = \delta(q - q')$ Dirac delta, $q' \to a$ analytic continuation

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Complexifier coherent states

A coherent state associated to a complexifier $C \in C^\infty(\mathcal{M})$ is given by

$$\psi_z(q) = \left[e^{-C/\hbar} \delta_{q'}(q) \right]_{q' \to z}$$

where $q' \to z$ denotes analytic continuation to $z = e^{-i\mathcal{L}_C} q$.

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Plan of the talk

A stability criterion

Definiton

If $\hat{U}(t, t_0)\,\psi_{z(t_0)} = e^{i\,\lambda(t)}\,\psi_{z(t)} \quad \forall\, t \in \mathbb{R}^+$ and $\forall\, z \in \mathcal{M}$

then the system of coherent states $\{\psi_z\}_{z\in\mathcal{M}}$ is called stable.

$\hat{U}(t, t_0)$ time evolution operator, $z(t)$ classical trajectory in phase space $\mathcal{M}$, $\lambda(t)$ arbitrary phase

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$\Leftrightarrow$ Time-evolution of $z(t)$ depends only on z

$$\Leftrightarrow \;\; e^{i\mathcal{L}_C} H = i\,p\,f(q) + g(q)$$

H Hamiltonian, f, g arbitrary functions of q, $\mathcal{L}_C$ Lie derivative along Hamiltonian vector field of C

First solutions

Want to solve

$$e^{i\mathcal{L}_C} H \stackrel{!}{=} i\, p\, f(q) + g(q)$$

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$$\sum_{n=0}^{\infty} \frac{i^n}{n!}\{C,H\}_{(n)} \overset{!}{=} i\,p\,f(q) + g(q)$$

C complexifier, H Hamiltonian, f, g real functions, $\{C,H\}_{(n+\mathbf{1})} = \{C, \{C,H\}_{(n)}\}$ multiple Poisson bracket

Ansatz

Assume H, C quadratic in momentum and $\{H,C\}_{(n)} = 0$ for some $n \in \mathbb{N}$.

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Radial oscillator

$$H = \frac{p^2}{2} + \frac{\omega^2}{2}q^2 + \lambda q^{-2} \qquad C = \frac{p^2}{2} + \frac{\lambda}{\omega}q^{-2}$$

$\lambda, \omega > 0$ some constants

A No-go theorem

After a tedious analysis of the

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One finds:

The radial oscillator and its symplectic transforms are the only solutions.

Plan of the talk

Generalized construction

Dynamics in action-angle coordinates (I, Θ)

$$\frac{dI}{dt} = 0 \quad \text{and} \quad \frac{d\Theta}{dt} = \omega(I)$$

Suggests to define

$$y := \sqrt{I} e^{i\Theta}$$

Dynamics: $\frac{d}{dt} y = i\,\omega(I)\,y$

Polar decomposition: $y =: \frac{1}{\sqrt{2}}(Q - iP)$

$$\rightsquigarrow \quad y = \frac{1}{\sqrt{2}} e^{-i\mathcal{L}_C} Q \quad \text{with} \quad C = P^2/2$$

$\omega(I)$ real function, (Q, P) canonically conjugated, $\mathcal{L}_C$ Lie derivative along Hamiltonian vector field of C

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Properties

Advantages

- complexifier straightforward to construct

Disadvantages

- generically complicated dependence of y on (p, q)
- y-parametrization defined only locally

Free particle

Action-angle coordinates

$$I = \frac{p^2}{2} \qquad \text{and} \qquad \Theta = \frac{q}{p}$$

y-Parametrization

$$y_\omega = \frac{p}{\sqrt{2}} e^{i\omega q/p}$$

Properties

- locally well-defined except for $p = 0$
- multi-valued $\rightsquigarrow$ motion appears to be periodically
- If $\omega << q/p$ then $P \approx p + \mathcal{O}(\omega^2)$ and $Q \approx \omega q + \mathcal{O}(\omega^3)$
- y_ω generators of quasi-periodic functions

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Summary and Outlook

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- ✓ stability criterion
- ✓ restricted set of solutions
- ✓ generalized construction

Quantum Theory

- ✎ which polarization
- ✎ global/local structure
- ✎ geometric quantization

Stability Criterion

- ✎ approximate stability
- ✎ several degrees of freedom

Application in LQG

- ✎ symmetry reduced models
- ✎ Loop/standard quantization

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