Noncommutative geometry, quantum symmetries and quantum gravity II

4 - 7 July 2016, Wroclaw, Poland

XXXVII Max Born Symposium & 2016 WG3 Meeting of COST Action MP1405

Cartan's structure equations and Levi-Civita connection in Noncommutative Geometry from Drinfeld Twist

Paolo Aschieri

Università del Piemonte Orientale, Alessandria, Italy

July 4, 2016

Noncommutative geometry, quantum symmetries and quantum gravity II

4 - 7 July 2016, Wroclaw, Poland

XXXVII Max Born Symposium & 2016 WG3 Meeting of COST Action MP1405

NC Geometry via Drinfeld twist and NC Gravity

Paolo Aschieri

Università del Piemonte Orientale, Alessandria, Italy

July 4, 2016

This presentation is therefore divided in 4 parts:

I) Introduction: motivations, approaches, applications

II) Drinfeld Twist and quantum Lie algebras

III) Differential geometry on NC manifolds

IV) Riemannian geometry on NC manifolds

Motivations

Classical Mechanics $\longrightarrow$ Quantum Mechanics observables becomes NC.

Classical Gravity $\longrightarrow$ Quantum Gravity spacetime coordinates becomes NC.

-Supported by impossibility to test (with ideal experiments) the structure of spacetime at infinitesimal distances. One is then lead to relax the usual assumtion of spacetime as a smooth manifold (a continuum of points) and to conceive a more general structure like a lattice or a noncommutative spacetime that naturally encodes a discretized or cell-like structure.

-In a noncommutative geometry a dynamical aspect of spacetime is encoded at a more basic kinematical level.

-It is interesting to formulate a consistent gravity theory on this spacetime. I see NC gravity as an effective theory. This theory may capture some aspects of a quantum gravity theory.

NC geometry approaches

• Algebraic: generators and relations. For example

$$[\widehat{x}^i, \widehat{x}^j] = i\theta^{ij} \qquad \textit{canonical}$$

$$[\widehat{x}^i, \widehat{x}^j] = i f^{ij}{}_k \widehat{x}^k \qquad \textit{Lie algebra}$$

$$\widehat{x}^i \widehat{x}^j - q \widehat{x}^j \widehat{x}^i = 0 \qquad \textit{quantum plane} \tag{1}$$

Quantum groups and quantum spaces are usually introduced in this way.

• $C^\star$-algebra completion; representation as bounded operators on Hilbert space. Spectral Triples.

• $\star$-product approach, usual space of functions, but we have a bi-differential operator $\star$, (noncommutative and associative) e.g.,

$$(f \star h)(x) = \mathrm{e}^{-\frac{i}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}} f(x)h(y)\Big|_{x=y} .$$

Notice that if we set

$$\mathcal{F}^{-1} = \mathrm{e}^{-\frac{i}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}}$$

then

$$(f \star h)(x) = \mu \circ \mathcal{F}^{-1}(f \otimes h)(x)$$

where μ is the usual product of functions $\mu(f \otimes g) = fh$.

The element

$$\mathcal{F} = \mathrm{e}^{\frac{i}{2}\lambda\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial y^\nu}} = 1\otimes 1+\frac{i}{2}\lambda\theta^{\mu\nu}\partial_\mu\otimes\partial_\nu-\frac{1}{8}\lambda^2\theta^{\mu_1\nu_1}\theta^{\mu_2\nu_2}\partial_{\mu_1}\partial_{\mu_2}\otimes\partial_{\nu_1}\partial_{\nu_2}+\dots$$

is an example of *Drinfeld Twist*.

In this presentation noncommutative spacetime will be spacetime equipped with a $\star$-product. We will not discuss when $f \star g$ is actually convergent. We will therefore work in the context of formal deformation quantization (Kontsevich 2003).

Convergence aspects can be studied [Rieffel], [Bieliavsky Gayral].

The method of constructing $\star$-products using *Drinfeld twists* is not the most general method, however it is quite powerful, and the class of $\star$-products obtained is quite wide.

Key point:

First deform a group in a quantum group. Then consider commutative algebras that carry a representation of the intitial group and deform these algebras so that they carry a representation of the quantum group.

Given a manifold M the group is (a subgroup of) that of Diffeomorphisms; the algebra is that of functions on M.
-This method allows to deform also the tensor algebra, the exterior algebra and the differential geometry.
-It fits very well the construction of gravity theories on a noncommutative manifold based on invariance under quantum diffeomeorphisms.

[P.A., Blohmann, Dimitrijevic, Meyer, Schupp, Wess]
[P.A., Dimitrijevic, Meyer, Wess]
[P.A., Castellani]

The work we present today is a further development of these previous works on NC gravity. We present the Cartan calculus, the Cartan structure equations for torsion and curvature, and the Levi-Civita connection in NC Riemannian geometry.

Physical applications

A noncommutative gravity theory is a modified gravity theory where the modification comes from an expected feature of spacetime at quantum gravity regimes. The theory can be coupled to matter.

Applications include *quantitative* studies in:

- Early universe cosmology near Planck scales. Here inflation, through its predictions for the primordial perturbations, provides a particularly suitable framework.

- Study of propagation of light in cuved NC spacetime. NC dispersion relations. Velocity of light depends on its frequency if spacetime is curved: *see talk by Anna Pachol*.
Results can be experimentally tested with gamma ray burst data from distant supernovae, and eventually NC gravity theory coupled to massless fields (light) could be verified or falsified.

Drinfeld Twist and quantum Lie algebras

Let g be a Lie algebra and Ug its universal enveloping algebra.
Ug is a Hopf algebra. On generators $u \in g$

$$\Delta(u) = u \otimes 1 + 1 \otimes u \ , \ \ \varepsilon(u) = 0 \ , \ \ S(u) = -u \ .$$

Definition [Drinfeld]. A twist $\mathcal{F}$ is an invertible element $\mathcal{F} \in Ug \otimes Ug$ such that

$$\mathcal{F} \otimes 1(\Delta \otimes id)\mathcal{F} = 1 \otimes \mathcal{F}(id \otimes \Delta)\mathcal{F} \qquad \text{in } \ Ug \otimes Ug \otimes Ug$$

Example: Let g be the (abelian) Lie algebra of translations $\partial_\mu = \frac{\partial}{\partial x^u}$ on $\mathbb{R}^4$.

$$\begin{aligned} \mathcal{F} &= e^{\frac{i}{2}\lambda\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} \\ &= 1 \otimes 1 + \frac{1}{2}\lambda\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu + O(\lambda^2) \end{aligned}$$

θ^{ab} is a constant (antisymmetric) matrix

Definition An algebra A is a (left) Ug-module algebra if it is a Ug-module (i.e. if there is an action of Ug on A) and, for all $\xi \in Ug$ and $a, b \in A$,

$$\mathcal{L}_\xi(ab) = \mathcal{L}_{\xi_1}(a)\mathcal{L}_{\xi_2}(b) .$$

Equivalently all $u \in g \subset Ug$ act as derivations of the algebra:

$$\mathcal{L}_u(ab) = \mathcal{L}_u(a)b + a\mathcal{L}_u(b) .$$

Example: ∂_μ is a derivation of the algebra $C^\infty(\mathbb{R}^4)$.

Definition A Ug**-module** A**-bimodule** Ω is both a Ug-module and an A-bimodule in a compatible way, for all $\xi \in Ug$, $a \in A, \omega \in \Omega$,

$$\mathcal{L}_\xi(a \cdot \omega) = \mathcal{L}_{\xi_1}(a)_{\xi_2}(\omega) \; , \;\; \mathcal{L}_\xi(\omega \cdot a) = \mathcal{L}_{\xi_1}(\omega) \cdot \mathcal{L}_{\xi_2}(a) .$$

${}^{H}{}_{A}\mathscr{M}_{A}$ denotes the category of Ug-modules A-bimodules; we write $\Omega \in {}^{Ug}{}_{A}\mathscr{M}_{A}$.

Example: The bimodule Ω of forms over $\mathbb{R}^4$.

Theorem [Drinfeld]
i) Given a Hopf algebra Ug, i.e., $(Ug, \mu, \Delta, S, \varepsilon)$, and a twist $\mathcal{F} \in Ug \otimes Ug$ then we have a new Hopf algebra $Ug^{\mathcal{F}}$:

$$(Ug^{\mathcal{F}}, \mu, \Delta^{\mathcal{F}}, S^{\mathcal{F}}, \varepsilon) \; ; \tag{2}$$

with triangular R-matrix $\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1}$. The new coproduct $\Delta^{\mathcal{F}}$ is, for all $\xi \in H$,

$$\Delta^{\mathcal{F}}(\xi) = \mathcal{F}\Delta(\xi)\mathcal{F}^{-1} \; .$$

ii) Given an H-module algebra A, then we have the $H^{\mathcal{F}}$-module algebra $A_\star$ (or $A_{\mathcal{F}}$) where, setting $\mathcal{F}^{-1} = \bar{\mathsf{f}}^{\alpha} \otimes \bar{\mathsf{f}}_{\alpha}$,

$$a \star b := \mathcal{L}_{\bar{\mathsf{f}}}^{\alpha}(a)\, \mathcal{L}_{\bar{\mathsf{f}}_{\alpha}}(b) \; .$$

Notation: we frequently omit writing the action $\mathcal{L}$ and simply write

$$a \star b = \bar{\mathsf{f}}^{\alpha}(a)\, \bar{\mathsf{f}}_{\alpha}(b) = \mu \circ \mathcal{F}^{-1}(a \otimes b) \; .$$

iii) Given a module $\Omega \in {}^{Ug}{}_{A}\mathscr{M}_{A}$ then we have $\Omega_\star \in {}^{Ug^{\mathcal{F}}}{}_{A_\star}\mathscr{M}_{A_\star}$,

$$a \star v = \cdot \circ \mathcal{F}^{-1} \triangleright (a \otimes v) = (\bar{\mathsf{f}}^{\alpha} \triangleright a) \cdot (\bar{\mathsf{f}}_{\alpha} \triangleright v) \; ,$$

$$v \star b = \cdot \circ \mathcal{F}^{-1} \triangleright (v \otimes b) = (\bar{\mathsf{f}}^{\alpha} \triangleright v) \cdot (\bar{\mathsf{f}}_{\alpha} \triangleright b) \; .$$

iv) The categories ${}^{Ug}{}_{A}\mathscr{M}_{A}$ and ${}^{Ug^{\mathcal{F}}}{}_{A_\star}\mathscr{M}_{A_\star}$ are equivalent.

Ug acts on itself via the adjoint action

$$\begin{aligned} \mathcal{L} : Ug \otimes Ug &\longrightarrow Ug \\ \xi \otimes \zeta &\longmapsto \mathcal{L}_\xi(\zeta) \equiv \xi(\zeta) := \xi_1 \zeta S(\xi_2) \end{aligned}$$

This action is compatible with the product in Ug, $\xi(\zeta\gamma) = \xi_1(\zeta)\,\xi_2(\gamma)$, hence, Ug as an algebra is a Ug-module algebra and can be deformed as in **ii)**.
Product in $Ug_\star$

$$\xi \star \zeta := \bar{\mathrm{f}}^\alpha(\xi)\,\bar{\mathrm{f}}_\alpha(\zeta)\ .$$

As algebras $Ug_\star$ and $Ug^{\mathcal{F}}$ are isomorphic via

$$D\ : Ug_\star \to Ug^{\mathcal{F}}\ \ , \quad D(\xi) := \bar{\mathrm{f}}^\alpha(\xi)\,\bar{\mathrm{f}}_\alpha$$

$$D(\xi \star \zeta) = D(\xi)D(\zeta)\ , \quad \text{for all } \xi, \zeta \in Ug_\star$$

$Ug_\star$ is a triangular Hopf algebra equivalent to $Ug^{\mathcal{F}}$ by defining

$$\begin{aligned} &\Delta_\star = (D^{-1} \otimes D^{-1}) \circ \Delta^{\mathcal{F}} \circ D \\ &S_\star = D^{-1} \circ S^{\mathcal{F}} \circ D \\ &\mathcal{R}_\star = (D^{-1} \otimes D^{-1})(\mathcal{R}) \end{aligned}$$

Corollary Every $Ug^{\mathcal{F}}$-module algebra $A_\star$ with $Ug^{\mathcal{F}}$-action $\mathcal{L} : Ug^{\mathcal{F}} \otimes A \to A$ is a $Ug_\star$-module algebra with $Ug_\star$-action

$$\begin{aligned} \mathcal{L}^\star := \mathcal{L} \circ (D \otimes id) : Ug_\star \otimes A_\star &\longrightarrow A_\star \\ \xi \otimes a &\longmapsto \mathcal{L}^\star_\xi(a) := \mathcal{L}_{D(\xi)}(a) \ . \end{aligned} \tag{3}$$

Similarly every module $\Omega_\star \in {}^{Ug^{\mathcal{F}}}{}_{A_\star}\mathscr{M}_{A_\star}$, is a module $\Omega_\star \in {}^{Ug_\star}{}_{A_\star}\mathscr{M}_{A_\star}$.

If $A_\star = Ug_\star$, then $\mathcal{L}^\star$ is the $Ug_\star$-adjoint action: $\mathcal{L}^\star_\xi(\zeta) = \xi_{1_\star} \star \zeta \star S_\star(\xi_\star)$.

Definition The quantum Lie algebra $g_\star$ of $Ug_\star$ is the vecotr space g with the bracket

$$[u,v]_\star = [\bar{\mathsf{f}}^{\alpha}(u), \bar{\mathsf{f}}_{\alpha}(v)] \; . \tag{4}$$

Theorem The quantum Lie algebra $g_\star$ is the subspace of $Ug_\star$ of braided primitive elements

$$\Delta_\star(u) = u \otimes 1 + \bar{R}^\alpha_\star \otimes \mathcal{L}^\star_{\bar{R}_{\star\alpha}}(u)$$

The Lie bracket $[\;,\;]_\star$ is the $Ug_\star$-adjoint action

$$[u,v]_\star = u_{1_\star} \star v \star S_\star(u_{2_\star}) \; . \tag{5}$$

Furthermore the Lie bracket $[\;,\;]_\star$ satisfies

$$[u,v]_\star = -[\bar{R}^\alpha(v), \bar{R}_\alpha(u)]_\star \qquad \text{Braided-antisymm. prop.}$$

$$[u,[v,z]_\star]_\star = [[u,v]_\star, z]_\star + [\bar{R}^\alpha(v), [\bar{R}_\alpha(u), z]_\star]_\star \qquad \text{Braided-Jacoby identity ,}$$

$$[u,v]_\star = u \star v - \bar{R}^\alpha(v) \star \bar{R}_\alpha(u) \; . \qquad \text{Braided-commutator}$$

$g_\star$ is the quantum Lie algebra of the quantum enveloping algebra $Ug_\star$ in the terminology of Woronowicz.

[P.A., Dimitrijevic, Meyer, Wess]

Differential geometry on NC manifolds

Let g be a subalgebra of the Lie algebra Ξ of vector fields on a manifold M.

A twist $\mathcal{F} \in Ug \otimes Ug$ is automatically a twist $\mathcal{F} \in U\Xi \otimes U\Xi$.

$$
\begin{array}{llcl}
\text{Function algebra} & A = C^\infty(M) & \xrightarrow{\mathcal{F}^{-1}} & A_\star = C^\infty(M)_\star\,, \quad \star = \mu \circ \mathcal{F}^{-1} \\
\text{Tensor algebra} & (\mathcal{T}, \otimes) & \longrightarrow & \mathcal{T}_\star = (\mathcal{T}, \otimes_\star)\,, \quad \otimes_\star = \otimes \circ \mathcal{F}^{-1} \\
\text{Exterior algebra} & (\Omega^\bullet, \wedge) & \longrightarrow & \Omega^\bullet_\star = (\Omega^\bullet, \wedge_\star)\,, \quad \wedge_\star = \wedge \circ \mathcal{F}^{-1}
\end{array}
$$

Explicitly

$a \star b = \bar{\mathrm{f}}^\alpha(a)\bar{\mathrm{f}}_\alpha(b), \quad \tau \otimes_\star \tau' = \bar{\mathrm{f}}^\alpha(\tau) \otimes \bar{\mathrm{f}}_\alpha(\tau'), \quad \theta \wedge_\star \theta' = \bar{\mathrm{f}}^\alpha(\theta) \wedge \bar{\mathrm{f}}_\alpha(\theta')$.

$$
\begin{array}{rlcl}
\text{Bimodule of 1-forms} & \Omega \in {}^{U\Xi}{}_A\mathscr{M}_A & \longrightarrow & \Omega_\star \in {}^{U\Xi_\star}{}_{A_\star}\mathscr{M}_{A_\star} \\
\text{Bimodule of vector fields} & \Xi \in {}^{U\Xi}{}_A\mathscr{M}_A & \longrightarrow & \Xi_\star \in {}^{U\Xi_\star}{}_{A_\star}\mathscr{M}_{A_\star}.
\end{array}
$$

Remark $\Xi_\star$ is both the quantum Lie algebra of $Ug_\star$ and an $A_\star$-bimodule.

Nondegenerate $\star$-pairing between the $\star$-bimodules $\Xi_\star$ and $\Omega_\star$

$$\begin{array}{rcl} \langle\ ,\ \rangle_\star :\ \Xi_\star \times \Omega_\star & \longrightarrow & A_\star\ , \\ (u,\omega) & \longmapsto & \langle \xi,\omega\rangle_\star := \langle \bar{\mathsf{f}}^{\,\alpha}(u), \bar{\mathsf{f}}_{\,\alpha}(\omega)\rangle\ . \end{array} \tag{6}$$

Extends to the $\star$-contraction operator on tensor fields

$$i^\star_u = i^\alpha_{\bar{\mathsf{f}}}(u) \circ \mathcal{L}_{\bar{\mathsf{f}}_{\,\alpha}}$$

Theorem (Cartan calculus)
-The exterior derivative d is compatible with the $\wedge_\star$-product and gives a $\star$-differential calculus $(\Omega^\bullet_\star, \mathsf{d})$.

-The contraction operator on $\Omega^\bullet_\star$ is a braided derivation:

$$i^\star_u(\theta \wedge_\star \theta') = i^\star_u(\theta) \wedge_\star \theta' + (-1)^n \bar{R}^\alpha(\theta') \wedge_\star i^\star_{\bar{R}_\alpha(u)}(\theta')$$

-We have the braided Cartan calculus equalities

$$[\mathcal{L}^\star_u, \mathcal{L}^\star_v]_\star = \mathcal{L}^\star_{[u,v]_\star}\ ,\ \ [\mathcal{L}^\star_u, i^\star_v]_\star = i^\star_{[u,v]_\star}\ ,\ \ [i^\star_u, i^\star_v]_\star = 0\ ,$$

$$[\mathcal{L}^\star_u, \mathsf{d}]_\star = 0\ ,\ \ [i^\star_u, \mathsf{d}]_\star = \mathcal{L}^\star_u\ ,\ \ [\mathsf{d}, \mathsf{d}]_\star = 0\ ;$$

where $[A,B]_\star = A \circ B + (-1)^{\mathsf{deg}(A)\,\mathsf{deg}(B)} \bar{R}^\alpha(B) \circ \bar{R}_\alpha(A)$ is the graded braided commutator of linear maps A, B on $\Omega^\bullet_\star$.

Connections

As usual **right connection** on $V_\star \in {}_{A_\star}\mathscr{M}_{A_\star}$ is a linear map $\nabla^\star : V_\star \to V_\star \otimes_{A_\star} \Omega_\star$, satisfying the right Leibniz rule, for all $v \in V_\star$ and $a \in A_\star$,

$$\nabla^\star(v \star a) = (\nabla v) \star a + v \otimes_{A_\star} \mathrm{d}a \,.$$

Theorem (not necessarily equivariant connections)
If $V \in {}^{Ug}{}_{A}\mathscr{M}_{A}$, then $V_\star \in {}^{Ug_\star}{}_{A_\star}\mathscr{M}_{A_\star}$ and the isomorphism $D : Ug^{\mathcal{F}} \to Ug_\star$ can be lifted to $V_\star$ so that there is a 1-1 correspondence between right connections on V and on $V_\star$.

[P.A., Schenkel]

The connection on $V_\star$ extends to a connection on form valued sections:

$$d_\nabla : V_\star \otimes \Omega^\bullet_\star \to V_\star \otimes \Omega^\bullet_\star$$

$$v \otimes \theta \mapsto \nabla^\star(v) \wedge_\star \theta + v \otimes_\star d\theta$$

Curvature $R_{\nabla^\star} := d_{\nabla^\star} \circ d_{\nabla^\star}$ is a right $A_\star$-linear map.

Theorem Since V is a commutative A-bimodule the right connection ∇ on $V_\star$ is also a braided left connection:

$$\nabla^\star(a \star w) = \bar{R}^\alpha(a) \star \bar{R}_\alpha(\nabla^\star)(w) + \bar{R}^\alpha(w) \otimes_\star \bar{R}_\alpha(\mathrm{d}a) \ . \qquad (7)$$

Rmk. If ∇ is $Ug_\star$-equivariant we recover the notion of A-bimodule connection:

$$\nabla(a \star w) = a \star \nabla(w) + R_\alpha(w) \otimes_\star R^\alpha(\mathrm{d}a) \ .$$

[Mourad], [Dubois-Violette Masson]

In particular, for $V = \Omega$, a right connection $\nabla : \Omega \to \Omega \otimes \Omega$ is deformed to a NC right connection

$$\nabla^* : \Omega_\star \to \Omega_\star \otimes_\star \Omega_\star$$

that is then uniquely extended to

$$d_{\nabla^*} : \Omega_\star \otimes \Omega_\star^\bullet \to \Omega_\star \otimes_\star \Omega_\star^\bullet$$

Next we introduce the connection on vector fields $\Xi_\star$, considering vector fields dual to 1-forms $\Omega_\star$ via the pairing $\langle\ ,\ \rangle_\star$ and therefore inducing on this dual bimodule the dual connection:

$$\langle {}^*\nabla v, \omega\rangle_\star = \mathrm{d}\langle v, \omega\rangle_\star - \langle v, \nabla^* \omega\rangle_\star$$

This is a **left**-connection

$$^*\nabla : \Xi_\star \to \Omega_\star \otimes - \star \, \Xi_\star \quad , \quad ^*\nabla(av) = da \otimes_\star v + a^*\nabla(v)$$

It is easily lifted to a connection

$$d_{^*\nabla} : \Omega^\bullet_\star \otimes_\star \Xi_\star \to \Omega^\bullet_\star \otimes \Xi_\star$$

that satisfies the Leibnitz rule

$$d_{^*\nabla}(\theta \wedge_\star \psi) = d\theta \wedge_\star \psi + (-1)^{|\theta|} \, \theta \wedge_\star d_{^*\nabla}\psi \ ,$$

with $\psi = \eta \otimes_\star v$ (vector field valued in the exterior algebra).

From now on, for ease of notation the index $\star$ is omitted but on the connections

Define covariant derivative along a vector field

$$d_{\star\nabla_u} := i_u d_{\star\nabla} + d_{\star\nabla} i_u$$

then we have the Cartan relation:

$$i_u d_{\star\nabla_v} - d_{\star\nabla_{\alpha_v}} i_{\alpha u} = i_{[u,v]}$$

The definitions of curvatures on the module Ξ and the dual module $|Omega$ are related by **Theorem**

$$\langle d^2_{\star\nabla} z, \theta\rangle = -\langle z, d^2_{\nabla\star}\theta\rangle$$

Moreover, define as in [P.A., Dimitrijevic, Meyer, Wess]

$$\mathsf{R}(u,v,z) := {}^\star\nabla_u {}^\star\nabla_v z - {}^\star\nabla_{\bar{R}^\alpha(v)} {}^\star\nabla_{\bar{R}_\alpha(u)} z - {}^\star\nabla_{[u,v]} z$$

then

$$\mathsf{R}(u,v,z) = -i_u i_v d^2_{\star\nabla} z \ , \qquad \langle \mathsf{R}(u,v,z), \theta\rangle = \langle u\otimes v\otimes z, d^2_{\nabla\star}\omega\rangle$$

This last equality is Cartan second structure equation in coordinate free notation.

Torsion

Let $I \in \Omega \otimes \Xi$ be the canonical form such that $i_v(I) = v$.

Locally in a basis $I = \theta^i \otimes e_i$, with $\langle e_i, \theta^j \rangle = \delta_i^j$.

Define $T(u, v) :=^\star \nabla_u v - {}^\star\nabla_{\bar{R}^\alpha(v)} \bar{R}_\alpha(u) - [u, v]$

Theorem

$$T(u, v) = -i_u i_v d_{*\nabla} I$$

In a basis $\nabla^\star \theta^i = \theta^j \otimes \omega_j{}^i$, then ${}^\star\nabla e_i = -\omega_i{}^j \otimes e_j$

$d^2_{\nabla^\star} \theta^i = \theta^k \otimes \Omega_k{}^i$ with

$$\Omega_k{}^l = d\omega_k{}^l - \omega_k{}^m \wedge \omega_m{}^l$$

$$d_{\star\nabla} I = d_{\star\nabla}(\theta^i \otimes e_i) = (d\theta^i - \theta^j \wedge \omega_j{}^i) \otimes e_j$$

Propedeutical to the study of Levi-Civita connections is the study of

Tensor product structure

If $V, W \in {}^{Ug}\mathscr{M}$ then $V \otimes W \in {}^{Ug}\mathscr{M}$ by defining, for all $\xi \in Ug$, $v \in V$ and $w \in W$,

$$\xi(v \otimes w) := \xi_1(v) \otimes \xi_2(w) \ ,$$

Given linear maps $V \xrightarrow{P} \widetilde{V}$ and $W \xrightarrow{Q} \widetilde{W}$, then $V \otimes W \xrightarrow{P \otimes Q} \widetilde{V} \otimes \widetilde{W}$ is defined by

$$(P \otimes Q)(v \otimes w) = P(v) \otimes Q(w)$$

we have $\xi \blacktriangleright (P \otimes Q) \neq (\xi_1 \blacktriangleright P) \otimes (\xi_2 \blacktriangleright Q)$ in general.

The tensor product compatible with the Ug-action is given by **Definition of** $\otimes_{\mathcal{R}}$ [Majid 94]

$$(id \otimes_R Q) = \tau_{\mathcal{R}} \circ (Q \otimes id) \circ \tau_{\mathcal{R}}^{-1} \, ;$$

where

$$\tau_{\mathcal{R}\,W,V} : W \otimes V \to V \otimes W \; , \quad w \otimes v \mapsto \tau_{\mathcal{R}\,W,V}(w \otimes v) = \bar{R}^{\alpha}(v) \otimes \bar{R}_{\alpha}(w) \; ,$$

is the braiding isomorphism and where we used the notation $\mathcal{R}^{-1} = \bar{R}^{\alpha} \otimes \bar{R}_{\alpha}$.

Proposition $\otimes_{\mathcal{R}}$ is associative and compatible with the Ug-action.

Sum of connections (Connections on tensor product modules)

Let $\nabla : V \to V \otimes \Omega$ and $\tilde{\nabla} : W \to W \otimes \Omega$

$$\nabla \oplus_{\mathcal{R}} \tilde{\nabla} : V \otimes_A W \to V \otimes_A W \otimes_A \Omega \ ,$$

defined by: $\nabla \oplus_{\mathcal{R}} \tilde{\nabla} := \nabla \otimes id + id \otimes_{\mathcal{R}} \tilde{\nabla}$.

Associativity: $\left(\nabla \oplus_{\mathcal{R}} \tilde{\nabla}\right) \oplus_{\mathcal{R}} \check{\nabla} = \nabla \oplus_{\mathcal{R}} \left(\tilde{\nabla} \oplus_{\mathcal{R}} \check{\nabla}\right)$.

Ug-action compatibility: $\xi \blacktriangleright (\nabla \oplus_{\mathcal{R}} \tilde{\nabla}) = (\xi \blacktriangleright \nabla) \oplus_{\mathcal{R}} (\xi \blacktriangleright \tilde{\nabla})$.

$\star$-Riemaniann geometry

$\star$-symmetric elements:

$$\omega \otimes_\star \omega' + \bar{R}^\alpha(\omega') \otimes_\star \bar{R}_\alpha(\omega) \ .$$

Any symmetric tensor in $\Omega \otimes \Omega$ is also a $\star$-symmetric tensor in $\Omega_\star \otimes_\star \Omega_\star$, proof: expansion of above formula gives $\bar{\mathsf{f}}^\alpha(\omega) \otimes \bar{\mathsf{f}}_\alpha(\omega') + \bar{\mathsf{f}}_\alpha(\omega') \otimes \bar{\mathsf{f}}^\alpha(\omega)$.

Requiring the right connection $\nabla^\star$ to vanish on the metric, we obtain, similarly to the classical case, a condition for the torsion free left connection ${}^\star\nabla$ on vector fields. Considering the ciclically permuted equations, adding and subtracting we obtain the Levi-Civita connection ${}^\star\nabla_g$.

$$\begin{aligned} 2\langle {}^\alpha v \otimes_\star {}^\star\nabla_{\alpha u} z, g\rangle &= \mathcal{L}^\star_u \langle v \otimes_\star z, g\rangle_\star - \mathcal{L}^\star_{\alpha_v} \langle {}_\alpha u \otimes_\star z, g\rangle_\star + \mathcal{L}^\star_{\alpha\beta_z} \langle {}_\alpha u \otimes_\star {}_\beta v, g\rangle_\star \\ &\quad + \langle [u,v]_\star \otimes_\star z, g\rangle_\star + \langle u \otimes_\star [v,z]_\star, g\rangle_\star + \langle [u, {}^\beta z]_\star \otimes_\star {}_\beta v, g\rangle_\star \end{aligned}$$

were ${}^\alpha v := \bar{R}^\alpha(v)$ and ${}_\alpha u := \bar{R}_\alpha(u)$. Now, since u, v, z are arbitrary, the pairing is nondegenerate and the metric is also nondegenerate, knowledge of the l.h.s. uniquely defines the connection.

This result generalizes to any Drinfeld twist previous ones found for abelian Drinfeld twist NC geometry.