# Kinematics of particles with q-de Sitter-inspired symmetries


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partially based on:
Barcaroli, Gubitosi, Phys.Rev. D93 (2016)

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# Motivations

✦ Hopf algebras provide consistent framework to introduce an invariant energy scale (as expected in QG) consistent with (deformed) spacetime symmetries

✦ k-Poincaré was used to develop phenomenology associated to deformed Poincaré symmetry, in particular focussing on energy-dependent time of travel of relativistic particles

*•Amelino-Camelia, Kowalski-Glikman, Mandanici, Procaccini, Int. J. Mod. Phys. A20 (2005)*

✦ opportunities for phenomenology arise in contexts where spacetime curvature is actually non-negligible (early universe, propagation of photons from Gamma-ray Bursts..)

*•Amelino-Camelia, Ellis, Mavromatos, Nanopoulos, Sarkar, Nature 393 (1998)*
*•M. Ackermann et al. (Fermi GBM/LAT), Nature 462(2009)*
*•Amelino-Camelia, Fiore, Guetta, Puccetti, Adv. High Energy Phys. 2014*

✦ extension of results fond in kP to curved spacetime is non-trivial, as one would in general expect some sort of interplay between effects of curvature and of quantum symmetry deformation (alternatively, curvature in momentum space)

*•Amelino-Camelia, Smolin, Starodubtsev, Class. Quant. Grav. 21(2004)*
*•Marciano, Amelino-Camelia, Bruno, Gubitosi, Mandanici, Melchiorri, JCAP 1006 (2010)*

✦ q-de Sitter provides a model for quantum-deformed symmetries associated to a curved (de Sitter) spacetime

# de Sitter algebra

✦ 1+1 dimensional de Sitter manifold can be described as the 2-dim hypersurface embedded in a 3-dim Minkowski manifold

$$(z^0)^2 - (z^1)^2 - (z^2)^2 = -H^{-2}$$

✦ line element in flat-slicing coordinates (comoving coordinates)

$$ds^2 = (dx^0)^2 - e^{2Hx^0}(dx^1)^2$$

✦ algebra of symmetries (first order in H)

$$\begin{aligned} \{\mathcal{P}_0, \mathcal{P}_1\} &= H\,\mathcal{P}_1 \\ \{\mathcal{P}_0, \mathcal{N}\} &= \mathcal{P}_1 - H\,\mathcal{N} \\ \{\mathcal{P}_1, \mathcal{N}\} &= \mathcal{P}_0 \end{aligned}$$

✦ mass Casimir $\mathcal{C}_{dS} = \mathcal{P}_0^2 - \mathcal{P}_1^2 + 2H\mathcal{N}\mathcal{P}_1$

✦ representation on phase space:

$$\begin{aligned} \{x^\mu, x^\nu\} &= 0\,, \\ \{x^\mu, p_\nu\} &= -\delta^\mu_\nu\,, \\ \{p_\mu, p_\nu\} &= 0\,. \end{aligned} \longrightarrow \begin{aligned} \mathcal{P}_0 &= p_0 - Hx^1 p_1\,, \\ \mathcal{P}_1 &= p_1\,, \\ \mathcal{N} &= p_1\,x^0 + p_0\,x^1 - H\left(p_1 (x^0)^2 + \frac{1}{2} p_1\,(x^1)^2\right) \end{aligned}$$

# de Sitter particle kinematics

- evolution of phase space coordinates is given by Hamilton equations with the Hamiltonian given by the Casimir

$$\begin{aligned} \dot{x}^0 &\equiv \{\mathcal{C}_{dS}, x^0\} = 2p_0\,, \\ \dot{x}^1 &\equiv \{\mathcal{C}_{dS}, x^1\} = -2p_1(1-2H\,x^0)\,, \\ \dot{p}_0 &\equiv \{\mathcal{C}_{dS}, p_0\} = -2H\,p_1^2\,, \\ \dot{p}_1 &\equiv \{\mathcal{C}_{dS}, p_1\} = 0\,, \end{aligned}$$

- the massless condition $\mathcal{C}_{dS} = 0$ relates energy and spatial momentum:

$$p_0 = -p_1(1 - Hx^0)$$

- coordinate velocity: $v \equiv \dfrac{\dot{x}^1}{\dot{x}^0} = -\dfrac{p_1}{p_0}(1 - 2Hx^0) = 1 - Hx^0$

- worldline

$$x^1 - \bar{x}^1 \equiv \int_0^\tau \dot{x}^1\,d\tau = \int_{\bar{x}^0}^{x^0} v\,dx^0 = x^0 - \bar{x}^0 - \frac{1}{2}H\left((x^0)^2 - (\bar{x}^0)^2\right)$$

# k-Poincaré algebra - bicrossproduct basis

✦ algebra of symmetries (first order in $\ell$ )

•*Lukierski, Nowicki, Ruegg, Phys. Lett. B 293 (1992)*
•*Lukierski, Ruegg, Nowicki, Tolstoi, Phys. Lett. B 264 (1991)*
•*Majid, Ruegg, Phys.Lett. B334 (1994)*

$$\begin{aligned}
\{\mathcal{P}_0, \mathcal{P}_1\} &= 0\,, \\
\{\mathcal{P}_0, \mathcal{N}\} &= \mathcal{P}_1\,, \\
\{\mathcal{P}_1, \mathcal{N}\} &= \mathcal{P}_0 - \ell\left(\mathcal{P}_0^2 + \frac{1}{2}\mathcal{P}_1^2\right)\,,
\end{aligned}$$

✦ mass Casimir

$$\mathcal{C}_\ell = \mathcal{P}_0^2 - \mathcal{P}_1^2 - \ell\,\mathcal{P}_0\mathcal{P}_1^2$$

✦ coproducts and antipodes

$$\begin{aligned}
\Delta(\mathcal{P}_0) &= \mathcal{P}_0 \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{P}_0\,, \\
\Delta(\mathcal{P}_1) &= \mathcal{P}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{P}_1 - \ell\,\mathcal{P}_0 \otimes \mathcal{P}_1\,, \\
\Delta(\mathcal{N}) &= \mathcal{N} \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{N} - \ell\,\mathcal{P}_0 \otimes \mathcal{N}\,.
\end{aligned}$$

$$\begin{aligned}
S(\mathcal{P}_0) &= -\mathcal{P}_0\,, \\
S(\mathcal{P}_1) &= -(1 + \ell\,\mathcal{P}_0)\mathcal{P}_1\,, \\
S(\mathcal{N}) &= -(1 + \ell\,\mathcal{P}_0)\mathcal{N}\,,
\end{aligned}$$

✦ momenta live on a (portion of) de Sitter manifold

•*Kowalski-Glikman, Nowak, Class. Quant. Grav. 20 (2003)*
•*Kowalski-Glikman Phys. Lett. B 547 (2002)*
•*Gubitosi, Mercati, Class.Quant.Grav. 30 (2013)*
•*Amelino-Camelia, Arzano, Kowalski-Glikman, Rosati, Trevisan, Class. Quant. Grav. 29 (2012)*

# Semiclassical approximation

✦ we are interested in studying the propagation of a free relativistic particle, without purely quantum effects (quantum correlations, fuzziness..)

✦ the Planck-scale regime is in this case defined as the limit where $\hbar \to 0$ while keeping the Planck energy ( $E_P = \sqrt{\frac{\hbar c^5}{G}}$ ) finite

✦ in this semiclassical approximation, the symmetries of phase space are described by Poisson brackets satisfying the same relations as the commutators of the Hopf algebra under consideration

✦ spacetime is defined via a classical phase-space construction - commutative coordinates are related to the non-commutative ones via an energy-dependent redefinition

✦ the coproducts and antipodes enter in the analysis only to define finite translation, as discussed later (no multi-particle states)

# k-Poincaré particle kinematics

✦ representation on phase space:

$$\begin{aligned} \{x^\mu, x^\nu\} &= 0\,, \\ \{x^\mu, p_\nu\} &= -\delta^\mu_\nu\,, \\ \{p_\mu, p_\nu\} &= 0\,. \end{aligned} \longrightarrow \begin{aligned} \mathcal{P}_0 &= p_0\,, \\ \mathcal{P}_1 &= p_1\,, \\ \mathcal{N} &= p_1 x^0 + p_0 x^1 - \ell\left(x^1(p_0)^2 + \frac{x^1(p_1)^2}{2}\right)\,, \end{aligned}$$

✦ evolution of phase space coordinates is given by Hamilton equations

$$\begin{aligned} \dot{x}^0 &\equiv \{\mathcal{C}_\ell, x^0\} = 2\,p_0 - \ell\,p_1^2\,, \\ \dot{x}^1 &\equiv \{\mathcal{C}_\ell, x^1\} = -2\,p_1(1+\ell\,p_0)\,, \\ \dot{p}_0 &\equiv \{\mathcal{C}_\ell, p_0\} = 0\,, \\ \dot{p}_1 &\equiv \{\mathcal{C}_\ell, p_1\} = 0\,. \end{aligned}$$

✦ the massless condition $\mathcal{C}_\ell = 0$ relates energy and spatial momentum:

$$p_0 = -p_1(1 - \frac{1}{2}\ell\,p_1)$$

✦ coordinate velocity: $v \equiv \dfrac{\dot{x}^1}{\dot{x}^0} = 1 - \ell\,p_1$

✦ particle worldline

$$x^1 - \bar{x}^1 \equiv \int_0^\tau \dot{x}^1\,d\tau = \int_{\bar{x}^0}^{x^0} v\,dx^0 = \left(x^0 - \bar{x}^0\right)(1 - \ell p_1)$$

# duality between dS and kP

- de Sitter worldline (starting from the origin):

$$x^1 = x^0 - \frac{1}{2} H (x^0)^2$$

- de Sitter energy-momentum relation:

$$p_0 = -p_1 (1 - H x^0)$$

- k-Poincaré worldline (starting from the origin):

$$x^1 = x^0 (1 + \ell p_0)$$

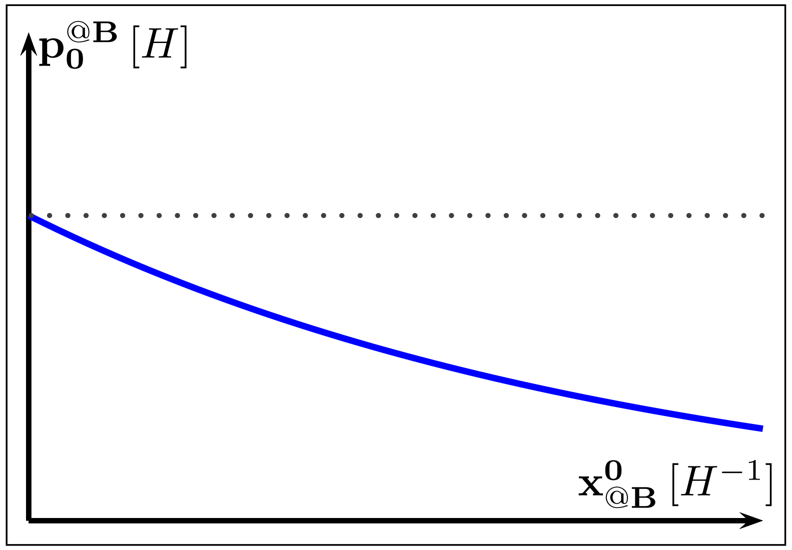
- k-Poincaré energy-momentum relation:

$$p_0 = -p_1 (1 - \frac{1}{2} \ell \, p_1)$$

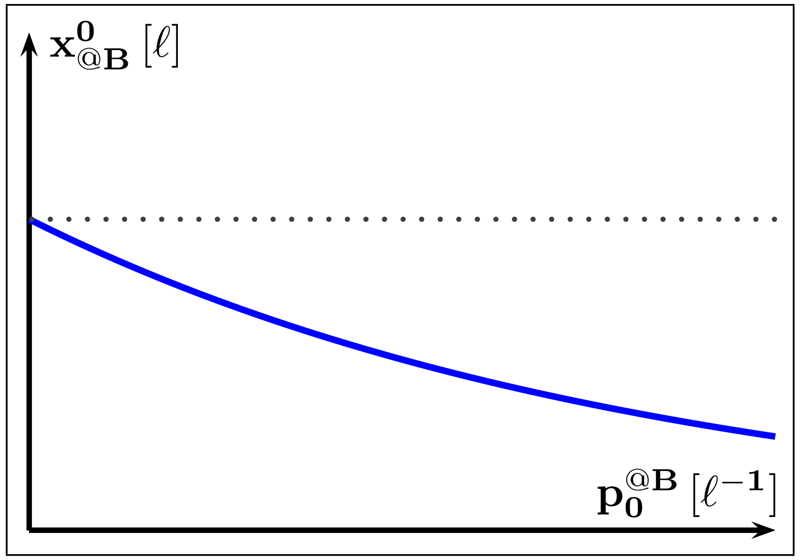
*•Amelino-Camelia, Barcaroli, Gubitosi, Loret, Class.Quant.Grav. 30 (2013)*

# duality between dS and kP

✦ correlation between time of detection and energy, for fixed energy and time of emission (de Sitter spacetime, all orders in H) - *redshift*




✦ correlation between time of detection and energy, for fixed energy and time of emission (de Sitter momentum space, all orders in $\ell$ ) - *time delay*





•*Amelino-Camelia, Barcaroli, Gubitosi, Loret, Class.Quant.Grav. 30 (2013)*

# q-de Sitter algebra [SO$_q$(3, 1)]

•*Lukierski, Ruegg, Nowicki , Tolstoi, Phys. Lett. B264 (1991)*
•*Lukierski, Nowicki,, Ruegg, Phys. Lett. B293 (1992)*
•*Ballesteros, Herranz, del Olmo, Santander, Journal of Physics A: Mathematical and General 26(1993)*
•*Ballesteros, Bruno, Herranz, Czech. J. Phys. 54 (2004)*

✦ algebra of symmetries

$$\begin{aligned} \{\mathcal{P}_0, \mathcal{P}_1\} &= H\,\mathcal{P}_1\,, \\ \{\mathcal{P}_0, \mathcal{N}\} &= \mathcal{P}_1 - H\,\mathcal{N}\,, \\ \{\mathcal{P}_1, \mathcal{N}\} &= \cosh(w/2)\frac{1-e^{-2\frac{w\,\mathcal{P}_0}{H}}}{2w/H} - \frac{1}{H}\sinh(w/2)e^{-\frac{w\,\mathcal{P}_0}{H}}\Theta\,, \end{aligned}$$

$$\Theta \equiv e^{wP_0/2H}(P-HN)e^{wP_0/2H}(P-HN) - H^2 e^{wP_0/2H} N e^{wP_0/2H} N$$

✦ mass Casimir

$$\mathcal{C} = H^2\frac{\cosh(w/2)}{w^2/4}\sinh^2\left(\frac{w\,\mathcal{P}_0}{2H}\right) - \frac{\sinh(w/2)}{w/2}\Theta$$

✦ coproducts and antipodes

$$\begin{aligned} \Delta(\mathcal{P}_0) &= \mathcal{P}_0\otimes\mathbb{I} + \mathbb{I}\otimes\mathcal{P}_0\,, & S(\mathcal{P}_0) &= -\mathcal{P}_0\,, \\ \Delta(\mathcal{P}_1) &= \mathcal{P}_1\otimes\mathbb{I} + e^{-w\frac{\mathcal{P}_0}{H}}\otimes\mathcal{P}_1\,, & S(\mathcal{P}_1) &= -e^{w\frac{\mathcal{P}_0}{H}}\mathcal{P}_1\,, \\ \Delta(\mathcal{N}) &= \mathcal{N}\otimes\mathbb{I} + e^{-w\frac{\mathcal{P}_0}{H}}\otimes\mathcal{N}\,. & S(\mathcal{N}) &= -e^{w\frac{\mathcal{P}_0}{H}}\mathcal{N}\,, \end{aligned}$$

(same structure as the coalgebra of k-Poincaré)

# qdS - contraction to dS and kP

the dimensionless parameter w can be constructed as a combination of the two relevant scales of the model, H and $\ell$. In particular an interesting choice is $w = H\ell$. In this case:

✦ the $H \rightarrow 0$ limit gives the contraction to k-Poincaré algebra

✦ the $\ell \rightarrow 0$ limit gives the contraction to de Sitter algebra algebra

This specific case allows to study the phenomenology of a model where curvature on both spacetime and momentum space is present and to investigate the effects of their interplay


*•Amelino-Camelia, Smolin, Starodubtsev, Class. Quant. Grav. 21(2004)*

*•Marciano, Amelino-Camelia, Bruno, Gubitosi, Mandanici, Melchiorri, JCAP 1006 (2010)*

# qdS - continued

if w= H $\ell$ and at the first order in H, $\ell$ and H $\ell$

- algebra of symmetries (first order in $\ell$ )

$$
\begin{aligned}
\{\mathcal{P}_0, \mathcal{P}_1\} &= H\,\mathcal{P}_1\,, \\
\{\mathcal{P}_0, \mathcal{N}\} &= \mathcal{P}_1 - H\,\mathcal{N}\,, \\
\{\mathcal{P}_1, \mathcal{N}\} &= \mathcal{P}_0 - \ell\,\left(\mathcal{P}_0^2 + \frac{\mathcal{P}_1^2}{2}\right) + \ell\,H\,\mathcal{N}\,\mathcal{P}_1\,,
\end{aligned}
$$

- mass Casimir

$$
\mathcal{C}_{qdS} = \mathcal{P}_0^2 - \mathcal{P}_1^2 - \ell\,\mathcal{P}_0\mathcal{P}_1^2 + 2H\,\mathcal{N}\,\mathcal{P}_1 + 2\ell H\,\mathcal{N}\mathcal{P}_0\mathcal{P}_1
$$

- coproducts and antipodes reduce to the ones of k-Poincaré

$$
\begin{aligned}
\Delta(\mathcal{P}_0) &= \mathcal{P}_0 \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{P}_0\,, \\
\Delta(\mathcal{P}_1) &= \mathcal{P}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{P}_1 - \ell\,\mathcal{P}_0 \otimes \mathcal{P}_1\,, \\
\Delta(\mathcal{N}) &= \mathcal{N} \otimes \mathbb{I} + \mathbb{I} \otimes \mathcal{N} - \ell\,\mathcal{P}_0 \otimes \mathcal{N}\,.
\end{aligned}
\qquad
\begin{aligned}
S(\mathcal{P}_0) &= -\mathcal{P}_0\,, \\
S(\mathcal{P}_1) &= -(1 + \ell\,\mathcal{P}_0)\mathcal{P}_1\,, \\
S(\mathcal{N}) &= -(1 + \ell\,\mathcal{P}_0)\mathcal{N}\,,
\end{aligned}
$$

# qdS - representation on phase space

✦ phase space defined by Poisson brackets:

$$\{x^\mu, x^\nu\} = 0\,,$$
$$\{x^\mu, p_\nu\} = -\delta^\mu_\nu\,,$$
$$\{p_\mu, p_\nu\} = 0\,.$$

✦ representation of generators

$$\begin{aligned}
\mathcal{P}_0 &= p_0 - Hx^1 p_1\,,\\
\mathcal{P}_1 &= p_1\,,\\
\mathcal{N} &= p_1 x^0 + p_0 x^1 - H\left(p_1 (x^0)^2 + \frac{p_1 (x^1)^2}{2}\right) - \\
&\quad -\ell x^1\left((p_0)^2 + \frac{(p_1)^2}{2}\right) + H\ell p_1 x^1\left(p_1 x^0 + \frac{3}{2} p_0 x^1\right)\,.
\end{aligned}$$

✦ representation of mass Casimir

$$\mathcal{C}_{qdS} = p_0^2 - p_1^2 - \ell p_0 p_1^2 + 2Hp_1^2 x^0 + 2\ell H\, p_0 p_1^2 x^0\,.$$

# Worldlines in phase space

✦ evolution of phase space coordinates is given by Hamilton equations with the Hamiltonian given by the Casimir

$$
\begin{aligned}
\dot{x}^0 &\equiv \{\mathcal{C}_{qdS}, x^0\} = 2p_0 - \ell p_1^2(1 - 2H\, x^0)\,, \\
\dot{x}^1 &\equiv \{\mathcal{C}_{qdS}, x^1\} = -2p_1(1 + \ell\, p_0)(1 - 2H\, x^0)\,, \\
\dot{p}_0 &\equiv \{\mathcal{C}_{qdS}, p_0\} = -2H\, p_1^2(1 + \ell\, p_0)\,, \\
\dot{p}_1 &\equiv \{\mathcal{C}_{qdS}, p_1\} = 0\,.
\end{aligned}
$$

✦ the massless condition $\mathcal{C}_{qdS} = 0$ relates energy and spatial momentum:

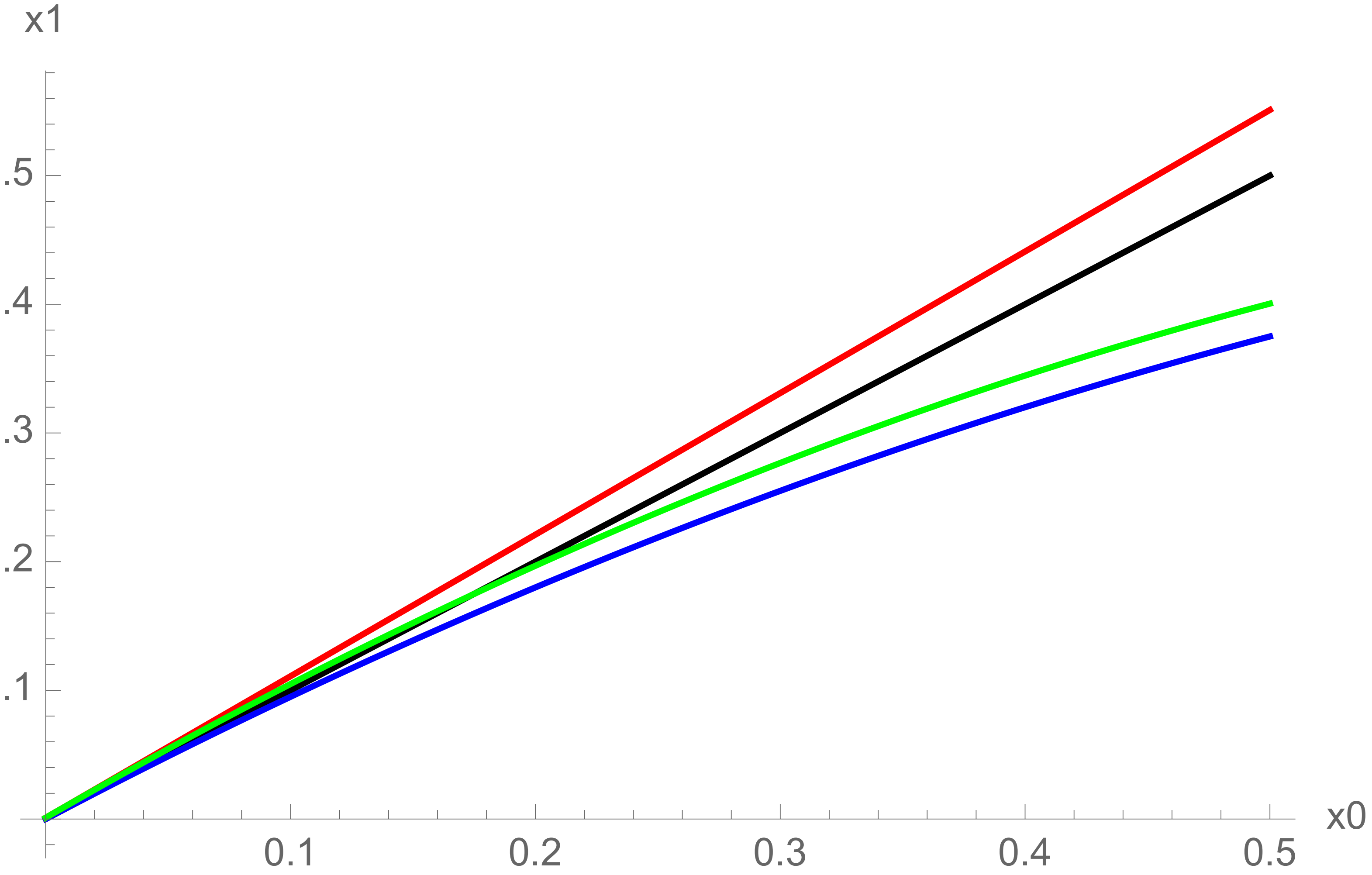
$$p_0 = -p_1 \left(1 - Hx^0 - \ell p_1 \left(\frac{1}{2} - Hx^0\right)\right)$$

✦ coordinate velocity: $v \equiv \dfrac{\dot{x}^1}{\dot{x}^0} = 1 - Hx^0 - \ell p_1(1 - 2Hx^0)$

✦ worldline

$$x^1 - \bar{x}^1 \equiv \int_0^\tau \dot{x}^1\, d\tau = \int_{\bar{x}^0}^{x^0} v\, dx^0 = \left(x^0 - \bar{x}^0\right)(1 - \ell p_1) - \frac{1}{2}H\left((x^0)^2 - (\bar{x}^0)^2\right)(1 - 2\ell p_1)$$

# Worldlines in spacetime




[ $H=1$, $\ell=1$, $p_0 = 0.1$ ]

BLACK: Minkowski
RED: k-Poincaré
BLUE: de Sitter
GREEN: q-de Sitter

# qdS - finite translations

✦ action of finite spacetime translations

$$\mathcal{T}_{\{a^0,a^1\}} \triangleright F = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{\{-a^\mu \mathcal{P}_\mu, \{\ldots, \{-a^\mu \mathcal{P}_\mu}_{n\,\mathrm{times}}, F\} \ldots\}$$

For a Hopf algebra one would in general use the adjoint action. However the adjoint action for k-Poincaré (in bicrossproduct basis) reduces to ordinary action via commutators and the same holds for qdS since the coproducts and antipodes have same structure as in kP.

✦ explicit form:

$$\begin{aligned} x_B^0 &\equiv \mathcal{T}_{\{a^0,a^1\}} \triangleright x_A^0 = x_A^0 - a^0 \,, \\ x_B^1 &\equiv \mathcal{T}_{\{a^0,a^1\}} \triangleright x_A^1 = x_A^1(1 + Ha^0) - a^1(1 + \frac{1}{2}Ha^0) \,, \\ p_0^B &\equiv \mathcal{T}_{\{a^0,a^1\}} \triangleright p_0^A = p_0^A \,, \\ p_1^B &\equiv \mathcal{T}_{\{a^0,a^1\}} \triangleright p_1^A = p_1^A(1 - Ha^0) \,. \end{aligned}$$

(same expression as standard de Sitter translations)

# Energy-dependent redshift

the redshift is found comparing the energy of a particle measured by an observer local at emission (Alice) and the energy measured by an observer local at detection (Bob)

$$z \equiv \frac{p_0^{A@A} - p_0^{B@B}}{p_0^{B@B}}$$

✦ evolution of energy along the particle's worldline as inferred by Alice:

$$p_0^A - p_0^{A@A} = H p_1^A x_A^0 \left(1 - \ell p_1\right) = -H p_0^{A@A} x_A^0 \left(1 + \frac{\ell}{2} p_0^{A@A}\right)$$

✦ the energy measured by Bob is found by applying a translation to the one inferred by Alice (the translation is such that it connects Alice and Bob)

$$p_0^{B@B} = \mathcal{T}_{a^0, a^1} \triangleright p_0^{A@B} = p_0^{A@B} = p_0^{A@A} \left(1 - H x_{A@B}^0 \left(1 + \frac{\ell}{2} p_0^{A@A}\right)\right)$$

✦ redshift:

$$z = H x_{A@B}^0 \left(1 + \frac{\ell}{2} p_0^{A@A}\right) = H a^0 \left(1 + \frac{\ell}{2} p_0^{A@A}\right)$$

# Time delay

compare the times of arrival at the observer Bob of two photons emitted simultaneously by Alice in the origin of her reference frame, with different energies

✦ worldlines of the two particles as seen by Alice:

$$\begin{aligned} x_A^1 &= x_A^0 \left(1 + \ell p_0^{A@A}\right) - \frac{1}{2} H (x_A^0)^2 \left(1 + 2\ell p_0^{A@A}\right) \,, \\ \tilde{x}_A^1 &= \tilde{x}_A^0 \left(1 + \ell \tilde{p}_0^{A@A}\right) - \frac{1}{2} H (\tilde{x}_A^0)^2 \left(1 + 2\ell \tilde{p}_0^{A@A}\right) \end{aligned}$$

✦ the worldlines seen by Bob are found by applying a translation to the ones above and asking that one particle intercepts Bob's spacetime origin (this relates time and space translation parameters) $a^1 = a^0 + \ell a^0 p_0^{A@A} - \frac{1}{2} H \ell (a^0)^2 p_0^{A@A}$

$$\begin{aligned} x_B^1 &= x_B^0 \left(1 + \ell p_0^{A@A}\right) - H x_B^0 \left(\frac{1}{2} x_B^0 + \ell p_0^{A@A} \left(x_B^0 + a^0\right)\right) \,, \\ \tilde{x}_B^1 &= \tilde{x}_B^0 \left(1 + \ell \tilde{p}_0^{A@A}\right) + \ell a^0 (\tilde{p}_0^{A@A} - p_0^{A@A}) - H \tilde{x}_B^0 \left(\frac{1}{2} \tilde{x}_B^0 + \ell \tilde{p}_0^{A@A} \left(\tilde{x}_B^0 + a^0\right)\right) \,. \end{aligned}$$

✦ time at which second photon intercepts Bob's spatial origin:

$$\tilde{x}^0_{B@B} = \ell a^0 \left[ (p_0^{B@B} - \tilde{p}_0^{B@B})(1 + H a^0) \right]$$