Sigma models from Courant algebroids and non-geometric string backgrounds

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Quantum Structure of Spacetime

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Noncommutative geometry, quantum symmetries and quantum gravity II

- Gauge symmetries
  - All successful theories have gauge symmetries.
  - Conventionally, Lie algebra $\mathfrak{g}$ with $\mathfrak{g}$-valued fields A and minimal coupling.

  Q: What happens when we have dynamical space-time?

Noncommutative geometry, quantum symmetries and quantum gravity II

- Gauge symmetries
  - All successful theories have gauge symmetries.
  - Conventionally, Lie algebra $\mathfrak{g}$ with $\mathfrak{g}$-valued fields A and minimal coupling.

  Q: What happens when we have dynamical space-time?

- Dualities
  - Also pretty much everywhere in physics.
  - String theory dualities: very profound properties.
  - e.g. T-duality introduces minimal length, small-large equivalence, emergent spacetime.

  Q: Only for backgrounds with isometries?

An underlying mathematical theme

Groupoids and Algebroids

- Whenever you have a local symmetry there is an algebroid (or groupoid) behind it
- T-duality is an isomorphism between (Courant) algebroids
- Hitchins generalized geometry has its basis on algebroid theory

Noncommutative geometry, quantum symmetries and quantum gravity II

- "Non-geometric" string backgrounds
  - Appear after application of T-duality transformations.
  - Require generalized geometric concepts.
  - Thought to be important for construction of phenomenologically interesting string vacua.

  Q: World-volume description?

Appropriate physical setting

Sigma models

- World-volume description of non-geometric backgrounds in string theory.
- Gauged sigma models are utilised to construct Buscher T-duality rules.
- Courant algebroid has associated 3D sigma model.

In this talk

- Using ideas of generalized geometry introduce the structure of Courant algebroid.
- Construct membrane sigma model using Courant algebroid data (Courant/AKSZ sigma model).
- Extend the membrane sigma model to provide a setting appropriate for description of genuinely non-geometric backgrounds.

Tangent bundle TM

TM is a prototype of a vector bundle.

Sections $X \in \Gamma(TM)$ are vector fields $\rightsquigarrow$ Lie algebra with the ordinary Lie bracket.

The Lie bracket satisfies Jacobi identity and Leibniz rule:

$$[X, fY]_{Lie} = f[X, Y]_{Lie} + (Xf)Y$$

Now we are going to generalize this notion.

Lie algebroids

Lie algebroid: A triple $(L, [\cdot,\cdot]_L, \rho)$ of a vector bundle L over a manifold M, a Lie bracket on $\Gamma(L)$ and a bundle map (anchor) $\rho : L \to \mathrm{TM}$, such that:

- Homomorphism: $\rho([X, Y]_L) = [\rho(X), \rho(Y)]_{\mathrm{Lie}}\ ,\ X, Y \in \Gamma(L)\ .$
- Leibniz rule: $[X, fY]_L = f[X, Y]_L + \rho(X)(f)Y,\ f \in C^\infty(\mathrm{M})\ .$

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Simple examples:

- Lie algebra. When M is a point.
- Standard LA over tangent bundle. $(\mathrm{TM}, [\cdot,\cdot]_{\mathrm{Lie}}, \mathrm{id})$.
- Trivial LA over cotangent bundle. $(\mathrm{T}^\star\mathrm{M}, [\cdot,\cdot]_{\mathrm{T}^\star\mathrm{M}} = 0, 0)$.
- Poisson manifolds (M, β). $(\mathrm{T}^\star\mathrm{M}, [\cdot,\cdot]_{\mathrm{KS}}, \beta^\sharp)$.
  Koszul-Schouten bracket:

$$[\eta,\xi]_{\mathrm{KS}} = \mathcal{L}_{\beta(\eta,\cdot)}\xi - \mathcal{L}_{\beta(\xi,\cdot)}\eta - \mathrm{d}(\beta(\eta,\xi))\ ,\ \eta,\xi \in \Gamma(\mathrm{T}^\star\mathrm{M})\ .$$

  Note: $[\mathrm{d}f, \mathrm{d}g]_{\mathrm{KS}} = \mathrm{d}\{f,g\}$ with $\{f,g\} = \beta(\mathrm{d}f, \mathrm{d}g)$ the Poisson bracket.

Basic differential calculus on L

How to take derivatives on an arbitrary Lie algebroid L:

- Exterior derivative $\mathrm{d}_L : \Gamma(\wedge^p L^\star) \to \Gamma(\wedge^{p+1} L^\star)$:

$$\mathrm{d}_L\omega(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i+1}\rho(X_i)\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1}) + \\ + \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j]_L,X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{p+1}) \ .$$

  e.g. for $L = \mathrm{TM}$: the standard de Rham differential acting on p-forms.

- Lie derivative $\mathcal{L}_X : \Gamma(\wedge^p L^\star) \to \Gamma(\wedge^p L^\star)$:

$$\mathcal{L}_X(\omega)(X_1,\ldots,X_p) = \rho(X)(\omega(X_1,\ldots,X_p)) - \\ - \sum_1^p \omega(X_1,\ldots,[X,X_i],\ldots,X_p) \ .$$

Standard properties. E.g. $\mathrm{d}_L^2 = 0$, $\mathcal{L}_X = \mathrm{d}_L\iota_X + \iota_X\mathrm{d}_L$, etc.

Courant algebroids

Liu, Weinstein, Xu '97

We saw that Lie algebroids "interpolate" between TM and $T^\star M$.

Courant algebroids generalize $\mathbb{T}M = TM \oplus T^\star M$, as Lie algebroids generalize TM.

Also they generalize $\mathfrak{g} \oplus \mathfrak{g}^*$ talk by A. Ballesteros.

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Also they generalize $\mathfrak{g} \oplus \mathfrak{g}^*$, as Lie algebroids do with Lie algebras.

The simplest such structure: standard Courant algebroid $(\mathbb{T}M, [\cdot,\cdot]_C, \langle\cdot,\cdot\rangle, \rho)$ with

- Sections: $\mathfrak{X} \in \Gamma(\mathbb{T}M)$: $\mathfrak{X} = X + \eta$,
- Courant bracket:
  $[\mathfrak{X}_1, \mathfrak{X}_2]_C = [X_1, X_2]_{\text{Lie}} + \mathcal{L}_{X_1}\eta_2 - \mathcal{L}_{X_2}\eta_1 - \frac{1}{2}\mathrm{d}\big(X_1(\eta_2) - X_2(\eta_1)\big)$.
- Pairing: $\langle\mathfrak{X}_1, \mathfrak{X}_2\rangle = \frac{1}{2}\big(X_1(\eta_2) + X_2(\eta_1)\big)$.
  The transformations that leave the bilinear invariant are $O(d,d)$ transformations.
- Anchor: $\rho(\mathfrak{X}) = X$.

Courant algebroids

Courant algebroid: A quadruplet $(E, [\cdot,\cdot], \langle\cdot,\cdot\rangle, \rho)$ of a vector bundle E over M, a skew-symmetric bracket on $\Gamma(E)$, a non-degenerate symmetric bilinear form on E and an anchor $\rho : E \to \mathrm{TM}$, such that for $\mathfrak{X}_i \in \Gamma(E)$:

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- $[[\mathfrak{X}_1, \mathfrak{X}_2], \mathfrak{X}_3] + \text{c.p.} = \mathcal{D}\mathcal{N}(\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3) \ , \qquad 3\mathcal{N} = \langle [\mathfrak{X}_1, \mathfrak{X}_2], \mathfrak{X}_3\rangle + \text{c.p.}$
- $\rho\big([\mathfrak{X}_1, \mathfrak{X}_2]\big) = [\rho(\mathfrak{X}_1), \rho(\mathfrak{X}_2)]$
- $[\mathfrak{X}_1, f\mathfrak{X}_2] = f[\mathfrak{X}_1, \mathfrak{X}_2] + (\rho(\mathfrak{X}_1)f)\mathfrak{X}_2 - \langle \mathfrak{X}_1, \mathfrak{X}_2\rangle \mathcal{D}f$
- $\langle \mathcal{D}f, \mathcal{D}g\rangle = 0$
- $\rho(\mathfrak{X})\langle \mathfrak{X}_1, \mathfrak{X}_2\rangle = \langle [\mathfrak{X}, \mathfrak{X}_1] + \mathcal{D}\langle \mathfrak{X}, \mathfrak{X}_1\rangle, \mathfrak{X}_2\rangle + \langle \mathfrak{X}_1, [\mathfrak{X}, \mathfrak{X}_2] + \mathcal{D}\langle \mathfrak{X}, \mathfrak{X}_2\rangle\rangle$

with $\mathcal{D} : C^\infty(\mathrm{M}) \to \Gamma(E)$ such that $\langle \mathcal{D}f, \mathfrak{X}\rangle_E = \frac{1}{2}\rho(\mathfrak{X})f$.

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all the structures are compatible.

Construction of Courant algebroids

Prototype: Take two dual Lie algebroids L and $L^\star$.
Liu, Weinstein, Xu '97

Then $E = L \oplus L^\star$ with bracket (generalization of Courant bracket...):

$$\begin{aligned}[X+\eta, Y+\xi]_E \quad &= \quad [X,Y]_L + \mathcal{L}_X\xi - \mathcal{L}_Y\eta - \tfrac{1}{2}\mathrm{d}_L(X(\xi) - Y(\eta)) \\ &\quad\; +[\eta,\xi]_{L^\star} + \mathcal{L}_\eta Y - \mathcal{L}_\xi X + \tfrac{1}{2}\mathrm{d}_{L^\star}(X(\xi) - Y(\eta))\ ,\end{aligned}$$

anchor $\rho_E(X+\eta) = \rho(X) + \rho_\star(\eta)$ and pairing $\langle X+\eta, Y+\xi\rangle = \frac{1}{2}(X(\xi) + Y(\eta))$.

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$$\begin{aligned}[X+\eta, Y+\xi]_E \quad &= \quad [X,Y]_L + \mathcal{L}_X\xi - \mathcal{L}_Y\eta - \tfrac{1}{2}\mathrm{d}_L(X(\xi)-Y(\eta)) \\ &\quad\ +[\eta,\xi]_{L^\star} + \mathcal{L}_\eta Y - \mathcal{L}_\xi X + \tfrac{1}{2}\mathrm{d}_{L^\star}(X(\xi)-Y(\eta))\ ,\end{aligned}$$

anchor $\rho_E(X+\eta) = \rho(X) + \rho_\star(\eta)$ and pairing $\langle X+\eta, Y+\xi\rangle = \frac{1}{2}(X(\xi)+Y(\eta))$.

Generic: Twist the structures by generalized 3-forms $\phi \in \Gamma(\wedge^3 L^\star)$ and $\psi \in \Gamma(\wedge^3 L)$. Roytenberg '99

In that case none of L and $L^\star$ is a LA, but E is a CA.

The bracket simply changes to

$$[X+\eta, Y+\xi]_E - \phi(X,Y,\cdot) - \psi(\eta,\xi,\cdot)\ .$$

Dirac structures

Courant '90

$$\text{Lie algebroid} \xrightarrow{\textit{doubling}} \text{Courant algebroid} \xrightarrow{\textit{polarization}} \text{Dirac structure}$$

Dirac structures

Courant '90

$$\text{Lie algebroid} \xrightarrow{\textit{doubling}} \text{Courant algebroid} \xrightarrow{\textit{polarization}} \text{Dirac structure}$$

Dirac structure: A maximally isotropic and integrable subbundle $L \subset E$:

- $\text{rk } L = \frac{1}{2}\text{rk } E$
- $\langle \mathfrak{X}_1, \mathfrak{X}_2 \rangle = 0 \ , \quad \mathfrak{X}_1, \mathfrak{X}_2 \in \Gamma(L)$
- $[\mathfrak{X}_1, \mathfrak{X}_2]_E \in \Gamma(L)$

A Dirac structure is a Lie algebroid.

From Courant algebroids to AKSZ σ-models

Every Courant algebroid has an associated 3D AKSZ sigma model. Roytenberg '06

AKSZ: Unification of a large class of TFTs in the spirit of BV quantization.
Alexandrov, Kontsevich, A. Schwarz, Zaboronsky '95

Turn out to be relevant in the description of non-geometric backgrounds.
Halmagyi '09, Mylonas, Schupp, Szabo '12, Chatzistavrakidis, L.J., Lechtenfeld '15

Membrane σ-models

Membrane action (3D, bosonic fields):

$$S_{\Sigma_3} = \int_{\Sigma_3} \left(F_a \wedge \mathrm{d}X^a + \tfrac{1}{2}\eta_{IJ} A^I \wedge \mathrm{d}A^J - \rho^a_I A^I \wedge F_a + \tfrac{1}{6} T_{IJK} A^I \wedge A^J \wedge A^K \right) .$$

$X^a : \Sigma_3 \to M$ worldvolume scalars. F_a: auxiliary worldvolume 2-form. ρ: the anchor
A^I: generalized 1-form, $I = 1, \dots, 2d$. η_{IJ}: $O(d,d)$-invariant metric. T: generalized 3-form.

The 3-form T systematically includes all types of 3-elements.

Add general topological boundary term

$$S_{\partial\Sigma_3,\mathrm{top}} = \int_{\partial\Sigma_3} \tfrac{1}{2}\mathcal{B}_{IJ}(X) A^I \wedge A^J .$$

In the boundary 2D theory, add dynamics, e.g. $S_{\partial\Sigma_3,\mathrm{kin}} = \int_{\partial\Sigma_3} \tfrac{1}{2} g_{ab} \mathrm{d}X^a \wedge \star \mathrm{d}X^b$.

A class of Courant algebroids

Consider as M a nilmanifold (geometric flux built-in).

$$\begin{aligned} \mathrm{TM} &= \mathrm{span}\{\theta_i = e_i^a(x)\partial_a = \delta_i^a(\partial_a + f_{ba}^c x^b \partial_c)\} \ , \\ \mathrm{T^*M} &= \mathrm{span}\{e^i = e_a^i(x)\mathrm{d}x^a = \delta_a^i(\mathrm{d}x^a - f_{bc}^a x^b \mathrm{d}x^c)\} \ . \end{aligned}$$

Take (non-closed) 2-form and (non-Poisson) 2-vector fields:

$$B = \tfrac{1}{2} B_{ij} e^i \wedge e^j \ , \quad \beta = \tfrac{1}{2}\beta^{ij}\theta_i \wedge \theta_j \ ,$$

and deform the bundles with the element $e^B e^\beta$:

$$\begin{aligned} L &= e^B e^\beta \mathrm{TM} = \mathrm{span}\{\Theta_i = \theta_i + B_{ij}e^j\} \ , \\ L^* &= e^B e^\beta \mathrm{T^*M} = \mathrm{span}\{E^i = e^i + \beta^{ik}B_{kj}e^j + \beta^{ij}\theta_j\} \ . \end{aligned}$$

Brackets, anchors, generalized 3-forms, via $e^B e^\beta$ deformations.

The twists

A generic Courant algebroid includes twists $\phi \in \Gamma(\wedge^3 L^*)$ and $\psi \in \Gamma(\wedge^3 L)$.

Consider the expansions:

$$
\begin{aligned}
\phi &= \tfrac{1}{6}\phi_{ijk} E^i \wedge E^j \wedge E^k \\
&= \tfrac{1}{6}\big((1+\beta B)^i_\rho (1+\beta B)^j_\sigma (1+\beta B)^k_\tau \phi_{ijk} e^\rho \wedge e^\sigma \wedge e^\tau \\
&\quad +3(1+\beta B)^i_\rho (1+\beta B)^j_\sigma \beta^{kl} \phi_{ijk} e^\rho \wedge e^\sigma \wedge \theta_l \\
&\quad +3(1+\beta B)^i_\rho \beta^{jl}\beta^{km} \phi_{ijk} e^\rho \wedge \theta_l \wedge \theta_m \\
&\quad +\beta^{il}\beta^{jm}\beta^{kn}\phi_{ijk}\theta_l \wedge \theta_m \wedge \theta_n\big)\ , \\
\psi &= \tfrac{1}{6}\psi^{ijk}\Theta_i \wedge \Theta_j \wedge \Theta_k \\
&= \tfrac{1}{6}\big(\psi^{ijk}\theta_i \wedge \theta_j \wedge \theta_k \\
&\quad +3B_{kn}\psi^{ijk}\theta_i \wedge \theta_j \wedge e^n \\
&\quad +3B_{jm}B_{kn}\psi^{ijk}\theta_i \wedge e^m \wedge e^n \\
&\quad +B_{il}B_{jm}B_{kn}\psi^{ijk} e^l \wedge e^m \wedge e^n\big)\ .
\end{aligned}
$$

$\rightsquigarrow$ From the twisted torus viewpoint *all* types of twists T_{ijk}, T_{ij}^k, T_i^{jk}, T^{ijk} are present.

The sigma model for $E = L \oplus L^\star$

For our class of CAs,

- $\rho_i^a = e_i^a(X)\ ,\ \rho^{ai} = \beta^{ij}(X) e_j^a(X)\ .$
- $A^I = (q^i, p_i)\ .$
- $T = f - \phi - \psi$, with $f = \frac{1}{2} f_{ij}^k q^i \wedge q^j \wedge p_k\ .$
- $\mathcal{B}_{ij} = B_{ij}(X)\ , \quad \mathcal{B}^{ij} = \beta^{ij}(X)\ , \quad \mathcal{B}_i^j = h_i^j(X)\ .$

The action is:

$$\begin{aligned} S \quad = & \quad \int_{\Sigma_3} \Big(F_a \wedge \mathrm{d}X^a + q^i \wedge \mathrm{d}p_i + p_i \wedge \mathrm{d}q^i - (e_i^a q^i + \beta^{ij} e_j^a p_i) \wedge F_a + f - \phi - \psi \Big) \\ & + \int_{\Sigma_2} \Big(\tfrac{1}{2} B_{ij}(X) q^i \wedge q^j + \tfrac{1}{2} \beta^{ij}(X) p_i \wedge p_j + \tfrac{1}{2} h_j^i(X) q^j \wedge p_i \Big)\ . \end{aligned}$$

For consistency, the boundary conditions should match the equations of motion on $\partial\Sigma_3$.

- $\rightsquigarrow$ vary the action w.r.t. X^a, q^i, p_i and determine BCs such that the variations vanish.
- $\rightsquigarrow$ ensure that 3D terms which did not reduce to the boundary vanish on it (classical master equation).

Boundary conditions and consistency

Bulk/boundary consistency conditions:

- for $\delta p_i|_{\Sigma_2} = 0$:
$$\mathcal{H}_{ijk} - \mathcal{F}^n_{[ij} B_{\underline{p}k]} \chi'^p_n + \mathcal{Q}^{mn}_{[i} B_{\underline{p}j} B_{\underline{q}k]} \chi'^p_m \chi'^q_n - \mathcal{R}^{lmn} B_{pi} B_{qj} B_{rk} \chi'^p_l \chi'^q_m \chi'^r_n = 0 \ ,$$
where $\chi' = 1 + \frac{1}{2} h$.
- for $\delta q^i|_{\Sigma_2} = 0$:
$$\mathcal{R}^{ijk} - \mathcal{Q}^{[ij}_n \beta^{\underline{p}k]} \chi^n_p + \mathcal{F}^{[i}_{mn} \beta^{\underline{p}j} \beta^{\underline{q}k]} \chi^m_p \chi^n_q - \mathcal{H}_{lmn} \beta^{pi} \beta^{qj} \beta^{rk} \chi^l_p \chi^m_q \chi^n_r = 0 \ ,$$
where $\chi = 1 - \frac{1}{2} h$.

$$\begin{aligned}
\mathcal{R}^{ijk} &= \psi^{ijk} - 3\beta^{[i\underline{l}}\theta_l \beta^{jk]} + \beta^{li}\beta^{mj}\beta^{nk}\phi_{lmn} \ , \\
\mathcal{Q}^{ij}_k &= -3\theta_k \beta^{ij} + 3\beta^{[i\underline{l}}\theta_l h^{j]}_k + 3B_{lk}\psi^{ijl} + 3(1+\beta B)^l_k \beta^{mi}\beta^{nj}\phi_{lmn} \ , \\
\mathcal{F}^i_{jk} &= -3\theta_{[j} h^i_{k]} - 3f^i_{jk} - 3\beta^{il}\theta_l B_{jk} + 3B_{lj}B_{mk}\psi^{lmi} + 3(1+\beta B)^l_j (1+\beta B)^m_k \beta^{ni}\phi_{lmn} \\
\mathcal{H}_{ijk} &= (1+\beta B)^l_i (1+\beta B)^m_j (1+\beta B)^n_k \phi_{lmn} - 3\theta_{[i} B_{jk]} + B_{li}B_{mj}B_{nk}\psi^{lmn} \ .
\end{aligned}$$

Relation to Dirac structures and integrability

The dictionary between these σ-models and Courant algebroids is:

Courant algebroid	Sigma model
Bracket twist $[\cdot,\cdot]_T$	Bulk term $-\int_{\Sigma_3} T$
Dirac structure deformation $L_{\mathcal{B}}$	Boundary term $\int_{\partial\Sigma_3} \mathcal{B}$
Integrability condition for Dirac structure	Bulk/boundary consistency condition

Notably, the b/b conditions are also integrability conditions for Dirac structures.

This generalizes previous results. Severa, Weinstein '01

The corresponding 2D field theories belong to the class of Dirac sigma models.

However although they may contain e.g. 3-vector fluxes, they are not the ones that appear in string theory via generalized T-duality. Halmagyi '09

AKSZ for R flux

Mylonas, Schupp, Szabo '12

The AKSZ membrane action:

$$\int_{\Sigma_3} \Big(F_a \wedge \mathrm{d}X^a + q^i \wedge \mathrm{d}p_i - \delta_i^a q^i \wedge F_a + \tfrac{1}{6} R^{ijk} p_i \wedge p_j \wedge p_k \Big) \ .$$

Integrating out the auxiliary 2-form F_a, the EOM for X^i on $\partial\Sigma_3$ for $R = \mathrm{const.}$ requires

$$\mathrm{d}p_a = 0 \quad \Rightarrow \quad p_a = \mathrm{d}P_a \ , \quad \text{locally} \ ,$$

with $P_a \in C^\infty(\Sigma_3, X^\star \mathrm{T}^\star \mathrm{M})$.

Very suggestive!

Generalized world volume coordinates:
$X^I = (X^a, P_a)$, or doubled formulation of closed string theory $X^I = (X^a, \tilde{X}_a)$.
Tseytlin '90

BUT, no CA yields this membrane action ...

One step beyond

Chatzistavrakidis, L.J., Lechtenfeld '15

Embed the membrane theory in extended space from the beginning.

Minimal generalization of AKSZ action:

$$S=\int_{\Sigma_3}\Big(F_a\wedge\mathrm{d}X^a+\tilde{F}^a\wedge\mathrm{d}P_a+\tfrac{1}{2}\eta_{IJ}A^I\wedge\mathrm{d}A^J-\rho^a_I A^I\wedge F_a-\tilde{\rho}_{aI}A^I\wedge\tilde{F}^a+\tfrac{1}{6}T_{IJK}A^I\wedge A^J\wedge A^K\Big)$$

with the same boundary action as before, but $\mathcal{B}=\mathcal{B}(X,P)$.

$\tilde{F}^a$ is also an auxiliary world volume 2-form; the map $\tilde{\rho}:E\to\mathrm{T}^\star\mathrm{M}$.

In more compact notation, writing $F^I=(F_a,\tilde{F}^a)$ and $\rho^J_I=(\rho^a_I,\tilde{\rho}_{aI})$

$$S_{\Sigma_3}=\int_{\Sigma_3}\Big(\delta_{IJ}F^I\wedge\mathrm{d}X^J+\tfrac{1}{2}\eta_{IJ}A^I\wedge\mathrm{d}A^J-\delta_{JK}\rho^J_I A^I\wedge F^K+\tfrac{1}{6}T_{IJK}A^I\wedge A^J\wedge A^K\Big)\,.$$

A similar analysis as before, yields extended bulk/boundary consistency conditions.

Note: only target undergoes a doubling, not world volume.

What have we found?

We can construct 3D sigma models which:

- Satisfy the extended bulk/boundary consistency conditions for the new σ-model,
- Contain all types of fluxes,
- Cannot be reduced to standard 2D theories in any (standard) duality frame (only to theories with both X and P).

However, we still need better understanding of the additional maps/structures introduced.

Gauging of Dirac σ-models

Chatzistavrakidis, Deser, L.J., Strobl, arXiv:1607.00342 [hep-th]

Consider 2D gauge σ -model

$$S_{\mathrm{WZ}}[X,A]=\int_\Sigma \tfrac{1}{2}g_{ij}(X)DX^i\wedge *DX^j+\int_{\hat{\Sigma}}H(X)+\int_\Sigma A^a\wedge\theta_a+\int_\Sigma \tfrac{1}{2}\gamma_{ab}A^a\wedge A^b\ ,$$

where $DX^i=dX^i-\rho^i_a(X)A^a$, the gauge fields $A\in\Gamma(L)$. Here

$$L\ \stackrel{\rho\oplus\theta}{\to}\ TM\oplus T^*M$$

This action functional is invariant under generalized gauged transformations provided that generalized vectors $\xi_a=\rho_a+\theta_a$ (taken as the local sections of $TM\oplus T^*M$) span (small) Dirac structure.
Upon identification

$$v=\rho(A)\Rightarrow v^i=\rho^i_a(X)A^a\quad\text{and}\quad \eta=\theta(A)\Rightarrow\eta_i=\theta_{ai}(X)A^a\ ,$$

we obtain Dirac σ-model Kotov, Schaller, Strobl '04

$$S_{\mathrm{DSM}}[X,v\oplus\eta]=\int_\Sigma \tfrac{1}{2}g_{ij}(X)DX^i\wedge *DX^j+\int_\Sigma\left(\eta_i\wedge \mathrm{d}X^i-\tfrac{1}{2}\eta_i\wedge v^i\right)+\int_{\hat{\Sigma}}H\ ,$$

where $v\oplus\eta\in\Omega^1(\Sigma,X^*D)$ and $DX^i=\mathrm{d}X^i-v^i$.

Main open issues

- Manifest T-duality for membrane sigma models?
- New classes of "non-geometric" string backgrounds?
- Quantization of Dirac σ-models?
  Cf. quantization of Poisson σ-model $\rightsquigarrow$ Kontsevich deformation formula...