Quantum Space-time and Metastrings

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based on a series of papers, past and future,
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Really quantum gravity

- **in string theory, the underlying theory is quantum mechanically consistent (the 2d worldsheet theory), and the fields are interpreted as coordinatizing a target space-time**
  - geometrical structures (metric, …) are interpreted as associated with *states* of the string
  - we use a built-in interpretation:
    - **there is an emergent target space-time theory that at low energies reproduces local effective quantum field theories, and includes (quantum) gravity**
- **this is nice, but it cannot be the whole story**
  - any theory of quantum gravity contains two parameters, $\hbar$ and G_N

$$[\hbar] = L \times E \qquad\qquad [G_N] = L/E$$

  - interested in probing the theory in all regimes

Locality

- **believe the key is to relax our notion of locality**

Absolute locality:

space-time is independent of probe,
an arena

Relative locality:

space-time depends on the nature
of a probe (e.g., its energy)

- **in metastring theory, relative locality is implemented by regarding space-time as a *subspace* of the target space of the string**
    - thought of as a choice of basis, i.e., a choice of polarization of phase space

Relative Locality and Metastrings

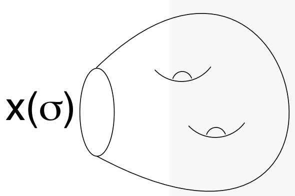
- **Born reciprocity can then be built in to the theory**
  - a given space-time is only one choice of polarization
    - **energy-momentum on the same footing as space-time**
  - within a fully consistent theory of quantum gravity
  - T-duality: in toroidal compactifications, string does not distinguish radius R from α'/R
    - **spectrum of string is simply relabeled, momentum and winding modes being exchanged**
  - T-duality can be thought of as acting non-trivially on a choice of polarization
    - **in fact, it can be thought of as a sort of "Fourier transform"**
  - short and long distance are equivalent
    - **or, more precisely, T-duality exchanges momentum and space**

Born Reciprocity and T-duality

- usually, in toroidal compactifications, we interpret short distance (radius) as long distance in a dual space-time

- however we can also directly relate this to a Fourier transform

- consider a string state



$$\Psi[x(\sigma)] = \int_{X|_{\partial\Sigma}=x} [DX\ \ Dg]\ e^{iS_P[X]/\lambda^2}$$

$$\alpha' p = \int_C *dX$$

$$\delta = \int_C dX$$

- define a Fourier transform of this state by

$$\tilde{\Psi}[y(\sigma)] \equiv \int [Dx(\sigma)] e^{i/\hbar \int_{\partial\Sigma} x^\mu dy_\mu} \Psi[x(\sigma)]$$

- extending $y(\sigma)$ to the world sheet, we integrate out X to obtain a *dual* Polyakov path integral

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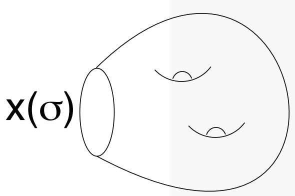

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δ and p exchange their roles

Metastrings

- **a reformulation of the Polyakov (bosonic) string theory, written in such a way that T-duality acts linearly on coordinates**
    - the interpretation will be that this is putting space-time on an equal footing with energy-momentum
- **this is a re-interpretation of the string path integral, and so is as well-defined as the usual quantum string theory**
    - the principle difference is a relaxation of boundary conditions, to allow general monodromies

$$X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau) + \delta^\mu$$

Metastring Path Integral

- the notation used by double field theory is useful here

$$\mathbb{X}^A \equiv \begin{pmatrix} X^\mu/\lambda \\ Y_\mu/\varepsilon \end{pmatrix}, \quad \eta_{AB} = \begin{pmatrix} 0 & \delta \\ \delta^T & 0 \end{pmatrix}, \quad H_{AB} \equiv \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, \quad \omega_{AB} \equiv \begin{pmatrix} 0 & \delta \\ -\delta^T & 0 \end{pmatrix}$$

$$\lambda\varepsilon = \hbar, \qquad \frac{\lambda}{\varepsilon} = \alpha'$$

$$S = \frac{1}{4\pi} \int \left(\partial_\tau \mathbb{X}^A (\eta_{AB} + \omega_{AB}) \partial_\sigma \mathbb{X}^B - \partial_\sigma \mathbb{X}^A H_{AB} \partial_\sigma \mathbb{X}^B \right)$$

- the ω term is a total derivative, but it must be kept, as maps are not single-valued

- this is the classical action of the 'flat metastring'
  - coordinates $\mathbb{X} \in \mathcal{P}$, and (η, ω, H) are geometric structures on $T\mathcal{P}$

Born geometry — expect that (η, ω, H) will become "curved"

Metastring Path Integral

- **this is a traditional sigma-model discussion, with some apparent relation with generalized geometry**
  - however, note that we speak in terms of $T\mathcal{P}$ not $TM \oplus T^*M$ as there is no preferred M
- **in fact, we now believe that the proper interpretation of $(\mathcal{P}; \omega, \eta, H)$ is through the *geometry of quantization***
  - a choice of polarization is a choice of a space-time within $\mathcal{P}$, but the most general such choice is a *modular polarization*
  - (ω, η, H) arise as a parameterization of such quantizations
    - such a quantization results in a notion of quantum space-time
    - large space-times result as a 'many-body' phenomenon, through a process we refer to as 'extensification'

Phase Space and the Heisenberg Group

- let's recall some familiar notions of quantization
- in fact, I will focus on the Heisenberg Group
  - generated by $\hat{q}^a, \hat{p}_a$ which satisfy the algebraic relation

$$[\hat{q}^a, \hat{p}_b] = i\hbar\delta^a{}_b\ \mathbf{1}$$

- it will be convenient to introduce a length scale λ and a momentum scale ε, with $\lambda\varepsilon = \hbar$
- then $\hat{x}^a \equiv \hat{q}^a/\lambda, \qquad \hat{\tilde{x}}_a = \hat{p}_a/\varepsilon, \qquad [\hat{x}^a, \hat{\tilde{x}}_b] = i\delta^a{}_b\ \mathbf{1}$
- or more compactly

$$\mathbb{X}^A \equiv \begin{pmatrix} x^a \\ \tilde{x}_a \end{pmatrix} \in \mathcal{P} \qquad [\hat{\mathbb{X}}^A, \hat{\mathbb{X}}^B] = i\omega^{AB}\ \mathbf{1}$$

$$\omega = \frac{1}{2}\omega_{AB}\, d\mathbb{X}^A \wedge d\mathbb{X}^B = \frac{1}{\hbar} dp_a \wedge dq^a \qquad \omega_{AB} = \begin{pmatrix} 0 & \delta \\ -\delta^T & 0 \end{pmatrix}$$

Heisenberg group

- recall Heisenberg group $H_{\mathcal{P}}$ generated by Weyl ops

$$W_{\mathbb{K}} \equiv e^{2\pi i\,\omega(\mathbb{K},\hat{\mathbb{X}})}$$

- these form a central extension of the translation algebra

$$W_{\mathbb{K}} W_{\mathbb{K}'} = e^{i\pi\,\omega(\mathbb{K},\mathbb{K}')} W_{\mathbb{K}+\mathbb{K}'}$$

- projection $\pi : H_{\mathcal{P}} \to \mathcal{P}$ (where $\pi : W_{\mathbb{K}} \mapsto \mathbb{K}$) defines a line bundle over $\mathcal{P}$
  - states are sections of degree one

$$W_{\mathbb{P}'}\Phi(\mathbb{P}) = e^{i\pi\omega(\mathbb{P},\mathbb{P}')}\Phi(\mathbb{P}+\mathbb{P}')$$

“prequantization”
$\Phi \in L^2(\mathcal{P})$

  - (geometric) quantization: take Lagrangian $L \subset \mathcal{P}$: states descend to $L^2(L)$
  - more generally, want a maximally commuting subalgebra, and the representation that diagonalizes it

Lagrangian Submanifolds

- a Lagrangian submanifold is a maximally isotropic subspace L with

$$\omega\big|_L = 0$$

  - e.g., $\{\partial/\partial q^a\} \subset T\mathcal{P}$ defines a Lagrangian submanifold, "space"
  - indeed $\omega(\partial/\partial q^a, \partial/\partial q^b) = 0$

- this is at least the classical characterization

- borrowing from notions of non-commutative algebra, we can say that *a Lagrangian submanifold is a maximally commutative subgroup of the Heisenberg group*

$$[\hat{\mathbb{X}}^A, \hat{\mathbb{X}}^B] = i\omega^{AB}\ \mathbf{1}$$

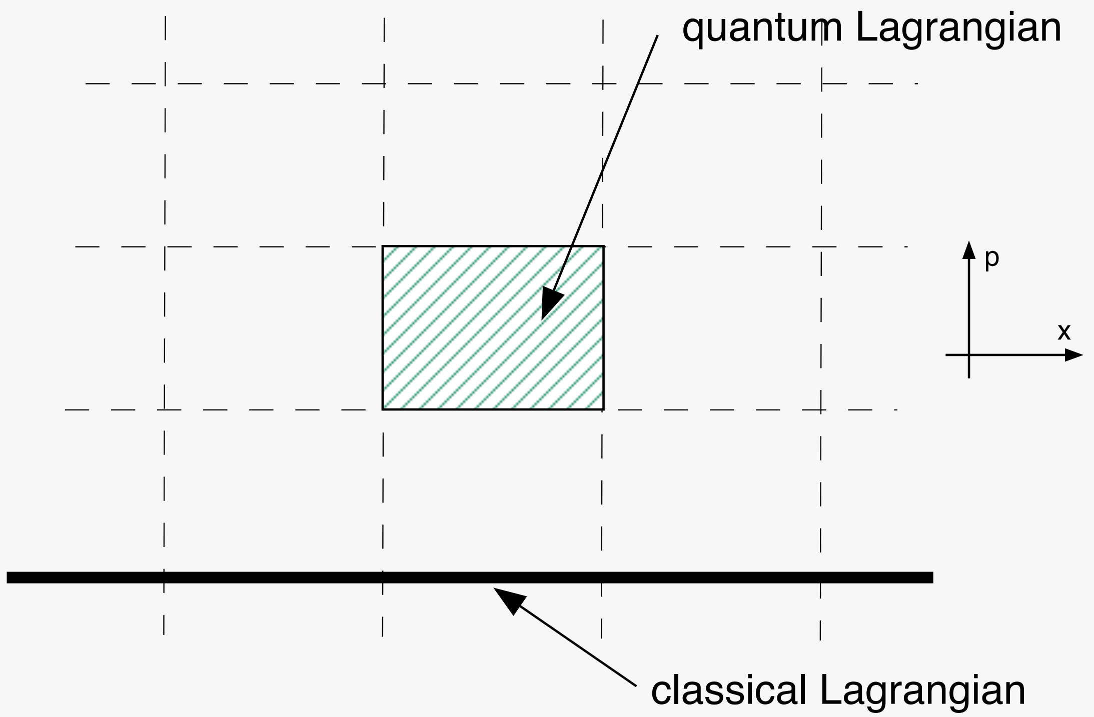
Quantum vs. Classical Lagrangians

- if we accept this notion of a Lagrangian, then the quantum regime is very different than the classical regime
  - e.g., $\{f(q), g(p)\} = 0$ requires either f or g to be constant
  - but $[f(\hat{q}), g(\hat{p})] = 0$ requires only that the functions be commensurately periodic
    - i.e., $e^{i\alpha\hat{p}}e^{i\beta\hat{q}} = e^{i\hbar\alpha\beta}e^{i\beta\hat{q}}e^{i\alpha\hat{p}}$ $\alpha\beta = 2\pi/\hbar$

- similar considerations led Aharonov to introduce *modular variables* to describe purely quantum phenomena
  - that is, introduce $[q] \sim [q] + 1/\alpha,$ $[p] \sim [p] + 1/\beta$
  - quantum (or modular) Lagrangian $q \to ([q], [p])$ ← torus

Quantum Lagrangians

- **the quantum Lagrangian is analogous to a Brillouin cell in CMT**
  - the volume and shape of the cell are given by λ, ε (i.e., $\hbar, \alpha'$)



  - uncertainty principle: can specify a point in modular cell, but if so, can't say *which* cell you are in

Modular Quantizations

- so this means that there is a more general notion of quantization, beyond that of geometric quantization
    - instead of selecting a classical polarization L, we can choose a modular polarization
- in terms of the Heisenberg group, all that is happening is that in order to have a commutative subgroup, we need only

$$\omega(\mathbb{K}, \mathbb{K}') \in 2\mathbb{Z}$$

$$W_{\mathbb{K}} W_{\mathbb{K}'} = e^{i\pi\, \omega(\mathbb{K},\mathbb{K}')} W_{\mathbb{K}+\mathbb{K}'}$$

- this defines a lattice $\Lambda \subset \mathcal{P}$
- finally, we specify a `lift' of the lattice from $\mathcal{P}$ to $H_{\mathcal{P}}$

Quantum Lagrangians

- maximally commuting subgroups $\hat{\Lambda}$ of the Heisenberg group correspond to lattices that are integral self-dual wrt ω

$$\mathcal{P} \ni \Lambda = \pi(\hat{\Lambda})$$

- given $W_\lambda, \ \lambda \in \Lambda$, there is a lift to $\hat{\Lambda}$ "modular polarization"

$$U_\lambda = \alpha(\lambda) W_\lambda$$

where $\alpha(\lambda)$ satisfies the co-cycle condition

$$\alpha(\lambda)\alpha(\mu)e^{i\pi\omega(\lambda,\mu)} = \alpha(\lambda+\mu), \qquad \lambda, \mu \in \Lambda$$

- one can parameterize a solution to the co-cycle condition by introducing a symmetric bilinear form η and setting

$$\alpha_\eta(\lambda) := e^{i\frac{\pi}{2}\eta(\lambda,\lambda)}$$

Hilbert Space and the Modular Vacuum

- **when we choose a classical Lagrangian L, there is a special state that we associate with the vacuum**
  - it is translation invariant
  - here, we interpret this translation invariant state as "empty space"
- **in modular quantization, there is no such translation invariant state**
- **the best we can do is to choose a state that minimizes an `energy'**
  - this invariably requires the introduction of another symmetric bilinear form, which we call H

Modular Quantization

- so modular quantization involves the introduction of three quadratic forms (ω, η, H)
- comments:
  - Stone- von Neumann theorem: all representations of $H_{\mathcal{P}}$ are unitarily equivalent
    - normally, we think of this as a choice of basis in phase space (a choice of polarization or classical Lagrangian), and all such choices are related by Fourier transform.
    - similarly, one can pass from a classical polarization to a modular polarization via Zak transform.
  - there is a connection on the line bundle over phase space that has unit flux through a modular cell
    - vacuum state must have at least one zero in cell —> theta functions

Quadratic forms

- symplectic form

$$\omega = \frac{1}{\hbar} dp_a \wedge dq^a$$

- 'polarization metric'

$$ds_\eta^2 = \frac{1}{\hbar}(dp_a \otimes dq^a + dq^a \otimes dp_a)$$

- 'quantum metric'

$$ds_H^2 = \frac{1}{\lambda^2}\left[h_{ab} dq^a \otimes dq^b + \frac{\lambda^2}{\varepsilon^2} h^{ab} dp_a \otimes dp_b\right]$$

- recall we identified $\hbar = \lambda\varepsilon$
- the ratio ε/λ defines a tension
  - if this is identified with c^3/G_N, it is enormous ($\sim 10^{32}\ kg \cdot /sec$)

Metastrings

- is there any real evidence that we should regard all this as connected to gravity in some way?
- in fact, the data $(\mathcal{P};\omega,\eta,H)$ is in one-to-one correspondence with the geometric data underlying metastrings, which indeed has the string length built in
- as well, the vertex operator algebra of the string gives copies of the Heisenberg group
  - in the zero-mode sector, with decoration by oscillator modes

- to reiterate, when the gravitational tension is large, the quantum metric reduces to a spatial metric

$$ds_H^2 = \frac{1}{\lambda^2}\left[h_{ab}dq^a \otimes dq^b + \frac{\lambda^2}{\varepsilon^2}h^{ab}dp_a \otimes dp_b\right] \to h_{ab}dq^a \otimes dq^b$$

- large space $\longleftrightarrow$ large tension (weak gravity)
- this suggests that we should regard modular quantization as a gravitization of the quantum
- the modular cell is a quantum unit of space-time
  - more generally, we can 'extensify' the modular cell by tensoring many together (resulting in large flux)
    - very similar to IQHE systems

Metastrings

- in the Polyakov string, the target space is interpreted as space-time
- in the metastring, the target is a phase space $(\mathcal{P}; \omega, \eta, H)$
  - space-time itself would be a Lagrangian submanifold
    - if we suppose this to be a modular Lagrangian, it is associated with a lattice which turns out to be unique!
    - this comes from a careful study of the operator algebra of the string, closely associated with the analysis of the Heisenberg group
    - mutual locality of physical states, etc., require the lattice to be Lorentzian even self-dual

$$\mathbb{P} \in \Lambda = \Pi_{1,25} \times \Pi_{1,25}$$

    - products of Borcherds algebras: all the usual string backgrounds (and presumably many more) are 'extensifications' of these algebras

-

Causality?

$$\mathbb{P} \in \Lambda = \Pi_{1,25} \times \Pi_{1,25}$$

- **NB: same as "Narain lattice of fully compactified space-time"**
  - this is the traditional interpretation of this lattice, but here it is non-sensical because it would have no apparent causal interpretation
- **so one way to view the modular quantization is as a potentially causal interpretation**
  - we know what causality is in local field theory
    - operators commute at space-like separation
    - (requires a 'large' space-time for the QFT to live in)
  - the metastring is causal from the worldsheet point of view
    - is it causal from the modular target space-time point of view?

Lorentz Invariance

$$\mathbb{X}^A \equiv \begin{pmatrix} X^\mu/\lambda \\ Y_\mu/\varepsilon \end{pmatrix}, \quad \eta_{AB} = \begin{pmatrix} 0 & \delta \\ \delta^T & 0 \end{pmatrix}, \quad H_{AB} \equiv \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, \quad \omega_{AB} \equiv \begin{pmatrix} 0 & \delta \\ -\delta^T & 0 \end{pmatrix}$$

d=26

$O(d,d) - invariant$ — 'polarization metric'

$O(2,2d-2) - invariant$ — 'quantum metric'

$Sp(2d) - invariant$ — 'symplectic form'

$$O(d,d) \cap O(2,2d-2) \cap Sp(2d) = O(1,d-1)$$

- **Lorentz preserves all these structures**

Qu: but doesn't a modular quantization break this to a discrete subgroup??

a fundamental conundrum for any QG:
how to make continuous symmetries consistent with the existence of a fundamental scale

Lorentz Invariance

- **relative locality means that the answer is no!**
- **a space-time is a choice of polarization**
  - each observer makes his own choice
  - each observer sees the discreteness
  - Lorentz acts to change the polarization, restoring Lorentz invariance
- **compare to spin**
  - choose a quantization axis, get a set of discrete spin states
    - **could regard as a discretization of sphere, breaking SO(3)**
  - but rotations act to rotate the quantization axis (choice of polarization)
    - **because the spin operators do not commute, the states of one axis are unitarily related to those of any other**
- **for Lorentz invariance to be unbroken, we need:**
  - non-commutative (√ — torus in phase space)
  - relative locality: space-time is NOT an arena shared by all observers; the arena is a 'many-body' interpretation/approximation

Final Remarks

- one of the most fundamental conundrums in quantum gravity is the lack of understanding of how continuous symmetries (such as Lorentz) can be made consistent with the existence of a fundamental scale (G_N)
- here this is manifested by the apparent breaking of Lorentz to a discrete subgroup by the presence of the lattice.
  - but in fact, in the modular interpretation, this is not so
    - each observer (i.e., each single free string) can 'make its own choice' of quantum Lagrangian
      - *closely analogous to choice of a quantization axis for quantum spins*
      - because it is a lattice in (non-commutative) phase space, each choice is unitarily equivalent
    - our usual notion of space-time as a fixed arena for all observers is a classical (many body) interpretation

'relative locality'

Classical and Quantum Space-times

- if the Born geometry is the target, where is space-time?
  - **Classical answer**: space-time is a chosen Lagrangian submanifold of $\mathcal{P}$
    - a choice of polarization
    - T-duality operations (canonical transformations) change this choice

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- **Quantum answer**: space-time is *any* commutative sub-algebra
  - we refer to this as 'modular space-time'
  - precise analogy to generic representations of the Heisenberg group in ordinary quantum mechanics
    - usual 'classical' space-times obtained by taking a certain limit, analogous to the Stone-von Neumann map to the Schrödinger representation
    - this is a process we refer to as 'extensification'
  - generally, we see space-time at finite $\hbar$ and α'

Born Geometry

- ω is closed and non-degenerate — suggests symplectic form
  - ω, H together then define a *complex structure* $\omega(\mathbb{X},\mathbb{Y}) = H(\mathbb{X}, I(\mathbb{Y}))$
- η is neutral
  - η, H together define a *chiral structure* $J = \eta^{-1}H$
- bi-Lagrangian structure: $T\mathcal{P} = L \oplus \tilde{L}$ with $\tilde{L} = J(L)$
  - η, ω together define an involution $K\Big|_L = Id, \; K\Big|_{\tilde{L}} = -Id$
- 'hyper-para-Kahler' (or just 'Born geometry')

$I^2 = -1$

$J^2 = +1$

$K^2 = +1$

$IJ = K$