Causality for probability measures

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Joint project with **Michał Eckstein** (UJ & CC, Cracow, Poland)

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Warsaw University of Technology & Copernicus Center (Cracow)

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Goal and motivations

- Causality (in the context of Lorentzian geometry) = certain binary relation between **events** (= spacetime points).
- Pointlike events are **not** directly observable!
    - Measuring apparatus' imperfection
    - Too precise measurement $\Longrightarrow$ black hole formation
    - Quantum effects – non-locality
- Goal: extend the causality relation to "nonlocal events" $\rightsquigarrow$ *(Borel) probability measures*.
    - Classical – given by densities $d\mu = \frac{\rho}{\int_{\mathcal{M}} \rho d^n x} d^n x$
    - Quantum – the 'modulus-squared' principle $d\mu = |\psi|^2 d^n x$

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The **causal precedence** relation $\preceq$ on a given spacetime $\mathcal{M}$

$$p \preceq q \;\Leftrightarrow\; \exists \text{ future-directed causal curve } \gamma \text{ from } p \text{ to } q \text{, or } p = q$$

Recall $J^+ := \{(p,q) \in \mathcal{M}^2 : \; p \preceq q\}, \quad J^+(p) := \{q \in \mathcal{M} : \; p \preceq q\}$

- Goal: Define $\mu \preceq \nu$ for $\mu, \nu \in \mathscr{P}(\mathcal{M})$

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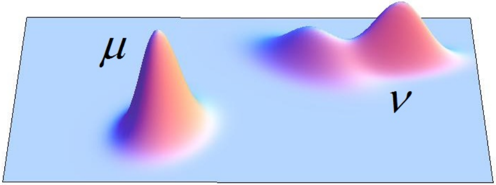

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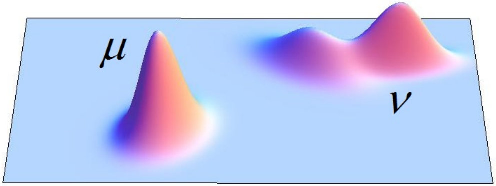

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Definition *[M. Eckstein, TM '15]*

Let $\mathcal{M}$ be a spacetime. For $\mu, \nu \in \mathscr{P}(\mathcal{M})$ we define

$\mu \preceq \nu \quad \underset{\text{def}}{\iff} \quad \exists \omega \in \mathscr{P}(\mathcal{M}^2)$ such that:

- $\forall_{A\text{ – Borel}} \quad \omega(A \times \mathcal{M}) = \mu(A), \;\; \omega(M \times A) = \nu(A)$,
- $\omega(J^+) = 1$.

Math caveat: J^+ must be a Borel set! Luckily, this is always true.

- ω can be called a causal coupling or a causal transference plan.
- For $\mu = \delta_p$, $\nu = \delta_q$, the only coupling is $\omega = \delta_{(p,q)}$ and so $\delta_p \preceq \delta_q$ iff $p \preceq q$.

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Causality for probability measures

Each infinitesimal part of the probability measure should travel along a future-directed causal curve.

ω couples μ with ν

$$\Rightarrow \exists \ \{\gamma_{p,q}\}_{p \in \operatorname{supp} \mu, \ q \in \operatorname{supp} \nu}$$

$\omega(J^+) = 1$

$$\Rightarrow \gamma_{p,q} \text{ – causal fut.-dir.}$$

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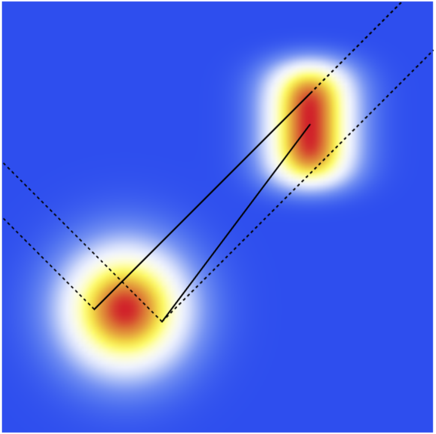

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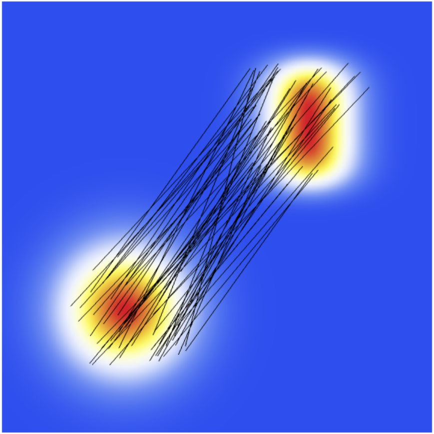
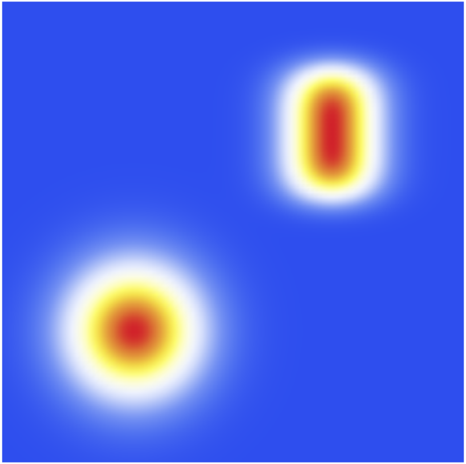


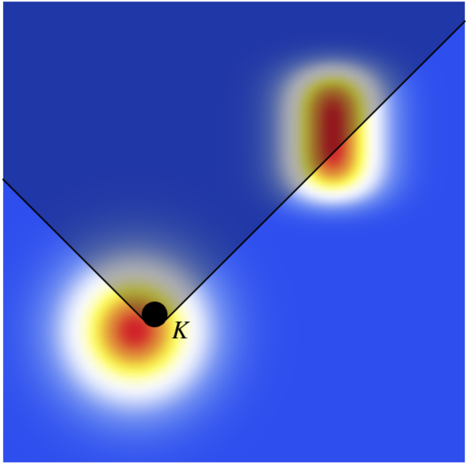
Characterisations of causality



Characterisations of causality

$\mathcal{M}$ – *causally simple* (i.e. no causal loops + J^+ closed)

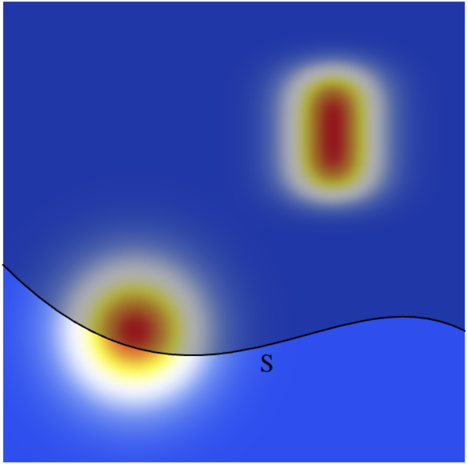
$$\mu \preceq \nu \quad \Longleftrightarrow \quad \forall \text{compact } \mathcal{K} \subseteq \operatorname{supp} \mu \quad \mu(\mathcal{K}) \leq \nu(J^+(\mathcal{K}))$$



Characterisations of causality

$\mathcal{M}$ – *globally hyperbolic* (i.e. admits a Cauchy hypersurface)

$\mu \preceq \nu \quad \Longleftrightarrow \quad \forall$ Cauchy hypersurface $\mathcal{S} \quad \mu(J^+(\mathcal{S})) \leq \nu(J^+(\mathcal{S}))$



Causal evolution in QM

- Wave packet formalism on $(n+1)$-dim Minkowski spacetime.
- The Schrödinger equation $i\hbar\partial_t\psi = \hat{H}\psi$.
- Any wave function ψ yields a measure-valued map

$$t \mapsto \mu_t \in \mathscr{P}(\mathbb{R}^{n+1}), \qquad d\mu_t = \delta_t \times \|\psi(t,x)\|^2 \, d^n x.$$

- Is the quantum evolution causal, i.e. $\mu_s \preceq \mu_t$ if $s < t$?

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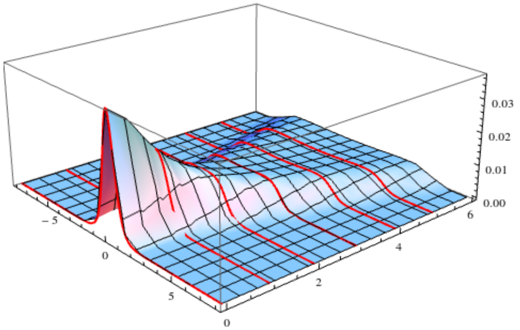

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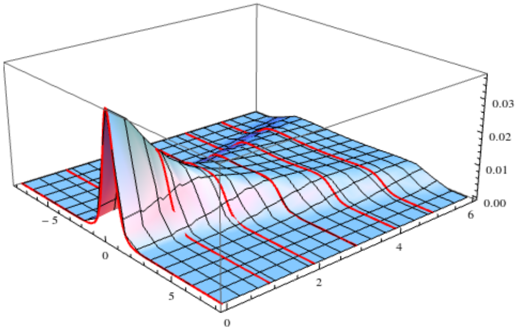

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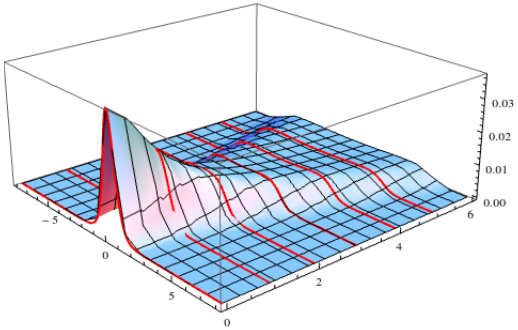

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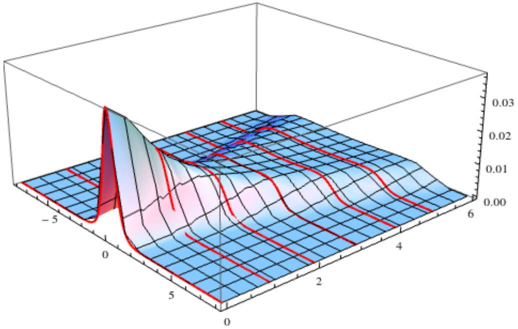

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  - **Hegerfeldt's theorem:** Evolution driven by $\hat{H} \geq 0$ of a *compactly supported* wave function breaks causality – infinite tails appear immediately.
  - Our formalism $\Rightarrow$ Acausality is a property of the system and not of a state.
- Beyond the wave packet formalism: QFT, gauge theories, QG?
- C^*-algebra of observables $\rightarrow$ causality in the space of states ("noncommutative spacetimes") [N. Franco, M. Eckstein 2013–2016]

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[R. Penrose, *Road to Reality*, 2004]

Thank you for your attention!

- M. Eckstein, T. Miller: Causality for nonlocal phenomena, arXiv:1510.06386.
- N. Franco, M. Eckstein: *An algebraic formulation of causality for noncommutative geometry*, Classical and Quantum Gravity **30** (2013) 135007, arXiv:1212.5171.
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