Observables and quantum spacetime

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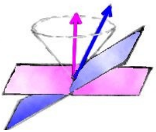
based on *work in preparation* with P. Aschieri and A. Borowiec

Plan:

1. Motivation and general framework
2. Poincaré Casimir and twisted observables
3. Dispersion relations : Flat spacetime
4. Twisted differential calculus
5. Dispersion relations: Curved spacetime

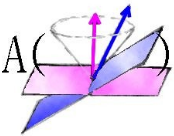
Non-commutative Geometry: origin of quantum space-times

points on manifold M → algebra of functions on M



http://visualrelativity.com

Minkowski spacetime (M, η) with the position of an event $p = (x^0, x^1, x^2, x^3)$



"Minkowski algebra": Abelian algebra with **"coordinate functions"** $x^\mu(p) \in C^\infty(M)$ satisfying
$[x^\mu, x^\nu] = 0$
$\mu, \nu = 0, 1, 2, 3.$

Planck scale

Classical Minkowski spacetime (=commutative algebra)
becomes "quantised" (deformed)
to noncommutative spacetime (=non-commutative algebra)

$$x^\mu \to \hat{x}^\mu$$

- Noncommutative geometry - generalised notion of geometry
- The noncommutative nature allows for obtaining quantum gravitational corrections to the classical solutions.
- Can be helpful in providing the phenomenological models quantifying the effects of quantum gravity.
- One of the mostly studied possible phenomenological effects of quantum gravity is the **modification in wave dispersion**. Such investigations were inspired by the observations of gamma ray bursts (GRBs).

Quantum symmetries

Deformed relativistic symmetries = **Hopf algebras**
quantum spacetimes = **Hopf module algebras**

- Hopf algebra $H(\mu, \eta, \Delta, \epsilon, S)$ is a structure composed by
  1. a (unital associative) algebra (H, μ, η)
  2. a (counital coassociative) coalgebra (H, Δ, ϵ)

  with $S : H \to H$ the antipode.

From any Lie algebra $\mathfrak{g}$ one can make a Hopf algebra

$$H = (U(\mathfrak{g}), \Delta_0, S_0, \epsilon)$$

Lie algebra of vector fields as Hopf algebra

1. $U\Xi$ as Hopf algebra $(U\Xi, \Delta_0, \epsilon, S_0)$, for $\xi \in \Xi$ (in the coordinate basis : $\xi = \xi^\mu \frac{\partial}{\partial x^\mu} = \xi^\mu \partial_\mu$):

$$\begin{aligned} [\xi, \eta] &= (\xi^\mu \partial_\mu \eta^\rho - \eta^\mu \partial_\mu \xi^\rho)\partial_\rho, \\ \Delta_0(\xi) &= \xi \otimes 1 + 1 \otimes \xi, \\ \varepsilon(\xi) &= 0, \quad S(\xi) = -\xi. \end{aligned}$$

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2. The **module algebra** $\mathcal{A} \ni f, g$ is an underlying spacetime of given symmetry:

$$\boxed{\xi \triangleright (f \cdot g) = (\xi_{(1)} \triangleright f) \cdot (\xi_{(2)} \triangleright g)}$$

where $\Delta(\xi) = \xi_{(1)} \otimes \xi_{(2)}$.

The algebra of functions on a manifold $\mathcal{A} = (C^\infty(M), \cdot)$ constitutes a $U\Xi$ (Hopf)- module algebra with a natural action $\triangleright$ of the vector fields on functions.

Twisting

"Classical" $(U\Xi, \mathcal{A}) \rightleftarrows (U\Xi^{\mathcal{F}}, \mathcal{A}^{\mathcal{F}})$ "Quantum"

The twist $\mathcal{F}$ is an invertible element of $U\Xi \otimes U\Xi$.

$$\mathcal{F} = 1 \otimes 1 + \mathcal{O}(h),$$

which provides an undeformed case at the zero-th order in the **deformation parameter** h.

Notation:

$$\mathcal{F} = \mathrm{f}^{\alpha} \otimes \mathrm{f}_{\alpha}, \quad \mathcal{F}^{-1} = \bar{\mathrm{f}}^{\alpha} \otimes \bar{\mathrm{f}}_{\alpha},$$

(sum over $\alpha = 1, 2, \ldots\infty$ assumed)
$\bar{\mathrm{f}}^{\alpha} \in U\Xi$ and $\mathrm{f}^{\alpha} \in U\Xi$

The twist changes the symmetry to twisted symmetry (as deformed Hopf algebra) $U\Xi^{\mathcal{F}}$

$$
\begin{aligned}
[\xi, \eta] &= (\xi^\mu \partial_\mu \eta^\rho - \eta^\mu \partial_\mu \xi^\rho)\partial_\rho, \\
\Delta^{\mathcal{F}}(\xi) &= \mathcal{F}\Delta_0(\xi)\mathcal{F}^{-1} \\
\varepsilon(\xi) &= 0, \quad S^{\mathcal{F}}(\xi) = \mathrm{f}^\alpha S_0(\mathrm{f}_\alpha) S_0(\xi) S_0(\bar{\mathrm{f}}^\beta)\bar{\mathrm{f}}_\beta
\end{aligned}
$$

- the algebra ($[\cdot, \cdot]$) remains undeformed;
- the deformation depends on formal parameter h;
- $\Delta^{\mathcal{F}}$ leads to the **deformed Leibniz rule** for the symmetry transformations when acting on product of fields:

$$
\boxed{\xi \triangleright (f \star g) = (\xi_{(1)^{\mathcal{F}}} \triangleright f) \star (\xi_{(2)^{\mathcal{F}}} \triangleright g)}
$$

where $\Delta^{\mathcal{F}}(\xi) = \xi_{(1)^{\mathcal{F}}} \otimes \xi_{(2)^F}$ and $\mathcal{A}^{\mathcal{F}} = (\mathcal{A}, \star)$.

Star-product

$$A = (C^\infty(M), \mu) \qquad \Longrightarrow \qquad A^{\mathcal{F}} = (C^\infty(M), \star)$$

the algebra of smooth functions becomes a **noncommutative spacetime** with the twisted $\star$-product

$$f \star g \;=\; \mu \mathcal{F}^{-1}(f \otimes g) = \bar{\mathrm{f}}^\alpha(f)\bar{\mathrm{f}}_\alpha(g)$$

$f, g \in C^\infty(M)$.

- such $\star$-product is noncommutative and associative.
- $A^{\mathcal{F}}$ can be represented by deformed, $\star$-commutators of noncommutative coordinates:

$$[\hat{x}^\mu, \hat{x}^\nu] = [x^\mu, x^\nu]_\star = x^\mu \star x^\nu - x^\mu \star x^\mu$$

Quantum (non-commutative) space-times

$\mathcal{A} = \{C^\infty(M), x^\mu : [x^\mu, x^\nu] = 0\}$ can be deformed into, e.g. :

1. Canonical (Moyal-Weyl) space-time: $[\hat{x}^\mu, \hat{x}^\nu] = ih\theta^{\mu\nu}$ $\mathcal{A}_\theta$

   with deformation parameter *h* of length² dimension.

   S. Doplicher, K. Fredenhagen, J. E. Roberts, Commun. Math. Phys. 172 (1995), [arXiv:hep-th/0303037].

2. Lie-algebraic type space-time: $[\hat{x}^\mu, \hat{x}^\nu] = ih\theta^{\mu\nu}_\rho \hat{x}^\rho$

   with deformation parameter *h* of mass dimension.

   Special case – the so-called κ-Minkowski space-time:

$$[\hat{x}^0, \hat{x}^k] = \frac{i}{\kappa}\hat{x}^k \quad , \quad [\hat{x}^i, \hat{x}^k] = 0 \qquad \mathcal{A}_\kappa$$

J.Lukierski, H. Ruegg, A. Nowicki, V.N. Tolstoy, Phys. Lett. B 264 (1991)
S. Majid, H. Ruegg Phys.Lett. B334 (1994)

Twisted generators

The action $\triangleright$ of the Hopf algebra $\mathcal{H}$ on $\mathcal{H}$-module V can be lifted to the algebra of endomorphisms $End_{\mathbb{C}}(V)$ via the adjoint action

$$\xi \blacktriangleright M := \xi_{(1)} \triangleright \circ M \circ S\left(\xi_{(2)}\right) \triangleright$$

for all $\xi \in \mathcal{H}$ and $M \in End_{\mathbb{C}}\left(V\right)$.

- twisted generators of Hopf algebra $\mathcal{H}^{\mathcal{F}} = \left(U\Xi^{\mathcal{F}}, \Delta^{\mathcal{F}}, \epsilon, S^{\mathcal{F}}\right)$

  $$\boxed{\xi^{\mathcal{F}} = \left(\bar{\mathrm{f}}^{\alpha} \blacktriangleright \xi\right) \cdot \bar{\mathrm{f}}_{\alpha}}$$

  for $\xi \in U\Xi$.
- the twisted commutator

  $$\left[\xi^{\mathcal{F}}, \chi^{\mathcal{F}}\right]_{\mathcal{F}} = ad^{\mathcal{F}}_{\xi^{\mathcal{F}}} \chi^{\mathcal{F}} = \xi^{\mathcal{F}} \blacktriangleright_{\mathcal{F}} \chi^{\mathcal{F}}$$

  $\xi, \chi \in U\Xi$.

Symmetry

[based on work with P. Aschieri and A. Borowiec]

- Poincaré-Weyl-Lie algebra

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\lambda}] &= i(\eta_{\mu\lambda} M_{\nu\rho} - \eta_{\nu\lambda} M_{\mu\rho} + \eta_{\nu\rho} M_{\mu\lambda} - \eta_{\mu\rho} M_{\nu\lambda}), \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu) \quad , \qquad [P_\mu, P_\lambda] = 0, \\
[D, P_\mu] &= i P_\mu \quad , \quad [D, M_{\mu\nu}] = 0.
\end{aligned}
$$

The differential representation of the generators of Poincaré-Weyl algebra is

$$
P_\mu = -i\partial_\mu \quad ; \quad M_{\mu\nu} = -i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \quad ; \quad D = -i x^\mu \partial_\mu
$$

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CLASSICAL

Universal enveloping algebra of Poincaré-Weyl algebra - as Hopf algebra :

$$\Delta_0(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}$$

$$\Delta_0(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu \quad \text{and} \quad \Delta_0(D) = D \otimes 1 + 1 \otimes D$$

with antipodes

$$S(M_{\mu\nu}) = -M_{\mu\nu}; \;\; S(P_\mu) = -P_\mu; \;\; S(D) = -D$$

and counits

$$\epsilon(M_{\mu\nu}) = \epsilon(P_\mu) = \epsilon(D) = 0$$

Jordanian twist

For the deformation we can use Jordanian twist

- $$\mathcal{F} = \exp\left(-iD \otimes \sigma\right) \quad ; \quad \sigma = \ln\left(1 + \frac{1}{\kappa}P_0\right)$$

- κ - deformation parameter (classical limit when $\kappa \to \infty$)

- it provides

$$[x^0, x^k]_\star = \frac{i}{\kappa}x^k \quad , \quad [x^i, x^k] = 0$$

κ-Minkowski spacetime

Jordanian twist

(with support in Poincaré-Weyl Hopf algebra)

- Poincaré Casimir $\Box = P_\mu P^\mu$ can be deformed through twist ($\Box^{\mathcal{F}} = \left(\bar{\mathrm{f}}^\alpha \blacktriangleright \Box\right) \bar{\mathrm{f}}_\alpha$) into:

$$\Box^{\mathcal{F}} = \frac{P_\mu P^\mu}{\left(1 + \frac{1}{\kappa} P_0\right)^2} = \Box e^{-2\sigma}$$

- This type of invariant on momentum space leading to deformed dispersion relation was already considered in DSR framework.

  [J. Magueijo and L. Smolin in Phys.Rev.Lett.88 (2002), hep-th/0112090; and in Phys.Rev.D67 (2003), gr-qc/0207085.]

Twisted generators

- Twisted generators of Poincaré-Weyl algebra:

$$P_\mu^{\mathcal{F}}\left(\bar{\mathrm{f}}^\alpha \blacktriangleright P_\mu\right)\bar{\mathrm{f}}_\alpha = P_\mu \frac{1}{1+\frac{1}{\kappa}P_0} = P_\mu e^{-\sigma}$$

$$M_{\mu\nu}^{\mathcal{F}} = M_{\mu\nu} \quad ; \quad D^{\mathcal{F}} = D$$

- Twisted Poincaré Casimir from $P_\mu^{\mathcal{F}}$

$$\Box^{\mathcal{F}} = P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}} = P_\mu P^\mu \frac{1}{\left(1+\frac{1}{\kappa}P_0\right)^2}$$

- Twisted commutation relations

$$\begin{aligned} \left[\Box^{\mathcal{F}}, P_\mu^{\mathcal{F}}\right] &= 0 = \left[\Box^{\mathcal{F}}, M_{\mu\nu}^{\mathcal{F}}\right] \\ \left[\Box^{\mathcal{F}}, D^{\mathcal{F}}\right]_{\mathcal{F}} &= -2i\Box^{\mathcal{F}} \end{aligned}$$

Twisted observables

Twisted generators $P^{\mathcal{F}}$ as the observables.

- They are the generators of $\mathfrak{g}^{\mathcal{F}} \in U_{\mathfrak{g}}^{\mathcal{F}}$ which close the Lie algebra $\mathfrak{g}^{\mathcal{F}}$ under the twisted commutator $[\cdot,\cdot]_{\mathcal{F}}$;
- $P_{\mu}^{\mathcal{F}}$ - have the interpretations of the (deformed) translations
- Additionally $P_{\mu}^{\mathcal{F}}$ are Hermitean

Dispersion relation: Flat spacetime

- Deformed wave equation: $\Box^{\mathcal{F}}\phi = P_\mu^{\mathcal{F}} P^{\mu\mathcal{F}}\phi$
- $P_\mu^{\mathcal{F}} e^{ik_\mu x^\mu} = k_\mu^{\mathcal{F}} e^{ik_\mu x^\mu}$
  where in four-vector notation : $k_\mu x^\mu = \omega t - \vec{k}\vec{x}$ and

$$k_0^{\mathcal{F}} = \omega^{\mathcal{F}} = \frac{\omega}{1 + \frac{i}{\kappa}\omega} \quad \text{and} \quad k_i^{\mathcal{F}} = \frac{k_i}{1 + \frac{i}{\kappa}\omega}$$

- $\left(\omega^{\mathcal{F}}\right)^2 - \left(k^{\mathcal{F}}\right)^2 = 0$
- The group velocity $v_g = \frac{d\omega}{dk} = c$ is as in the classical case due to the fact that the plane waves are the 'eigenvectors' of the twisted observables.

Twisted differential calculus - general framework

[S. Majid, R. Oeckl, Commun.Math.Phys. 205 (1999)
arXiv:math/9811054
P. Aschieri, M. Dimitrijevic, F. Meyer, J. Wess , Class.Quant.Grav. 23 (2006)
arXiv:hep-th/0510059]

The star-product between functions $h \in C^\infty(M)$ and 1-forms $\omega \in \Omega^r(M)$:

$$h \star \omega = \bar{\mathrm{f}}^\alpha(h)\bar{\mathrm{f}}_\alpha(\omega)$$

- the action of $\bar{\mathrm{f}}_\alpha$ - via the Lie derivative;
- The $\star$-wedge product on two arbitrary forms ω and ω' is

$$\omega \wedge_\star \omega' = \bar{\mathrm{f}}^\alpha(\omega) \wedge \bar{\mathrm{f}}_\alpha(\omega')$$

- The exterior derivative $\mathrm{d} : A \to \Omega$ satisfies:

$$
\begin{aligned}
\mathrm{d}(f \star g) &= \mathrm{d}f \star g + f \star \mathrm{d}g, \\
\mathrm{d}^2 &= 0, \\
\mathrm{d}f &= (\partial_\mu f)\mathrm{d}x^\mu
\end{aligned}
$$

- The usual exterior derivative d commutes with the Lie derivative which enters in the definition of the $\star$-product.

Differential calculus deformed with Jordanian twist

- For the **twisted differential calculus** we use the coordinate basis where the basis 1-forms are denoted as $\mathrm{d}x^\mu$.
- The action of a vector fields in the twist is via Lie derivative:

$$\mathcal{L}_{P_\mu}(\mathrm{d}x^\nu) = 0, \quad \mathcal{L}_D(\mathrm{d}x^\mu) = -i\mathrm{d}x^\mu$$

- Using these relations one can show that the basis 1-forms anticommute:

$$\mathrm{d}x^\mu \wedge_\star \mathrm{d}x^\nu = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$$

Therefore we have:

$$\mathrm{d}x^\mu \wedge_\star \mathrm{d}x^\nu = \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu = -\mathrm{d}x^\nu \wedge_\star \mathrm{d}x^\mu$$

But the basis 1-forms do not $\star$-commute with functions:

$$\begin{aligned} f \star \mathrm{d}x^\mu &= f \mathrm{d}x^\mu \\ \mathrm{d}x^\mu \star f &= \mathrm{d}x^\mu (1 + \frac{1}{\kappa} P_0) f \end{aligned}$$

Therefore:

$$\boxed{[f, \mathrm{d}x^\mu]_\star = \frac{i}{\kappa} \mathrm{d}x^\mu \partial_0 f}$$

- Generalization to higher order forms.

Wave equation in curved commutative spacetime

- The Laplace-Beltrami operator is a generalization to curved spacetime of the D'Alembert operator and on a scalar field φ we have (using local coordinates)

$$\Box_{LB}\varphi = *\mathrm{d} * \mathrm{d}\varphi = \frac{1}{\sqrt{g}}\partial_\nu \left[\sqrt{g} g^{\nu\mu}\partial_\mu\varphi\right]$$

where classically, for (M, g) in n-dimensions, we have:

$$*\omega = \frac{\sqrt{g}}{r!\,(n-r)!}\omega_{\mu_1 \dots \mu_r}\epsilon^{\mu_1 \dots \mu_r}{}_{\nu_{r+1} \dots \dots \nu_n} dx^{\nu_{r+1}} \wedge \dots \wedge dx^{\nu_n}$$

where $\omega \in \Omega^r(M)$.

Hodge star deformed

P. Aschieri, A. Schenkel, Adv. Theor. Math. Phys. 18 3 (2014), arXiv:1210.0241

- "quantum map" - can be also used to quantize maps (morphisms):

  $m \to \mathcal{D}(m) := \left(\bar{\mathrm{f}}^{\alpha} \blacktriangleright m\right) \circ \bar{\mathrm{f}}_{\alpha} \triangleright = \bar{\mathrm{f}}^{\alpha}_{(1)} \triangleright \circ m \circ S\left(\bar{\mathrm{f}}^{\alpha}_{(2)}\right) \triangleright \circ \bar{\mathrm{f}}_{\alpha} \triangleright \quad , \quad m \in Hom_C\left(V_1, V_2\right)$

  where $Hom_{\mathbb{C}}\left(V_1, V_2\right)$ is the algebra of homomorphisms.

- The deformation of the Hodge $*$ operation is explicitly dependent on the twist form:

$$*^{\mathcal{F}} = \bar{\mathrm{f}}^{\alpha}_{(1)} \triangleright \circ * \circ S\left(\bar{\mathrm{f}}^{\alpha}_{(2)}\right) \triangleright \circ \bar{\mathrm{f}}_{\alpha} \triangleright$$

Hodge star deformed with Jordanian twist

- From the form of the Jordanian twist and the fact that $\mathcal{L}_{P_\nu}(\mathrm{d}x^\mu) = 0$ the non vanishing is only the zero-th order:

$$*^{\mathcal{F}}(dx^{\mu_1} \wedge ... \wedge dx^{\mu_s}) = *(dx^{\mu_1} \wedge ... \wedge dx^{\mu_s})$$

  $\omega \in \Omega^s(M)$.

- In particular for $\varphi \in \Omega^0(M)$ in $n-$dimensions:

$$*^{\mathcal{F}}(\varphi) = *^{\mathcal{F}}(1 \star \varphi) = *^{\mathcal{F}}(1) \star \varphi = *(1) \star \varphi = (dx^{\mu_1} \wedge ... \wedge dx^{\mu_n}) \star \varphi$$

Deformed Laplace-Beltrami

The deformed Casimir $\Box^{\mathcal{F}}$ is compatible with Laplace-Beltrami operator coming from this deformation of the Hodge star:

$$*^{\mathcal{F}}\left(\partial_\mu\varphi \star \mathrm{d}x^\mu\right) = *^{\mathcal{F}}\left(\mathrm{d}x^\mu\right) \star \frac{1}{1+\frac{1}{\kappa}P_0}\partial_\mu\varphi$$

$$\mathrm{d} *^{\mathcal{F}} \mathrm{d}\varphi = *\left(\mathrm{d}x^\mu\right) \star \partial_\nu \left(\frac{1}{1+\frac{1}{\kappa}P_0}\partial_\mu\varphi\right) \mathrm{d}x^\nu$$

Finally:

$$*^{\mathcal{F}}\mathrm{d} *^{\mathcal{F}} \mathrm{d}\varphi \quad = \quad \eta^{\mu\nu}\frac{1}{\left(1+\frac{1}{\kappa}P_0\right)^2}\partial_\nu\partial_\mu\varphi$$

Curved spacetimes

The wave equation is now governed by the Laplace-Beltrami operator:

$$\Box_{LB}\varphi = \frac{1}{\sqrt{g}}\left(\partial_\mu\left(\sqrt{g}g^{\mu\rho}\right)\partial_\rho\varphi\right)$$

Laplace-Beltrami operator twisted with Jordanian twist:

- Deformation of the Laplace-Beltrami operator

$$\boxed{\Box^{\mathcal{F}}_{LB}\varphi = *^{\mathcal{F}} d *^{\mathcal{F}} d\varphi}$$

- The wave equation for the scalar field in terms of twisted momenta:

$$\Box^{\mathcal{F}}_{LB}\varphi = \frac{1}{\sqrt{g}} \star \frac{\partial^{\mathcal{F}}_\rho}{\left(1+\frac{i}{\kappa}\partial^{\mathcal{F}}_0\right)^{1-n}}\left(\left(\sqrt{g}g^{\mu\rho}\right)\star\frac{\partial^{\mathcal{F}}_\mu}{\left(1+\frac{i}{\kappa}\partial^{\mathcal{F}}_0\right)^{n-1}}\varphi\right) = 0$$

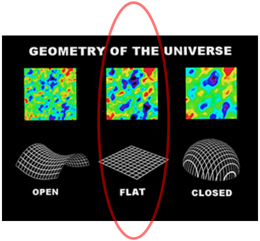
Solutions of wave eq. for FRWL metric

Friedman-Robertson-Walker-Lemaitre (FRWL) metric

(for simplicity in 2 dimensions)

$$g = -dt^2 + a^2(t)\, dx^2$$

where $a(t)$ - scale factor



2-dim twisted wave equation

$$-\partial_0^{\mathcal{F}} e^{-\tilde{\sigma}}(a) \star \partial_0^{\mathcal{F}} \varphi - e^{-2\tilde{\sigma}}(a) \star \partial_0^{\mathcal{F}} \partial_0^{\mathcal{F}} \varphi + e^{-2\tilde{\sigma}}\left(a^{-1}\right) \star \partial_i^{\mathcal{F}} \partial_i^{\mathcal{F}} \varphi = 0$$

where $e^{\tilde{\sigma}} = \left(1 + \frac{i}{\kappa}\partial_0^{\mathcal{F}}\right)^{-1}$.

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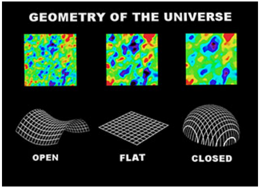

2-dim twisted wave equation

$$-\partial_0^{\mathcal{F}} e^{-\tilde{\sigma}}(a) \star \partial_0^{\mathcal{F}} \varphi - e^{-2\tilde{\sigma}}(a) \star \partial_0^{\mathcal{F}} \partial_0^{\mathcal{F}} \varphi + e^{-2\tilde{\sigma}}\left(a^{-1}\right) \star \partial_i^{\mathcal{F}} \partial_i^{\mathcal{F}} \varphi = 0$$

where $e^{\tilde{\sigma}} = \left(1 + \frac{i}{\kappa}\partial_0^{\mathcal{F}}\right)^{-1}$.
In the classical limit it reduces to:

$$-\dot{a}\partial_0\varphi - a\partial_0^2\varphi + \frac{1}{a}\partial_i^2\varphi = 0 \text{ where } \dot{a} = \partial_0 a(t)$$

Classical equation

$$-a\partial_0^2\varphi - \dot{a}\partial_0\varphi + \frac{1}{a}\partial_i^2\varphi = 0$$

- separation of variables: $\varphi = \lambda\,(t)\,e^{ikx}$
- $$a\ddot{\lambda} + \dot{\lambda}\dot{a} + k^2\lambda\frac{1}{a} = 0$$
- it corresponds (in conformal time) to harmonic oscillator type equation

$$(\partial_\eta^2 + k^2)\lambda = 0$$

Twisted wave equation

$$-\partial_0^{\mathcal{F}} e^{-\tilde{\sigma}}(a) \star \partial_0^{\mathcal{F}} \varphi - e^{-2\tilde{\sigma}}(a) \star \partial_0^{\mathcal{F}} \partial_0^{\mathcal{F}} \varphi + e^{-2\tilde{\sigma}}\left(a^{-1}\right) \star \partial_i^{\mathcal{F}} \partial_i^{\mathcal{F}} \varphi = 0$$

- In the noncommutative case in 2 dimensions we assume the solution of the form: $\varphi = \lambda(t) \star e^{ikx} = \lambda(t)\, e^{ikx}$
- $\partial_i^{\mathcal{F}} e^{i\vec{k}\vec{x}} = \left(ik_i^{\mathcal{F}}\right) e^{i\vec{k}\vec{x}}$
  ($e^{i\vec{k}\vec{x}}$ are still the eigenvectors of the deformed operator $\partial_i^{\mathcal{F}}$ with eigenvalues: $k_i^{\mathcal{F}}$)
- using $\partial_\mu^{\mathcal{F}} = \partial_\mu \frac{1}{1-\frac{i}{\kappa}\partial_0} = \partial_\mu e^{-\sigma}$

We simplify the equation as:

$$a \star \partial_0^2 \lambda + \partial_0(a) \star e^{\sigma} \partial_0 \lambda + a^{-1} \star k^2 \lambda = 0$$

Finding the solution

- As in the classical case - change the coordinates into conformal time η, and $' = \partial_\eta$
- Introduce simplified notation $s = \ln a; s' = \frac{a'}{a}; \frac{a''}{a} = s'' + (s')^2;$
- expand star-product in the first order of κ;

Finding the solution

- As in the classical case - change the coordinates into conformal time η, and $' = \partial_\eta$
- Introduce simplified notation $s = \ln a$; $s' = \frac{a'}{a}$; $\frac{a''}{a} = s'' + (s')^2$;
- expand star-product in the first order of κ;
- Look for the solution of the type:

$$\lambda = \exp\left(i\omega\eta + \frac{i}{\kappa}F\right) = \exp\left(i\Omega_{tot}t\right)$$

term responsible for classical group velocity

quantum correction

- Classical part remains:

$$\left(\omega^2 - k^2\right)\lambda = 0$$

- And equation on $F(\eta)$ is:

$$F'' + 2i\omega F' =$$
$$\frac{i\omega t\left(\eta\right)}{a^2}\left(2\left(s'\right)^3 - 2s's'' - s'k^2 + i\omega\left(s'' - 3\left(s'\right)^2\right) - s'\omega^2\right) - \frac{i\omega}{a}s'\left(s' - i\omega\right)$$

Scale factor

Solution of Einstein equation with perfect barotropic fluid leads to the relation between the Hubble parameter and the scale factor of FRWL metric and has the form: $\boxed{H^2 = H_0^2 a^{-3(1+w)}}$.

The scale factor, in some cases, has the form:

$$\boxed{a(\eta) = \left[1 + H_0(\eta - \eta_0)\right]^\delta}$$

where $\delta = \frac{2}{1+3w}$ is related with type of the Universe and can be, e.g.:
$w = -1 = \delta$ dark energy (cosmological constant) -
$w = 0$ dust -
$w = \frac{1}{3}$ radiation -
$w = 1$ stiff matter -
- dominated.

- Approximation for the scale factor

$$a(\eta) = 1 + \delta H_0 \tau + \frac{\delta(\delta - 1)}{2} H_0^2 \tau^2$$

where $\tau = \eta - \eta_0$;

- Eq. on $F(\eta)$ - linear terms in H_0 :

$$\left[F'' + 2i\omega F'\right]|_{H_0} = -i\omega t(\eta)\, \delta H_0 \left(k^2 + \omega^2\right) - \omega^2 H_0 \delta$$

we expand the correction function of the wave solution as:

$$F = F_0 + H_0 F_1 + H_0^2 F_2$$

where due to initial conditions (flat case) we have $F_0 = 0$.

The solution of this second order differential equation is the following:

$$F_1 = -\frac{\delta}{2}\left(t_0 + \frac{1}{2}\tau\right)\left(k^2 + \omega^2\right)\tau$$

- Hence:

$$v_g = \frac{d\Omega_{tot}}{dk} = V_{g(cl)} - 2\omega\frac{\delta H_0}{\kappa}\left(\frac{t_0}{t} + \frac{1}{2}\frac{\tau}{t}\right)\tau$$

and τ is related with the redshift z.

classical group velocity in curved spacetime

Conclusions

- Twisted generators as observables?
- Framework is valid not only for the flat spacetimes, but allows for more general **curved background** as well.
- Curved spacetime (FRWL) leading to **correction** to group velocity of photons.
- The group velocity depends linearly on the frequency ω up to the first order of the deformation parameter κ .
- Dependence on the type of the Universe - form of $a(\eta)$ - for cosmological constant case correction is subtracted from the classical value.
- Deformation of the wave equation valid for different types of the twists and the metrics.

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Thank you!