Vector-like deformations Minkowski space

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Undeformed Heisenberg algebra and action $\triangleright$

Commutative coordinates x_μ and momenta p_μ generate undeformed Heisenberg algebra $\mathcal{H}(x,p)$:

$$[x_\mu, x_\nu] = 0$$
$$[x_\mu, p_\nu] = i\eta_{\mu\nu}$$
$$[p_\mu, p_\nu] = 0$$

Action $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$:

$$x_\mu \triangleright f(x) = x_\mu f(x)$$
$$p_\mu \triangleright f(x) = [p_\mu, f(x)] = i\frac{\partial f(x)}{\partial p^\mu}$$

Deformed Heisenberg algebra

$x_\mu \to \hat{x}_\mu$
Realization:

$$\hat{x}_\mu = x_\alpha {\varphi^\alpha}_\mu(p) + \chi_\mu(p)$$

Coordinates $\hat{x}_\mu$ and momenta p_μ generate deformed Heisenberg algebra $\hat{\mathcal{H}}(\hat{x}, p)$:

$$[\hat{x}_\mu, \hat{x}_\nu] = i x_\beta \left({\varphi^\alpha}_\mu \frac{\partial {\varphi^\beta}_\nu}{\partial p^\alpha} - {\varphi^\alpha}_\nu \frac{\partial {\varphi^\beta}_\mu}{\partial p^\alpha} \right) + i \left({\varphi^\alpha}_\mu \frac{\chi_\nu}{\partial p^\alpha} - {\varphi^\alpha}_\nu \frac{\chi_\mu}{\partial p^\alpha} \right)$$

$$[\hat{x}_\mu, p_\nu] = i\varphi_{\nu\mu}$$

$$[p_\mu, p_\nu] = 0$$

where $x_\mu = (\hat{x}_\alpha - \chi_\alpha){(\varphi^{-1})^\alpha}_\nu$

General vector-like deformations

The most general (NC) coordinates realized with vector $u_\mu \in \mathcal{M}_{1,n-1}$, $u^2 \in \{-1, 0, 1\}$ κ-Minkowski space:

$$\hat{x}_\mu = x_\mu f_1 + \frac{1}{M} u_\mu (x \cdot p) f_2 + \frac{1}{M} u_\mu (u \cdot x)(u \cdot p) f_3 + \frac{1}{M} (u \cdot x) p_\mu f_4 + \frac{1}{M^2} (x \cdot p) p_\mu f_5 + \frac{1}{M^2} u_\mu (u \cdot x) p^2 f_6 + \frac{1}{M} u_\mu f_7 + \frac{1}{M^2} p_\mu f_8$$

where $f_{1,\ldots,8}$ are functions of $A \equiv u \cdot p/M$ and $B \equiv p^2/M^2$.

General vector-like deformations

Commutator of coordinates:

$$[\hat{x}_\mu, \hat{x}_\nu] = i(u_\mu \hat{x}_\nu - u_\nu \hat{x}_\mu)\frac{1}{M}F_1 + i(\hat{x}_\mu p_\nu - \hat{x}_\nu p_\mu)\frac{1}{M^2}F_2$$
$$+ i(u \cdot \hat{x})(u_\mu p_\nu - u_\nu p_\mu)\frac{1}{M^2}F_3 + i(\hat{x} \cdot p)(u_\mu p_\nu - u_\nu p_\mu)\frac{1}{M^3}$$
$$+ i(u_\mu p_\nu - u_\nu p_\mu)\frac{1}{M^3}F_5$$

where $F_{1,\ldots,5}$ are expressed in terms of $f_{1,\ldots,8}$ and their derivatives.

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Includes: commutative space, κ-Minkowski, Snyder, generalizations of Snyder (e.g. κ-Snyder)...
Does not include: Moyal space, quadratic deformations...

Linear realizations

Linear realizations in p_μ and vector u_μ, with $\chi_\mu = 0$:

$$\begin{aligned}\hat{x}_\mu &= x_\mu + \frac{c_1}{M} x_\mu (u\cdot p) + \frac{c_2}{M} u_\mu (x\cdot p) \\ &\quad + \frac{c_3}{M} u_\mu (u\cdot x)(u\cdot p) + \frac{c_4}{M}(u\cdot x) p_\mu \\ &= x_\mu + K_{\beta\mu\alpha} x^\alpha p^\beta\end{aligned}$$

The twist $\mathcal{F}^{-1}$ is given by:

$$\mathcal{F}^{-1} = e^{ip^W_\alpha \otimes (\hat{x}^\alpha - x^\alpha)}$$

where p^W_μ is a function of momenta p_μ and vector u_μ with property

$$(p^W_\mu - k_\mu) e^{ik\cdot\hat{x}} \triangleright 1 = 0, \qquad \forall k_\mu \in \mathcal{M}_{1,n-1}$$

Realization from the twist

$$\hat{x}_\mu = m\left[\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)\right]$$

Hopf algebra

Coproduct of momenta Δp_μ:

$$\Delta p_\mu = \mathcal{F}^{-1}\Delta_0 p_\mu \mathcal{F} = p_\mu \otimes 1 + (e^{-\mathcal{K}})_{\alpha\mu} \otimes p^\alpha$$

where $\Delta_0 p_\mu = p_\mu \otimes 1 + 1 \otimes p_\mu$ and $\mathcal{K}_{\mu\nu} = -K_{\mu\alpha\nu}(p^W)^\alpha$

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Antipode:

$$S(p_\mu) = -(e^{\mathcal{K}})_{\alpha\mu} p^\alpha$$

Counit is trivial:

$$\epsilon(p_\mu) = 0$$

Star product and deformed addition of momenta

Star product from the twist:

$$(f \star g)(x) = m\left[\mathcal{F}^{-1}(\triangleright \otimes \triangleright)(f \otimes g)\right]$$

Deformed addition of momenta

$$e^{ik\cdot x} \star e^{iq\cdot x} = e^{i\mathcal{D}(k,q)\cdot x}, \qquad k_\mu, q_\mu \in \mathcal{M}_{1,n-1}$$

$$(k \oplus q)_\mu = \mathcal{D}_\mu(k,q)$$

$$\Delta p_\mu = \mathcal{D}_\mu(p \otimes 1, 1 \otimes p)$$

$[\hat{x}_\mu, \hat{x}_\nu]$ closed in $\hat{x}_\lambda$ $\rightarrow$ star product is associative and coproduct is coassociative.

General Lorentz transformations

Generally, Lorentz generators $M_{\mu\nu}$ are defined by

$$M_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$$

where X_μ and P_μ generate Heisenberg algebra $\mathcal{H}(X, P)$:

$$[X_\mu, X_\nu] = 0$$
$$[X_\mu, P_\nu] = i\eta_{\mu\nu}$$
$$[P_\mu, P_\nu] = 0$$

Generators X_μ and P_μ are related to x_μ and p_μ by similarity transformations.

Similarity transformations

Similarity transformations

$$P_\mu = \Sigma_\mu(p)$$
$$X_\mu = x_\alpha \psi^\alpha{}_\mu(p)$$

where functions $\Sigma_\mu(p)$ and $\psi_{\mu\nu}(p)$ satisfy:

$$\frac{\partial \Sigma_\mu(p)}{\partial p_\alpha} \psi_{\alpha\nu} = \eta_{\mu\nu}$$

Generators X_μ and P_μ transform vector-like under $M_{\mu\nu}$:

$$[M_{\mu\nu}, X_\lambda] = i(X_\mu \eta_{\nu\lambda} - X_\nu \eta_{\mu\lambda})$$
$$[M_{\mu\nu}, P_\lambda] = i(P_\mu \eta_{\nu\lambda} - P_\nu \eta_{\mu\lambda})$$

Coproduct of Lorentz generators

In terms of x_μ and p_μ, Lorentz generators $M_{\mu\nu}$ are given by:

$$M_{\mu\nu} = x_\alpha \left(\psi^\alpha{}_\mu(p)\Sigma_\nu(p) - \psi^\alpha{}_\nu(p)\Sigma_\mu(p) \right)$$

Coproduct $\Delta M_{\mu\nu}$ is given by

$$\Delta M_{\mu\nu} = \mathcal{F}^{-1}\Delta_0 M_{\mu\nu}\mathcal{F}$$

where $\Delta_0 M_{\mu\nu}$ is undeformed coproduct.

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where $\Delta_0 M_{\mu\nu}$ is undeformed coproduct.
However, $\Delta_0 M_{\mu\nu} \neq M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu}$ since $M_{\mu\nu} \neq x_\mu p_\nu - x_\nu p_\mu$.

Lorentz transformations of momenta

Lorentz transformations with respect to M_{01} acting on momenta P_μ are simply given by

$$P_0' = P_0 \cosh\xi + P_1 \sinh\xi$$
$$P_1' = P_0 \sinh\xi + P_1 \cosh\xi$$

where ξ is the rapidity.

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where ξ is the rapidity.
Obtainig boost transformations on p_μ:

- Express $p_\mu = \Sigma_\mu^{-1}(P)$
- Perform boost transformations $p'_\mu = \Sigma_\mu^{-1}(P')$
- Express P'_μ in terms of P_μ and then P_μ in terms of p_μ

Introducing backreaction

p_μ transforms $p_\mu \to p'_\mu = \Lambda_\mu(\xi, p)$
However, generally

$$\Lambda_\mu(\xi, k \oplus q) = \Lambda_\mu(\xi, \mathcal{D}(k,q)) \neq \mathcal{D}_\mu(\Lambda(\xi,k), \Lambda(\xi,q)) = [\Lambda(\xi,k) \oplus \Lambda(\xi,q)]_\mu$$

The backreaction on ξ has to be introduced:

$$\Lambda_\mu(\xi, k \oplus q) = [\Lambda(\xi_1, k) \oplus \Lambda(\xi_2, q)]_\mu$$

where $\xi_1 = \xi \triangleleft q$ and $\xi_2 = \xi \triangleleft k$.

Perturbative results

Up to the first order in $1/M$:

$$[\hat{x}_\mu, \hat{x}_\nu] = i(a_\mu \hat{x}_\nu - a_\nu \hat{x}_\mu) + \mathcal{O}\left(\frac{1}{M^2}\right)$$

where $a_\mu = \frac{c_1 - c_2}{M} u_\mu$.
Deformed addition of momenta:

$$(k \oplus q)_\mu = k_\mu + q_\mu + c_1 k_\mu (u \cdot q) + c_2 (u \cdot k) q_\mu + c_3 u_\mu (u \cdot k)(u \cdot q) + c_4 u_\mu (k \cdot q)$$

Perturbative results

Lorentz generators are $M_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$ where

$$P_\mu = p_\mu + d_1 + (u \cdot p) p_\mu d_2 u_\mu p^2 + d_3 (u \cdot p)^2 u_\mu$$
$$X_\mu = x_\mu - d_1 x_\mu (u \cdot p) - d_1 u_\mu (u \cdot x)(u \cdot p) - 2 d_2 (u \cdot x) p_\mu$$

Therefore, in terms of x_μ and p_μ:

$$M_{\mu\nu} = (x_\mu p_\nu - x_\nu p_\mu) - (u_\mu x_\nu - u_\nu x_\mu)[d_2 p^2 + d_3 (u \cdot p)^2] - [d_1 (x \cdot p) + 2 d_3 (u \cdot x)(u \cdot p)](u_\mu p_\nu - u_\nu p_\mu)$$

Generally, coefficients $c_{1,\dots 4}$ and $d_{1,2,3}$ are independent.

κ-Poincaré

For κ-Poincaré (Hopf) algebra, it holds:

$$[P_\mu, \hat{x}_\nu] = -i[\eta_{\mu\nu}(1 + a \cdot P) - a_\mu P_\nu] + \mathcal{O}\left(\frac{1}{M^2}\right)$$

which gives:

$$d_1 = -c_2, \qquad d_2 = -\frac{c_1 - c_2 + c_4}{2}, \qquad d_3 = -\frac{c_3}{2}$$

Commutator of Lorentz generators with NC coordinates is:

$$[M_{\mu\nu}, \hat{x}_\lambda] = \hat{x}_\mu \eta_{\nu\lambda} - \hat{x}_\nu \eta_{\mu\lambda} + a_\mu M_{\nu\lambda} - a_\nu M_{\mu\lambda}$$

κ-Poincaré Hopf algebra structure

Coproducts:

$$\Delta P_\mu = \Delta_0 P_\mu + P_\mu \otimes a \cdot P + a_\mu P_\alpha \otimes P^\alpha$$
$$\Delta M_{\mu\nu} = \Delta_0 M_{\mu\nu} + (\delta_\mu^\alpha a_\nu - \delta_\nu^\alpha a_\mu)$$

Antipodes:

$$S(P_\mu) = -P_\mu + (a \cdot P)P_\mu - a_\mu P^2$$
$$S(M_{\mu\nu}) = -M_{\mu\nu} - P^\alpha (a_\mu M_{\nu\alpha} - a_\nu M_{\mu\alpha})$$

Perturbative results for the backreaction factors

Backreaction factors in first order of rapidity and deformation in 1+1 dimensions:

$$\xi_1 = \xi \left(1 - \frac{c_2 + d_1}{M} u \cdot q\right)$$
$$\xi_2 = \xi \left(1 - \frac{c_1 + d_1}{M} u \cdot k\right)$$

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For κ-Poincaré, this simplifies to:

$$\xi_1 = \xi$$
$$\xi_2 = \xi\left(1 - \frac{c_1 - c_2}{M} u \cdot k\right) = \xi(1 - a \cdot k)$$

Conclusion

- Realizations of (non-)commutative coordinates, that can be constructed with vector $u_\mu \in \mathcal{M}_{1,n-1}$ and deformation parameter $1/M$, cover a wide class of spaces.
- In Lorentz transformations of deformed addition of momenta, the backreation has to be taken into account.
- Backreaction factors are calculated in 1+1 dimensions in the first order of deformation and rapidity.

Thank you for attention!