An extension of R-matrix approach to construction of integrable systems to (3+1) dimensions

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Integrable systems

- nonlinear
- tractable:
  * infinitely many explicit exact solutions
  * rich symmetry algebras
  * infinitely many conservation laws

Integrable systems usually arise as compatibility conditions for overdetermined linear systems (Lax pairs)

A *dispersionless* (or *hydrodynamic-type*) system in d independent variables $x^1, \dots, x^d$ and N dependent variables $u^1, \dots, u^N$ is a first-order homogeneous quasilinear system

$$A_1(\boldsymbol{u})\boldsymbol{u}_{x^1} + A_1(\boldsymbol{u})\boldsymbol{u}_{x^2} + \dots + A_d(\boldsymbol{u})\boldsymbol{u}_{x^d} = 0, \quad (1)$$

where A_i are $M \times N$ matrices, $M \geqslant N$, $\boldsymbol{u} \equiv (u^1, \dots, u^N)^T$.

Nearly all known today (classical bosonic) integrable systems with $d \geqslant 4$ are dispersionless, e.g. (anti-)-self-dual Yang–Mills equations and (anti-)-self-dual vacuum Einstein equations with vanishing cosmological constant.

Integrable systems: 3D vs 4D

	Dispersive	**Dispersionless**
3D	systematic construction (central extension)	systematic construction (Hamiltonian vect. fields; central extension)
4D	exceptional	exceptional

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	Dispersive	Dispersionless
3D	systematic construction (central extension)	systematic construction (Hamiltonian vect. fields; central extension)
4D	exceptional	systematic construction* (contact geometry)

*See A. Sergyeyev, A new class of (3+1)-dimensional integrable systems related to contact geometry, arXiv:1401.2122

Let $\mathfrak{g}$ be an (infinite-dimensional) Lie algebra.
The Lie bracket $[\cdot,\cdot]$ defines the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$:

$$\mathrm{ad}_a\, b = [a, b].$$

An $R \in \mathrm{End}(\mathfrak{g})$ is a *(classical) R-matrix* if the R-bracket

$$[a, b]_R := [Ra, b] + [a, Rb] \qquad (2)$$

is a new Lie bracket on $\mathfrak{g}$. The skew symmetry of (2) is obvious. As for the Jacobi identity for (2), a sufficient condition for it to hold is the *classical modified Yang–Baxter equation* for R,

$$[Ra, Rb] - R[a, b]_R - \alpha[a, b] = 0, \qquad \alpha \in \mathbb{R}. \qquad (3)$$

A simple construction of commuting flows

Theorem 1

Suppose that R is an R-matrix on $\mathfrak{g}$ which satisfies

$$(RL)_{t_n} = RL_{t_n}, \quad n \in \mathbb{N}, \tag{4}$$

and obeys the classical modified Yang–Baxter equation (3) for $\alpha \neq 0$. Let $L_i \in \mathfrak{g}$, $i \in \mathbb{N}$ satisfy

$$(L_n)_{t_r} = [RL_r, L_n], \qquad r, n \in \mathbb{N}. \tag{5}$$

Then the following conditions are equivalent:

i) $$(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s] = 0, \quad r, s \in \mathbb{N} \tag{6}$$

ii) $$[L_i, L_j] = 0, \qquad i, j \in \mathbb{N}. \tag{7}$$

Moreover, if i) or ii) holds then the flows (5) commute:

$$((L_n)_{t_r})_{t_s} - ((L_n)_{t_s})_{t_r} = 0, \quad n, r, s \in \mathbb{N}. \tag{8}$$

Proof of Theorem 1

Using (5) and the assumption (4) we see that the left-hand side of (6) takes the form

$$\begin{aligned}&(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s]\\ &= R[RL_s, L_r] - R[RL_r, L_s] + [RL_r, RL_s]\\ &= [RL_r, RL_s] - R[L_r, L_s]_R \stackrel{(3)}{=} -\alpha[L_r, L_s]\end{aligned}$$

which establishes the equivalence of (7) and (6).

To complete the proof observe that the left-hand side of (7) can be written as

$$\begin{aligned}&((L_n)_{t_r})_{t_s} - ((L_n)_{t_s})_{t_r} = [RL_r, L_n]_{t_s} - [RL_s, L_n]_{t_r}\\ &= [(RL_r)_{t_s} - (RL_s)_{t_r}, L_n] + [RL_r, [RL_s, L_n]] - [RL_s, [RL_r, L_n]]\\ &= [(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s], L_n] \stackrel{(6)}{=} 0.\end{aligned}$$

Commutativity of extended flows

Assume that all elements of $\mathfrak{g}$ depend on an additional independent variable y not involved in the Lie bracket.

Theorem 2

Suppose that $\mathcal{L} \in \mathfrak{g}$ and $L_i \in \mathfrak{g}$, $i \in \mathbb{N}$ are such that the zero-curvature equations (6) hold for all $r, s \in \mathbb{N}$, the R-matrix R on $\mathfrak{g}$ satisfies

$$(RL)_{t_n} = RL_{t_n}, \quad n \in \mathbb{N}, \qquad (RL)_y = RL_y \tag{9}$$

and L_r satisfy the extended Lax equations

$$\mathcal{L}_{t_r} = [RL_r, \mathcal{L}] + (RL_r)_y, \qquad r \in \mathbb{N}. \tag{10}$$

Then the flows (10) commute, i.e.,

$$(\mathcal{L}_{t_r})_{t_s} - (\mathcal{L}_{t_s})_{t_r} = 0, \quad r, s \in \mathbb{N}. \tag{11}$$

Using equations (10) and the Jacobi identity for the Lie bracket we obtain

$$\begin{aligned}(\mathcal{L}_{t_r})_{t_s} - (\mathcal{L}_{t_s})_{t_r} &= [(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s], \mathcal{L}] \\ &\quad + ((RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s])_y \\ &= 0.\end{aligned}$$

The right-hand side of the above equation vanishes by virtue of the zero curvature equations (6).

R-matrices from the Lie algebra decomposition

If $\mathfrak{g}$ admits a decomposition into two Lie subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ such that

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-, \qquad [\mathfrak{g}_\pm, \mathfrak{g}_\pm] \subset \mathfrak{g}_\pm, \qquad \mathfrak{g}_+ \cap \mathfrak{g}_- = \emptyset,$$

the operator

$$R = \frac{1}{2}(P_+ - P_-) = P_+ - \frac{1}{2} \tag{12}$$

where $P_\pm$ are projectors onto $\mathfrak{g}_\pm$, satisfies the classical modified Yang–Baxter equation (3) with $\alpha = \frac{1}{4}$, i.e., R defined by (12) is a classical R-matrix.

Next, let us specify the dependence of L_j on y via the so-called Lax–Novikov equations

$$[L_j, \mathcal{L}] + (L_j)_y = 0, \qquad j \in \mathbb{N}. \tag{13}$$

Then, upon applying (7), (12) and (13), equations (5), (6) and (10) are readily seen to take the following form:

$$(L_s)_{t_r} = [B_r, L_s], \qquad r, s \in \mathbb{N}, \tag{14}$$

$$(B_r)_{t_s} - (B_s)_{t_r} + [B_r, B_s] = 0, \tag{15}$$

$$\mathcal{L}_{t_r} = [B_r, \mathcal{L}] + (B_r)_y, \qquad n, r \in \mathbb{N} \tag{16}$$

where $B_i = P_+ L_i$.

Reduction with respect to y

If upon the reduction to the y-independent case we put $\mathcal{L} = L_n$ for some $n \in \mathbb{N}$, then the hierarchies (10), i.e.,

$$\mathcal{L}_{t_r} = [RL_r, \mathcal{L}] + (RL_r)_y, \qquad r \in \mathbb{N},$$

boil down to hierarchies (5) and the Lax–Novikov equations (13), i.e.,

$$[L_j, \mathcal{L}] + (L_j)_y = 0, \qquad j \in \mathbb{N},$$

reduce to (a part of) the commutativity conditions (7), i.e., $[L_n, L_j] = 0$. In particular, if the bracket $[\cdot, \cdot]$ is such that equations (10) give rise to integrable systems in d independent variables, then equations (5) yield integrable systems in $d - 1$ independent variables.

Commutative subalgebras: a standard construction

A standard construction of a commutative subalgebra spanned by L_i whose existence by Theorem 1 ensures commutativity of the flows (10) is, in the case of Lie algebras which admit an additional associative multiplication $\circ$ which obeys the Leibniz rule

$$\mathrm{ad}_a(b \circ c) = \mathrm{ad}_a(b) \circ c + b \circ \mathrm{ad}_a(c), \tag{17}$$

as follows: the commutative subalgebra in question is generated by fractional powers of a given element $L \in \mathfrak{g}$.

Further remarks

In our setting, when we no longer assume existence of an associative multiplication on $\mathfrak{g}$ which obeys the Leibniz rule (17), the above construction does not work anymore. To circumvent this difficulty, instead of an explicit construction of commuting L_i we will *impose* the zero-curvature constraints (6), i.e.,

$$(RL_r)_{t_s} - (RL_s)_{t_r} + [RL_r, RL_s] = 0, \quad r, s \in \mathbb{N}$$

on chosen elements $L_i \in \mathfrak{g}$, $i \in \mathbb{N}$; in our setting ($[\cdot, \cdot]$ is the *contact bracket*, see below), this can be done in a consistent fashion. By Theorem 1 this guarantees the commutativity of L_i for any R-matrix which obeys the classical modified Yang–Baxter equation (3) with $\alpha \neq 0$.

The contact bracket setting

Consider a commutative and associative algebra A of formal series in p of the form $f = \sum_i u_i p^i$ with the standard multiplication

$$f_1 \cdot f_2 \equiv f_1 f_2, \qquad f_1, f_2 \in A. \tag{18}$$

The coefficients u_i of these series are assumed to be smooth functions of x, y, z and infinitely many times $t_1, t_2, \ldots$.
The contact bracket on A (see AS, arxiv:1401.2122):

$$\{f_1, f_2\} = \frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial x} - p\frac{\partial f_1}{\partial p}\frac{\partial f_2}{\partial z} + f_1\frac{\partial f_2}{\partial z} - (f_1 \leftrightarrow f_2). \tag{19}$$

The variable y is not involved in the bracket.

The contact bracket setting II

Note that A is not a Poisson algebra as the contact bracket (19) does not obey the Leibniz rule. However, it belongs to a more general class of the so-called *Jacobi algebras* that obey the following generalization of the Leibniz rule:

$$\{f_1 f_2, f_3\} = \{f_1, f_3\} f_2 + f_1 \{f_2, f_3\} - f_1 f_2 \{1, f_3\}. \qquad (20)$$

If the unity 1 belongs to the center of the Lie algebra in question, then (20) boils down to the usual Leibniz rule and the algebra under study is then just a Poisson algebra.

To relate to the R-matrix approach, we identify $\mathfrak{g}$ with A and the bracket $[\cdot,\cdot]$ in $\mathfrak{g}$ with the contact bracket (19). As for the choice of the splitting of $\mathfrak{g}$ into Lie subalgebras $\mathfrak{g}_\pm$ with $P_\pm$ being projections onto the respective subalgebras, we have two natural choices when the R's defined by (12) satisfy the classical modified Yang–Baxter equation (3) and thus are R-matrices. These two choices are

$$P_+ = P_{\geqslant k},$$

where $k=0$ or $k=1$, and by definition

$$P_{\geqslant k}\left(\sum_{j=-\infty}^{\infty} a_j p^j\right) = \sum_{j=k}^{\infty} a_j p^j.$$

Thus, in our approach we have $\mathcal{L}$ of a general form

$$\mathcal{L} = \sum_{i=-\infty}^{n} u_i p^i, \quad n > 0. \tag{21}$$

$$B_m := \sum_{j=k}^{m} v_{m,j} p^j, \quad m \in \mathbb{N}. \tag{22}$$

$\mathcal{L}$ and B_m are required to obey the following equations:

$$\mathcal{L}_{t_m} = \{B_m, \mathcal{L}\} + (B_m)_y, \quad m \in \mathbb{N}, \tag{23}$$

$$(B_n)_{t_m} - (B_m)_{t_n} - \{B_m, B_n\} = 0, \quad m, n \in \mathbb{N}. \tag{24}$$

Eqs.(23) define the time evolution (24) relate $v_{m,j}$ and $v_{n,i}$.
Below we will subject $\mathcal{L}$ to various constraints.

AS, arXiv:1401.2122: a zero-curvature equation

$$f_y - g_\tau - \{f, g\} = 0$$

holds if and only if the following linear system is compatible:

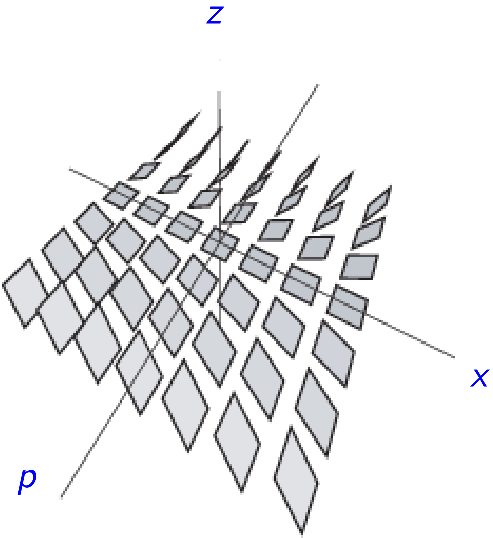
$$\psi_\tau = X_f(\psi), \quad \psi_y = X_g(\psi),$$

where $\psi = \psi(x, y, z, \tau, p)$ and

$$X_h = h_p\partial_x + (ph_z - h_x)\partial_p + (h - ph_p)\partial_z$$

is a contact vector field with the contact Hamiltonian h (so by definition there exists a function f: $L_{X_h}\alpha = f\alpha$, where $\alpha = dz + pdx$ is our contact form, and $i_{X_h}\alpha = h$).

$\ker\alpha$ for $\alpha = dz + pdx$
(original image credit: John Etnyre, arXiv:math/0111118)



Let $k = 0$ so $R = \frac{1}{2}(P_{\geq 0} - P_{<0})$. For $n > 0$ and $m > 0$ put

$$\mathcal{L} := u_n p^n + u_{n-1} p^{n-1} + \cdots + u_0 + u_{-1} p^{-1} + \cdots , \quad (25)$$

$$B_m := v_{m,m} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,0}, \quad (26)$$

where

$$u_i = u_i(\vec{t}, x, y, z),$$

$$v_{m,j} = v_{m,j}(\vec{t}, x, y, z),$$

and $\vec{t} = (t_1, t_2, \ldots)$.

Substituting $\mathcal{L}$ and B_m into the zero-curvature equations

$$\mathcal{L}_{t_m} = \{B_m, \mathcal{L}\} + (B_m)_y \tag{27}$$

yields a hierarchy of infinite-component systems ($m \in \mathbb{N}$)

$$\begin{aligned} (u_r)_{t_m} &= X_r^m[u, v_m], \qquad r \leq n+m, \quad r \neq 0, \dots, m, \\ (u_r)_{t_m} &= X_r^m[u, v_m] + (v_{m,r})_y, \qquad r = 0, \dots, m. \end{aligned} \tag{28}$$

where $u_r \equiv 0$ for $r > n$; for $r \leq m+n$ we put

$$\begin{aligned} X_r^m[u, v_m] = \sum_{s=0}^{m} [& s v_{m,s} (u_{r-s+1})_x - (r-s+1) u_{r-s+1} (v_{m,s})_x \\ & - (s-1) v_{m,s} (u_{r-s})_z + (r-s-1) u_{r-s} (v_{m,s})_z], \end{aligned}$$

$u = (u_n, u_{n-1}, \dots)$ and $v_m = (v_{m,0}, \dots, v_{m,m})$.

Constraints I

The first equation from the system (28), i.e., the one for $r = n + m$, takes the form

$$(n-1)u_n(v_{m,m})_z - (m-1)v_{m,m}(u_n)_z = 0,$$

and hence, for $n > 1, m > 1$, admits the constraint

$$v_{m,m} = (u_n)^{\frac{m-1}{n-1}}. \tag{29}$$

For $n = 1$ the constraint in question takes the form $u_1 = \mathrm{const}$.

Constraints II

Let $u_n = c_n,\ v_{m,m} = c_{m,m}$, where $c_n, c_{m,m} \in \mathbb{R}$.
Then, if $c_n = c_{m,m} = 1$, we have, for $n > 0$ and $m > 0$

$$\mathcal{L} = p^n + u_{n-1}p^{n-1} + \cdots + u_0 + u_{-1}p^{-1} + \cdots, \qquad (30)$$

$$B_m \equiv P_+L_m = p^m + v_{m,m-1}p^{m-1} + \cdots + v_{m,0}, \qquad (31)$$

and equations (27) take the form (28), where now
$r < n + m$ and

$$\begin{aligned} X_r^m[u, v_m] \ = \ & m(u_{r-m+1})_x - (m-1)(u_{r-m})_z \\ & + \sum_{s=0}^{m-1} [s v_{m,s}(u_{r-s+1})_x - (r-s+1)u_{r-s+1}(v_{m,s})_x \\ & -(s-1)v_{m,s}(u_{r-s})_z + (r-s-1)u_{r-s}(v_{m,s})_z]. \end{aligned}$$

Constraints III

Again, the first equation from the system (28), i.e., the one for $r = n + m - 1$, takes the form

$$(n-1)(v_{m,m-1})_z - (m-1)(u_{n-1})_z = 0,$$

so the system under study for $n > 1$ admits a further constraint

$$v_{m,m-1} = \frac{(m-1)}{(n-1)} u_{n-1}. \tag{32}$$

It is readily seen that for $n = 1$ the constraint (32) should be replaced by $u_0 = \mathrm{const}$.

First-order Lax operator

Upon taking $u_0 = 0$, the Lax equation (27) for

$$\mathcal{L} = p + u_{-1}p^{-1} + u_{-2}p^{-2} + \cdots, \qquad (33)$$

and for $m = 2$, with $B_2 = p^2 + v_1 p + v_0$ generates the following infinite-component system

$$\begin{aligned}
(v_1)_y &= (v_1)_x + (u_{-1})_z, \\
(v_0)_y &= (v_0)_x + (u_{-2})_z - 2(u_{-1})_x + 2u_{-1}(v_1)_z, \\
(u_r)_{t_2} &= 2(u_{r-1})_x - (u_{r-2})_z - (r+1)u_{r+1}(v_0)_x \\
&\quad + v_0(u_r)_z + (r-1)u_r(v_0)_z + v_1(u_r)_x \\
&\quad - ru_r(v_1)_x + (r-2)u_{r-1}(v_1)_z,
\end{aligned} \qquad (34)$$

where $r < 0$ and $v_{2,r} \equiv v_r$.

General case for $k=1$

Let $k=1$, when $P_+ = P_{\geqslant 1}$, $m>0, n>0$,

$$\begin{aligned} \mathcal{L} &= u_n p^n + u_{n-1} p^{n-1} + \dots + u_0 + u_{-1} p^{-1} + \dots, \\ B_m &= v_{m,m} p^m + v_{m,m-1} p^{m-1} + \dots + v_{m,1} p, \end{aligned} \tag{35}$$

$$\begin{aligned} (u_r)_{t_m} &= X_r^m[u, v_m], \ r \leq n+m, \ r \neq 1, \dots, m, \\ (u_r)_{t_m} &= X_r^m[u, v_m] + (v_{m,r})_y, \ r = 1, \dots, m, \end{aligned} \tag{36}$$

where $u_r \equiv 0$ for $r>n$, $u = (u_n, u_{n-1}, \dots)$,
$v_m = (v_{m,1}, \dots, v_{m,m})$, and for $r \leq m+n$

$$\begin{aligned} X_r^m[u, v_m] = \sum_{s=1}^{m} [& s v_{m,s} (u_{r-s+1})_x - (r-s+1) u_{r-s+1} (v_{m,s})_x \\ & - (s-1) v_{m,s} (u_{r-s})_z + (r-s-1) u_{r-s} (v_{m,s})_z]. \end{aligned}$$

For $n>1, m>1$ we again obtain the constraint (29), i.e., $v_{m,m} = (u_n)^{\frac{m-1}{n-1}}$, and for $n=1$ it is replaced by $u_1 = \mathrm{const}$.

Let $m > 1$. Put

$$\mathcal{L} = p + u_0 + u_{-1}p^{-1} + \cdots , \qquad (37)$$

$$B_m \equiv P_+ L_m = v_{m,m-1}p^m + v_{m,m-2}p^{m-1} + \cdots + v_{m,1}p. \qquad (38)$$

The first flow for $m = 2$, where $v_{2,r} \equiv v_r$, is

$$\begin{aligned}
(v_2)_y &= (v_2)_x + u_0(v_2)_z + v_2(u_0)_z,\\
(v_1)_y &= (v_1)_x + u_0(v_1)_z + v_2(u_{-1})_z\\
&\quad +2u_{-1}(v_2)_z - 2v_2(u_0)_x,\\
(u_r)_{t_2} &= v_1(u_r)_x - ru_r(v_1)_x + (r-2)u_{r-1}(v_1)_z\\
&\quad +2v_2(u_{r-1})_x - (r-1)u_{r-1}(v_2)_x\\
&\quad -v_2(u_{r-2})_z + (r-3)u_{r-2}(v_2)_z.
\end{aligned} \qquad (39)$$

Finite-component reductions for $k = 0$

We have natural reductions to finite-component systems by putting $u_r = 0$ for $r < 1$ or $r < 0$ in (25) and (30), i.e.,

$$\begin{aligned} \mathcal{L} &= u_n p^n + u_{n-1} p^{n-1} + \cdots + u_r p^r, \quad r = 0, 1, \\ B_m &= (u_n)^{\frac{m-1}{n-1}} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,0}, \end{aligned} \tag{40}$$

$$\begin{aligned} \mathcal{L} &= p^n + u_{n-1} p^{n-1} + \cdots + u_r p^r, \quad r = 0, 1, \\ B_m &= p^m + \frac{(m-1)}{(n-1)} u_{n-1} p^{m-1} + \cdots + v_{m,0}. \end{aligned} \tag{41}$$

The case (41) for $r = 0$ was considered for the first time in AS, arXiv:1401.2122. In (40) and (41) for $r = 0$ $\mathcal{L}$ has the same form as B_n, so y can be identified with t_n.

We now have two cases:

$$\begin{aligned} \mathcal{L} &= u_n p^n + u_{n-1} p^{n-1} + \cdots + u_r p^r, \\ B_m &= (u_n)^{\frac{m-1}{n-1}} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,1} p; \end{aligned} \tag{42}$$

$$\begin{aligned} \mathcal{L} &= p + u_0 + u_{-1} p^{-1} + \cdots + u_r p^r, \\ B_m &= v_{m,m} p^m + v_{m,m-1} p^{m-1} + \cdots + v_{m,1} p. \end{aligned} \tag{43}$$

Here $m > 1$ and $r = 0, 1, -1, \ldots$.

$$\mathcal{L} := p + u_0 + u_{-1} p^{-1} \tag{44}$$

and, with a slight variation of the earlier notation, put

$$B_2 = v_2 p^2 + v_1 p, \quad B_3 = w_3 p^3 + w_2 p^2 + w_1 p.$$

The member of the hierarchy associated with B_2 reads

$$\begin{aligned}
(u_{-1})_{t_2} &= u_{-1}(v_1)_x + v_1(u_{-1})_x,\\
(u_0)_{t_2} &= -2u_{-1}(v_1)_z + v_1(u_0)_x\\
&\quad + u_{-1}(v_2)_x + 2v_2(u_{-1})_x,\\
(v_1)_y &= (v_1)_x + 2u_{-1}(v_2)_z\\
&\quad + v_2(u_{-1})_z + u_0(v_1)_z - 2v_2(u_0)_x,\\
(v_2)_y &= (v_2)_x + u_0(v_2)_z + v_2(u_0)_z,
\end{aligned} \tag{45}$$

In turn, the flow associated with B_3 has the form

$$
\begin{aligned}
(u_{-1})_{t_3} &= u_{-1}(w_1)_x + w_1(u_{-1})_x, \\
(u_0)_{t_3} &= w_1(u_0)_x - 2u_{-1}(w_1)_z + u_{-1}(w_2)_x + 2w_2(u_{-1})_x, \\
(w_1)_y &= (w_1)_x + w_2(u_{-1})_z - u_{-1}(w_3)_x \\
&\quad -2w_2(u_0)_x + 2u_{-1}(w_2)_z \\
&\quad +u_0(w_1)_z - 3w_3(u_{-1})_x, \\
(w_2)_y &= (w_2)_x - 3w_3(u_0)_x + 2w_3(u_{-1})_z + w_2(u_0)_z \\
&\quad +u_0(w_2)_z + 2u_{-1}(w_3)_z, \\
(w_3)_y &= (w_3)_x + u_0(w_3)_z + 2w_3(u_0)_z,
\end{aligned}
$$

Finite-component reductions for $k = 1$: an example III

Commutativity of the flows associated with t_2 and t_3, i.e.,

$$((u_i)_{t_2})_{t_3} = ((u_i)_{t_3})_{t_2}, \quad i = 0, 1,$$

can be readily checked using the zero-curvature equation

$$(B_2)_{t_3} - (B_3)_{t_2} + \{B_2, B_3\} = 0. \tag{46}$$

The compatibility conditions

$$((v_i)_y)_z = ((v_i)_z)_y, \quad i = 1, 2,$$

are also satisfied by virtue of (45) and (46).

Eq.(46) is equivalent to the system

$$
\begin{aligned}
(v_1)_z &= -\frac{v_2}{w_3}(w_3)_x - \frac{v_2 w_2}{4 w_3^2}(w_3)_z + \frac{v_2}{2 w_3}(w_2)_z + \frac{3}{2}(v_2)_x,\\
(v_2)_z &= \frac{v_2}{2 w_3}(w_3)_z,\\
(w_1)_{t_2} &= v_1 (w_1)_x - w_1 (v_1)_x + (v_1)_{t_3},\\
(w_2)_{t_2} &= v_1 (w_2)_x - w_1 (v_2)_x + 2 v_2 (w_1)_x - 2 w_2 (v_1)_x + (v_2)_{t_3},\\
(w_3)_{t_2} &= \frac{v_2 w_2}{2 w_3}(w_2)_z - \frac{w_2}{2}(v_2)_x - \frac{v_2 w_2^2}{4 w_3^2}(w_3)_z\\
&\quad + \frac{(v_1 w_3 - v_2 w_2)}{w_3}(w_3)_x\\
&\quad - v_2 (w_1)_z + 2 v_2 (w_2)_x - 3 w_3 (v_1)_x.
\end{aligned}
$$

Finite-component reductions for $k = 1$: another example

The simplest nontrivial example of Lax pair (42) is given by

$$\begin{aligned}\mathcal{L} &= u_3 p^3 + u_2 p^2 + u_1 p,\\ B_2 &= v_2 p^2 + v_1 p,\end{aligned}$$

and the associated system reads

$$\begin{array}{rcl} 0 & = & 2u_3(v_2)_z - v_2(u_3)_z,\\ 0 & = & u_2(v_2)_z - v_2(u_2)_z + 2u_3(v_1)_z\\ & & +2v_2(u_3)_x - 3u_3(v_2)_x\\ (u_3)_{t_2} & = & v_1(u_3)_x + 2v_2(u_2)_x - 2u_2(v_2)_x\\ & & -3u_3(v_1)_x - v_2(u_1)_z + u_2(v_1)_z,\\ (u_2)_{t_2} & = & (v_2)_y + v_1(u_2)_x + 2v_2(u_1)_x\\ & & -2u_2(v_1)_x - u_1(v_2)_x,\\ (u_1)_{t_2} & = & (v_1)_y + v_1(u_1)_x - u_1(v_1)_x. \end{array} \tag{47}$$

The first two of the above equations impose constraints on the 'non-dynamical' fields v_1 and v_2. The first of these constraints is satisfied once we impose (29), i.e.,

$$v_2 = (u_3)^{\frac{1}{2}},$$

and then the second one boils down to

$$(v_1)_z = \left[\frac{1}{2}u_2(u_3)^{-\frac{1}{2}}\right]_z - \left[\frac{1}{2}(u_3)^{\frac{1}{2}}\right]_x,$$

or

$$v_1 = \frac{1}{2}u_2(u_3)^{-\frac{1}{2}} - \partial_z^{-1}\partial_x\left[\frac{1}{2}(u_3)^{\frac{1}{2}}\right].$$

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- **Open problem:** Is it possible to generalize our construction to the case of noncommutative independent and/or dependent variables?

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Dziękuję za uwagę