TWISTING REALITY

Andrzej Sitarz

Institute of Physics
Jagiellonian University
Kraków

Institute of Mathematics,
Polish Academy of Sciences
Warsaw

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Plan

1. From differential to noncommutative geometry
2. Softened reality.
3. Twisted reality
4. Conformally transformed Dirac operators
5. Quantum Twists and κ.
6. Conclusions

DIFFERENTIAL GEOMETRY

Classical differential geometry:

- an orientable manifold M, smooth functions, $C^\infty(M)$,
- differential algebra $\Omega(M)$, metric $g^{\mu\nu}$, Laplace operator Δ,
- spin^c structure(s), real spin structure, Dirac operator

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Problems are to calculate:

- the eigenvalues of the Dirac operator
- the invariants of the manifolds/structures

THE GEOMETRY ACCORDING TO CONNES

Spectral geometry:

- an algebra $\mathcal{A}$, its representation π on Hilbert space $\mathcal{H}$,
- an unbounded operator D such that $[D, \pi(a)]$ is bounded for any $a \in \mathcal{A}$,
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- regularity axioms
- finiteness axioms
- dimension axioms

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A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36, 6194, (1995)

Theorem [Connes]

If $\mathcal{A} = C^{\infty}(M)$, M a spin Riemannian compact manifold, $\mathcal{H} = L^2(S)$ (sections of spinor bundle) and D the Dirac operator on M then to $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple (with a real structure).

Commutative geometries, which satisfy Connes' axioms are in 1:1 correspondence with Riemannian spin manifolds with a given spin structure and metric.

A. Connes, *On the spectral characterization of manifolds*, J. Noncom. Geom. 7, 1–82 (2013)

Genuine noncommutative real spectral triples

Examples of real spectral geometries

- The Noncommutative Torus: $UV = e^{2\pi i\theta} VU$
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How to construct them?

There is so far no general method. Only examples.

Real structure and Poisson geometry

E. Hawkins, *Noncommutative rigidity*, Commun. Math. Phys. 246, 211–235 (2004)

In fact, with the help of Poisson geometry, it was showed by Eli Hawkins that if a noncommutative real spectral triple is a deformation of the real spectral triple of functions on a 2-dimensional smooth manifold, then the underlying Riemannian manifold can only be either the flat torus or the round sphere.

Commun. Math. Phys. 246, 211–235 (2004)
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Communications in Mathematical Physics

Noncommutative Rigidity

Eli Hawkins

Scuola Internazionale Superiore di Studi Avanzati, Via Beirut 4, 34014 Trieste, Italy.
E-mail: mrmuon@mac.com

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Abstract: Using very weak criteria for what may constitute a noncommutative geometry, I show that a pseudo-Riemannian manifold can only be smoothly deformed into noncommutative geometries if certain geometric obstructions vanish. These obstructions can be expressed as a system of partial differential equations relating the metric and the Poisson structure that describes the noncommutativity. I illustrate this by computing the obstructions for well known examples of noncommutative geometries and quantum groups. These rigid conditions may cast doubt on the idea of noncommutatively deformed space-time.

1. Introduction

One plausible way to try and construct examples of noncommutative geometry is to start with an ordinary, commutative manifold and deform it. One can try to construct noncommutative algebras that in some sense approximate the algebra of smooth functions on the manifold, and then to construct noncommutative geometries which approximate the geometry of the original manifold. There has been considerable success with the first step. Techniques of geometric quantization can be applied in many cases to construct a sequence of algebras which approximate the algebra of functions in a very strong sense. In a much weaker sense, the formal deformation quantization constructions of Fedosov [16] and Kontsevich [21] give noncommutative approximations to any manifold.

Another motive for considering deformations is physical. There are many reasons to suspect that pseudo-Riemannian geometry might not accurately describe the small scale structure of space-time. Noncommutative geometry is a plausible route toward a better description. However, the fact that pseudo-Riemannian geometry *is* a sufficient description of space-time for most purposes, suggests that noncommutativity might be treated as a perturbation.

If so, then this noncommutativity would be described in the leading order by a Poisson structure. Much optimism about this direction was generated by Kontsevich's remarkable

Are there any interesting NC Geometries ?

A softer version of *geometry*?

The facts:

1. for the examples of q-deformed algebras (Podleś spheres, $SU_q(2)$) - there are no spectral geometries in the exact sense – but – there are geometries in which some of the commutation relations are satisfied up to compact operators:

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Remark: Leads to nontrivial classical "triples".

Reality twisted by a linear automorphism

with T.Brzeziński, N.Ciccoli and L.Dąbrowski

Let A be a complex $*$-algebra and let (H, π) be a (left) representation of A on a complex vector space H. A linear automorphism ν of H defines an algebra automorphism

$$\bar{\nu} : \mathrm{End}(H) \to \mathrm{End}(H), \qquad \phi \mapsto \nu \circ \phi \circ \nu^{-1}.$$

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The inverse of $\bar{\nu}$ is $\phi \mapsto \nu^{-1} \circ \phi \circ \nu$. Since $\bar{\nu}$ is an algebra map,

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$$\pi^{\nu} : \ A \xrightarrow{\quad \pi \quad} \mathrm{End}(H) \xrightarrow{\quad \bar{\nu} \quad} \mathrm{End}(H)$$

is an algebra map too, and hence it defines a new representation (H, π^{ν}) of A. The map ν is an isomorphism that intertwines (H, π) with (H, π^{ν}).

We could also require that $\pi^{\nu}(a) \in \pi(A)$ so for faithful π the map $\bar{\nu}$ defines an (algebra) automorphism of A

Twisted reality

Definition (Twisted Real Spectral Triple)

Let A be a $*$-algebra, (H, π) a representation of A, D a linear operator on H, and let ν be a linear automorphism of H. We say that the triple (A, H, D) admits a ν*-twisted real structure* if there exists an anti-linear map $J : H \to H$ such that $J^2 = \epsilon\,\mathrm{id}$, and, for all $a, b \in A$,

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$$\nu J \nu = J,$$

Twisted reality

If (A, H, D) admits a grading operator $\gamma : H \to H$:

$$\gamma^2 = \mathrm{id}, \quad [\gamma, \pi(a)] = 0, \quad \gamma D = -D\gamma, \quad \nu^2\gamma = \gamma\nu^2,$$

then the twisted real structure J is also required to satisfy

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In case of H being a Hilbert space the automorphism ν is also assumed to be densely defined and selfadjoint, with the requirement that $\bar{\nu}$ maps $\pi(A)$ into bounded operators.

The signs $\epsilon, \epsilon', \epsilon''$ determine the *KO*-dimension modulo 8 in the usual way and the operator J is antiunitary.

Twisted real spectral triples

We shall say that a spectral triple admits a ν-twisted real structure, or simply that is a ν*-twisted real spectral triple*.

The commutant condition is called the *order-zero condition* and the one with the Dirac operator is called the twisted order-one condition. We shall call the modified condition the the twisted ϵ'-condition.

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Remark

This is an extension not a replacement. In the case of $\nu = \mathrm{id}$ we get the usual, well known, spectral triples.

On twisting real spectral triples by algebra automorphisms

by Giovanni Landi, Pierre Martinetti

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Abstract

We systematically investigate ways to twist a real spectral triple via an algebra automorphism and in particular, we naturally define a twisted partner for any real graded spectral triple. Among other things we investigate consequences of the twisting on the fluctuations of the metric and possible applications to the spectral approach to the standard model of particle physics.

Contents

1st January 2016

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ON TWISTING REAL SPECTRAL TRIPLES BY ALGEBRA AUTOMORPHISMS

by Giovanni Landi, Pierre Martinetti

Twisted and graded twisted spectral triples were defined in [7] by replacing the boundedness of the commutator $[D, a]$ with the requirement that the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D, \qquad (2.6)$$

for an automorphism $\rho \in \mathrm{Aut}(\mathcal{A})$, be bounded for any $a \in \mathcal{A}$. Furthermore, the automorphism ρ is not taken to be a $*$-automorphism, but rather to satisfy

$$\rho(a^*) = (\rho^{-1}(a))^*. \qquad (2.7)$$

Such an automorphism was named *regular* in [14]. The requirement (2.7) has origin in the additional assumption (coming from considerations in index theory in [7]) that the algebra $\mathcal{A}$ has a 1-parameter group of automorphisms $\{\rho_t\}_{t\in\mathbb{R}}$ and that ρ coincides with the value at $t = \mathrm{i}$ of the analytic extension of $\{\rho_t\}_{t\in\mathbb{R}}$. In typical examples (for instance the spectral triples associated to codimension 1 foliations) the 1-parameter group of automorphisms is the modular automorphism group of a twisted trace. Such twisted traces appear naturally with twisted spectral triples. Indeed, if $(\mathcal{A}, \mathcal{H}, D)$ is a ρ-twisted spectral triple with $D^{-1} \in \mathcal{L}^{n,\infty}$, the Dixmier ideal, from [7, Prop. 3.3] the functional

$$\mathcal{A} \ni a \mapsto \varphi(a) = ⨍ aD^{-n} := \mathrm{Tr}_\omega(aD^{-n}), \qquad (2.8)$$

[1]When possible we omit the representation symbol and identify $a \in \mathcal{A}$ with its representation $\pi(a) \in \mathcal{L}(\mathcal{H})$.

3

with Tr_ω the Dixmier trace, is a ρ^{-n}-trace, that is $\varphi(ab) = \varphi(b\rho^{-n}(a))$ for all $a, b \in \mathcal{A}$.

Now, the algebras $\mathcal{A}$ and $\mathcal{A}^\circ$ have isomorphic automorphism groups. An isomorphism is:

$$\mathrm{Aut}(\mathcal{A}) \ni \rho \to \rho^\circ \in \mathrm{Aut}(\mathcal{A}^\circ), \qquad \rho^\circ(b^\circ) := (\rho^{-1}(b))^\circ, \quad \forall b^\circ \in \mathcal{A}^\circ. \qquad (2.9)$$

The use of ρ^{-1} instead of ρ is to parallel condition (2.7). In a sense, the above means

$$\rho^\circ(Jb^*J^{-1}) = J(\rho^{-1}(b))^*J^{-1} = J\rho(b^*)J^{-1}, \qquad (2.10)$$

and the second equality is due to condition (2.7). We are then led to the following.

Definition 2.1. *A real twisted spectral triple of KO-dimension k is the datum of a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D;\, \rho)$ together with an antilinear isometry operator J satisfying the rule of signs (2.1), the zero-order condition (2.3), and the twisted first-order condition*

$$[[D, a]_\rho, Jb^*J^{-1}]_{\rho^\circ} = 0, \quad \forall a, b \in \mathcal{A}. \qquad (2.11)$$

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$$\Omega^1_D := \{\sum_i \pi(a_i)[D, \pi(b_i)] \,|\, a_i, b_i \in A\}.$$

The standard fluctuaction of a spectral triple (A, H, D) consist of adding to the Dirac operator D a selfadjoint one form $\alpha \in \Omega^1_D$.

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For our case of ν-twisted real spectral triple we set the fluctuated Dirac operator D_α to be:

$$D_\alpha := D + \alpha + \epsilon' \nu J \alpha J^{-1} \nu,$$

with the requirement that $\alpha + \epsilon' \nu J \alpha J^{-1} \nu$ is selfadjoint.

FLUCTUATIONS

PROPOSITION

If (A, H, D) with $J \in \mathrm{End}(H)$ is a ν-twisted real spectral triple, then (A, H, D_α) with (the same) J is also a ν-twisted real spectral triple. If (A, H, D) is even with grading γ, then (A, H, D_α) is even with (the same) grading γ. The composition of twisted fluctuations is a twisted fluctuation.

Fluctuations

Proposition

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Proof

As a perturbation of D by a bounded selfadjoint operator, the fluctuated Dirac operator D_α is selfadjoint, has bounded commutators with $\pi(a) \in A$ and has compact resolvent.
First, we shall demonstrate that a fluctuation of the fluctuated Dirac operator is also a fluctuation. In other words the bimodule of one forms is independent of the choice of α. Let $a \in A$ and $\alpha \in \Omega^1_D$.

Proof (continued)

We compute:

$$\begin{aligned}
[\alpha', \pi(a)] &= \nu J \alpha J^{-1} \nu \pi(a) - \pi(a) \nu J \alpha J^{-1} \nu \\
&= \nu J \alpha J^{-1} \nu \pi(a) - \nu \pi(\bar{\nu}^{-1}(a)) J \alpha J^{-1} \nu \\
&= \nu J \alpha J^{-1} \nu \pi(a) - \nu J \alpha J^{-1} \pi(\bar{\nu}(a)) \nu \\
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Therefore for any $\alpha \in \Omega^1_D$ and $a \in A$ we have:

$$[D_\alpha, \pi(a)] = [D, \pi(a)] + [\alpha, \pi(a)],$$

PROOF (CONTINUED)

To finish the proof it remains only to check that D_α satisfies the compatibility relation with J, that is:

$$D_\alpha J\nu = \epsilon' \nu J D_\alpha.$$

Since D itself satisfies it, we compute it for $\alpha + \epsilon' \alpha'$:

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$$\begin{aligned}(\alpha + \epsilon'\alpha')J\nu &= (\alpha J\nu + \epsilon'\nu J\alpha J^{-1}\nu J\nu \\ &= \alpha J\nu + \epsilon'\nu J\alpha \\ &= \epsilon'\nu J\left(\alpha + \epsilon' J^{-1}\nu^{-1}\alpha J\nu\right) \\ &= \epsilon'\nu J\left(\alpha + \epsilon'\alpha'\right).\end{aligned}$$

Proof (continued)

To finish the proof it remains only to check that D_α satisfies the compatibility relation with J, that is:

$$D_\alpha J\nu = \epsilon' \nu J D_\alpha .$$

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Like in the case of the real spectral triples the twisted one admit a family of fluctuated Dirac operators.

EXAMPLE 1: CONFORMAL PERTURBATIONS

Let us assume that we have a real spectral triple (A, H, D, J) with reality operator J and fixed signs ϵ, ϵ'. Let $k \in \pi(A)$ be a positive and invertible bounded operator such that k^{-1} is also bounded, and let us denote by $k' = JkJ^{-1}$.

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Proposition

If (A, H, D) with J is a real spectral triple, which satisfies order one condition, then for:

$$D_k = k' D k', \qquad \nu(h) = k^{-1} k' \, h,$$

the triple (A, H, D_k) with J is a ν-twisted real spectral triple. If furthermore (A, H, D) is even with grading γ, then (A, H, D_α) is even with (the same) grading γ.

Example 1: conformal perturbations

Proof

Since k and k' are bounded operators it is clear that $\bar{\nu}$ maps bounded operators to bounded operators, and we have for all $a \in A$:

$$\bar{\nu}(a) = k^{-1}ak.$$

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We show now that D_k satisfies the twisted order-one condition :

$$J\pi(b)J^{-1}[D_k, \pi(a)] = J\pi(b)J^{-1}JkJ^{-1}[D, \pi(a)]JkJ^{-1}$$
$$= k'[D, \pi(a)]k'J(k^{-2}\pi(b)k^2)J^{-1} = [D_k, \pi(a)]\, J\pi(\bar{\nu}^2(b))J^{-1}.$$

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Next we check compatibilities between J and ν and D:

$$\nu J \nu = k^{-1} J k J^{-1} J k^{-1} J k J^{-1} = J,$$
$$\nu J D_k = k'(k)^{-1} J k' J^{-1} J D k' = \epsilon' k' D J k' = \epsilon' D_k J \nu,$$

Example 2: twisted derivations

Assume:

A is a $*$ algebra, ν an automorphism, $\nu(a^*) = (\nu^{-1}(a))^*$ and δ is a twisted derivation:

$$\delta(ab) = \delta(a)\nu^2(b) + a\delta(b),$$

EXAMPLE 2: TWISTED DERIVATIONS

ASSUME:

A is a $*$ algebra, ν an automorphism, $\nu(a^*) = (\nu^{-1}(a))^*$ and δ is a twisted derivation:

$$\delta(ab) = \delta(a)\nu^2(b) + a\delta(b),$$

TWISTED ORDER ONE CONDITION:

$$[\delta, a]h = \delta(ah) - a\delta(h) = \delta(a)\nu^2(h),$$

$$Jh = h^*, \qquad b^o h = JbJ^{-1}h = hb^*,$$

$$[\delta, a]b^o\, h = \nu^{-2}(b)^o[\delta, a]\, h.$$

Twisted reality on quantum disc/cones

The coordinate algebra of the *quantum disc* $\mathcal{O}(D_q)$ is a complex $*$-algebra generated by z, subject to the relation

$$z^* z - q^2 z z^* = 1 - q^2,$$

where $q \in (0, 1)$. $\mathcal{O}(D_q)$ can be understood as a $\mathbb{Z}$-graded algebra

$$\mathcal{O}(D_q) = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}(D_q)_n,$$

with the degrees given on the generators by $|z| = -|z^*| = 1$.

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$\mathbb{Z}_N$ action:

$$z \mapsto e^{\frac{2\pi i}{N}} z, \qquad z^* \mapsto e^{-\frac{2\pi i}{N}} z^*.$$

Twisted reality on quantum cones

Using the $\mathbb{Z}$-grading of $\mathcal{O}(D_q)$ we define a degree counting algebra automorphism,

$$\nu : \mathcal{O}(D_q) \to \mathcal{O}(D_q), \qquad a \mapsto q^{|a|} a,$$

for all homogeneous elements of $\mathcal{O}(D_q)$, ν is also compatible with the $*$-structure:

$$\nu \circ * \circ \nu = *.$$

The maps $\partial_-, \partial_+ : \mathcal{O}(D_q) \to \mathcal{O}(D_q)$, defined on generators of the disc algebra by

$$\partial_-(z) = z^*, \quad \partial_-(z^*) = 0, \qquad \partial_+(z) = 0, \quad \partial_+(z^*) = q^2 z,$$

extend to the whole of $\mathcal{O}(C_q^N)$ as ν^2-skew derivations, i.e. by the twisted Leibniz rule,

$$\partial_\pm(ab) = \partial_\pm(a)\nu^2(b) + a\partial_\pm(b),$$

Twisted reality on quantum cones

Set

$$H_+ = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(D_q)_{nN+1}, \quad H_- = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}(D_q)_{nN-1}, \qquad H = H_+ \oplus H_-,$$

$$\pi(a)(h_\pm) = \nu^2(a)h_\pm, \qquad \text{for all } a \in \mathcal{O}(C_q^N),\ h_\pm \in H_\pm,$$

$$\bar{\nu}(\pi(a)) = \pi(\nu(a)), \qquad \text{for all } a \in \mathcal{O}(C_q^N).$$

$$D : H \to H, \qquad (h_+, h_-) \mapsto \left(-q^{-1}\partial_+(h_-),\ q\partial_-(h_+)\right),$$

$$J : H \to H, \qquad (h_+, h_-) \mapsto \left(-h_-^*, h_+^*\right),$$

$$\gamma : H \to H, \qquad (h_+, h_-) \mapsto (h_+, -h_-).$$

Proposition

With these definitions, $(\mathcal{O}(C_q^N), H, D)$ is an (algebraic) even spectral triple of KO-dimension two with ν-twisted real structure J and grading γ.

Some better deformations...

Possibly even related to physics...

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$$[x_0, x_i] = \frac{i}{\kappa} x_i,$$
$$[x_i, x_j] = 0,$$

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THE ALGEBRA

B.Durhuus and AS: J. Noncommut. Geom. 7 (2013), 605–645

If $f, g \in \mathcal{B}$ then f^* and $f * g$ also belong to $\mathcal{B}$ and are given by

$$(f * g)(\alpha, \beta) = \frac{1}{2\pi} \int dv \int d\alpha' \, f(\alpha + \alpha', \beta) g(\alpha, e^{-v}\beta) e^{-i\alpha' v},$$

and

$$f^*(\alpha, \beta) = \frac{1}{2\pi} \int dv \int d\alpha' \, \bar{f}(\alpha + \alpha', e^{-v}\beta) e^{-i\alpha' v}.$$

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The twisted derivations

$$[P, E] = [P, \mathcal{E}] = [E, \mathcal{E}] = 0,$$
$$\Delta E = E \otimes 1 + 1 \otimes E\,, \quad \Delta P = P \otimes 1 + \mathcal{E} \otimes P\,,$$
$$\Delta \mathcal{E} = \mathcal{E} \otimes \mathcal{E}\,,$$

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The action on the algebra

$$(T_\gamma f)(\alpha, \beta) = f(\alpha + i\gamma, \beta),$$

$$E \triangleright f = -i\frac{\partial f}{\partial \alpha}, \quad P \triangleright f = -i\frac{\partial f}{\partial \beta}, \quad \mathcal{E} \triangleright f = T_1 f\,.$$

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The real twisted spectral triple ?

Hint: use Hilbert space representation (etc) as in Marco Matassa construction (Journal of Geometry and Physics 76C (2014), pp. 136-157) THEN the above construction with twisted derivations.

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Is there a spectral triple ?

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THE REAL TWISTED SPECTRAL TRIPLE ?

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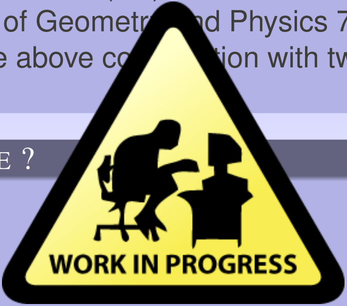

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Conclusions

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- Are further modifications possible ?

THANK YOU