The DFR approach to noncommutative spacetime, from flat to curved

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Introduction

- The DFR model: spacetime and fields
- Friedmann expanding backgrounds
- Friedmann expanding n.c. spacetimes
- Quantum fields and Friedmann equations

The DFR proposal

DFR (1995): “Grav. stability under localization experiments”: Determining the localization of a quantum field theoretic observable needs concentration of energy in a region of the size of the uncertainty; extreme precision should cause the formation of a black hole.

The following program was outlined:

- Derive physically meaningful uncertainty relations between coordinates of spacetime events from gravitational stability under localization experiments.
- Promote these coordinates to the status of operators and find commutation relations among them from which the uncertainty relations follow.
- Construct quantum fields over the resulting noncommutative spacetime.

Starting point: fixed classical background, to be recovered by some $L_P \to 0$ procedure.

Noncommutative Minkowski space (DFR)

Spacetime uncertainty relations (STUR) derived by the linear approximation:

$$c\Delta t\left(\Delta x^1 + \Delta x^2 + \Delta x^3\right) \geq L_P^2$$

$$\Delta x^1 \Delta x^2 + \Delta x^1 \Delta x^3 + \Delta x^2 \Delta x^3 \geq L_P^2$$

From here, commutation relations (in principle, highly non unique!):

$$[x_\mu, x_\nu] = iL_P^2 Q_{\mu\nu}, \qquad x_\mu = x_\mu^*$$

One can show that the STUR are satisfied using the "Quantum conditions"

$$[x_\mu, Q_{\nu\rho}] = 0, \qquad Q^{\mu\nu} Q_{\mu\nu} = 0, \qquad \left(Q^{\mu\nu}(*Q)_{\mu\nu}\right)^2 = 16I.$$

The x_μ generate a C^*-algebra E, (some of) its states are our n.c. Minkowski. Covariance is granted by the following action of the (full) Poincaré group P:

$$\alpha_{(\Lambda,a)}(x_\mu) = \Lambda_\mu^\nu x_\nu + a_\mu I, \qquad \alpha_{(\Lambda,a)}(Q_{\mu\nu}) = \Lambda_\mu^{\mu'} \Lambda_\nu^{\nu'} Q_{\mu'\nu'}.$$

Quantum fields on n.c. Minkowski space

A quantum field Φ on the quantum spacetime is defined by

$$\Phi(x) = \int_{R^4} dk\, e^{ikx} \otimes \hat{\Phi}(k).$$

It is a map from states on E to smeared field operators,

$$\omega \to \Phi(\omega) = \langle \omega \otimes I, \Phi(x) \rangle = \int_{R^4} dx\, \Phi(x)\psi_\omega(x).$$

The r.h.s. is a quantum field on the ordinary spacetime, smeared with ψ_ω defined by $\hat{\psi}_\omega(k) = \langle \omega, e^{ikx} \rangle$. If products of fields are evaluated in a state, the r.h.s. will in general involve non-local expressions. One has

$$[\Phi(\omega), \Phi(\omega')] = i \int d^4x d^4y\, \Delta(x-y)\psi_\omega(x)\psi_{\omega'}(y).$$

Thus (smeared) non commutative quantum fields are functions from a quantum spacetime to a C^*-algebra F (analogue to the one generated by ordinary fields) and are described by elements affiliated to $E \otimes F$. Knowledge of the classical commutator entails knowledge of its n.c. spacetime counterpart one.

Curved spacetimes and n.c. Einstein's equations

Problem: generalise the above construction to curved spacetimes. Big problem: make sense of "n.c. Einstein equations"

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}(\Phi) \qquad F(\Phi) = 0,$$
$$[x_\mu, x_\nu] = iQ_{\mu\nu}(g).$$

Friedmann flat expanding spacetimes with metric (comoving coordinates)

$$ds^2 = dt^2 - a(t)^2(dx_1^2 + dx_2^2 + dx_3^2).$$

Combination of mathematical simplicity (due to symmetry) and physical relevance (cosmological models).

Uncertainty relations for FFE spacetimes (comoving coordinates)

- Black holes do not form if the (positive) excess of proper mass-energy δE inside a two-surface S of proper area ΔA contained in a slice of constant universal time t_0 satisfies the inequality:

$$\sqrt{\Delta A}\left(\frac{1}{4\sqrt{\pi}}+\frac{H\sqrt{\Delta A}}{4\pi c}\right)\geq\frac{G}{c^4}\delta E.$$

where $H(t)=a'(t)/a(t)$ is the Hubble parameter $(a,a'>0)$.

For a box-like localisation region with comoving edges $\Delta x_1^c, \Delta x_2^c, \Delta x_3^c$,

$$\Delta A = a^2(t)(\Delta x_1^c\Delta x_2^c+\Delta x_1^c\Delta x_3^c+\Delta x_2^c\Delta x_3)=a^2(t)\Delta A^c.$$

Estimate δE making use of Heisenberg's uncertainty relations and get

$$a^2(t)\Delta A_c\left(\frac{1}{4\sqrt{3}}+\frac{a'(t)\sqrt{\Delta A_c}}{12c}\right)\geq\frac{\lambda_P^2}{2},$$

$$c\Delta t\cdot\sqrt{\Delta A_c}\min_{t\in\Delta t}\left\{a(t)\left(\frac{1}{4\sqrt{3}}+\frac{a'(t)\sqrt{\Delta A_c}}{12c}\right)\right\}\geq\frac{\lambda_P^2}{2}.$$

Solving the first inequality with respect to the comoving area ΔA_c gives

$$\Delta A_c \geq f\left(a(t), a'(t)\right),$$

with $f_1 = (x_0 - c\sqrt{3}a/a')^2$ and x_0 is the greatest solution of a certain cubic equation from which one has

$$c\Delta t \cdot \sqrt{\Delta A_c} \geq \frac{\lambda_P^2}{2} \max_{t \in \Delta t} \{a(t)\Delta A_c\} \geq \frac{\lambda_P^2}{2} \max_{t \in \Delta t} \{a(t) f\left(a(t), a'(t)\right)\}.$$

The corresponding *quantum uncertainty relations* are:

$$\Delta_\omega A_c \geq \frac{\lambda_P^2}{2} |\omega(f)|,$$

$$c\Delta_\omega t \left(\Delta_\omega x_1 + \Delta_\omega x_2 + \Delta_\omega x_3\right) \geq \frac{\lambda_P^2}{2} |\omega(af)|.$$

Uncertainty relations for FFE spacetimes (more general coordinates)

Consider new coordinates $\tau = g(t), \mathbf{x}' = \mathbf{x}$ with g a diffeomorphism. Then for a large class of g we can write

$$\Delta\tau = \frac{\Delta\tau}{\Delta t}\Delta t \simeq g'(t)\Delta t$$

so that

$$\Delta_\omega A_c \geq \frac{\lambda_P^2}{2}|\omega(f)|, \tag{1}$$

$$c\Delta_\omega\tau\left(\Delta_\omega x_1 + \Delta_\omega x_2 + \Delta_\omega x_3\right) \geq \frac{\lambda_P^2}{2}|\omega(af/g')|. \tag{2}$$

where the function f is the same as before. Thus we have a sound "quantisation" procedure in all these cases. In particular, we can choose $g' = a$ (conformal coordinates) and hence in the following we will speak of "the" function f on right hand side of the STUR.

Building n.c. FFE spacetimes

Definition. A C^*-algebra E of operators with generators x_μ, $\mu = 0, \ldots, 3$ affiliated to it is said to be a concrete covariant realisation of the n.c. spacetime M corresponding to the (classical) spacetime M with global isometry group G if:

1) the relevant STUR are satisfied;
2) there is a (strongly continuous) unitary representation of the global isometry group G under which the operators η transform as their classical counterparts (covariance);
3) there is some reasonable classical limit procedure for $L_P \to 0$ such that the η_μ become in an appropriate sense commutative coordinates on some space containing the manifold M as a factor.
4) For FFE, we should in some suitable sense recover the DFR model in the limit $a \to 1$.

For FFE (De Sitter exluded) $G = SO(3) \ltimes R^3$. By isotropy and homogeneity, we restrict attention to c.r. of the form ($x_0 = t$)

$$[x_\mu, x_\nu] = i\lambda_P^2 Q(t)_{\mu\nu}, \quad [t, Q(t)_{\mu\nu}] = 0, \quad [Q(t)_{\mu\nu}, Q(t)_{\rho\sigma}] = 0.$$

Here we do *not* assume that the $Q(t)_{\mu\nu}$'s are simple functional calculi of t.

The assumptions above and covariance have far reaching consequences. Set $e_i = Q_{0i}$, $m_i = \epsilon_{ikj} Q_{kj}$ (and pass to natural units), then

Proposition. Let the generators $t, \mathbf{x}$ satisfy 1) and 3) and the components of the two-tensor $Q(t)$ be as before. Then the corresponding commutation relations are of the form

$$[t, \mathbf{x}] = i h_1(t) \mathbf{e}, \qquad [\mathbf{x}, \mathbf{x}] = i \mathbf{m}_0 + i \mathbf{m}_\perp(t),$$

with $\mathbf{m}_0' = 0$, $\mathbf{m}_\perp(t) \cdot \mathbf{e} = 0$ and some function h_1. Moreover, the operators $\mathbf{e}, \mathbf{m}_0, \mathbf{m}_\perp(t)$ transform as vectors under the action of the automorphism α_R, $R \in SO(3)$.

The proof is obtained by asking that the Jacobi ientity be satisfied.

Next, in analogy with the minkowskian case we impose the following **Friedmann Quantum Conditions**:

$$\mathbf{e}^2 = I, \quad \mathbf{m}_0^2 = c \cdot I, \quad \mathbf{m}_\perp(t) = h_2(t)\mathbf{m}, \quad \mathbf{m}^2 = I, \quad h_i : \mathrm{sp}(t) \to R,\ h_1, h_2 > 0,$$

and $\mathbf{e}, \mathbf{m}_0, \mathbf{m}$ are all central. Moreover, $\mathbf{m}_0 \cdot \mathbf{m}_\perp = 0$

Proposition. Let $x_\mu, \mathbf{e}, \mathbf{m}_0, \mathbf{m}_\perp$ be as above and the quantum conditions be satisfied. Then, for any state $\omega \in E^*$ in the domain of $t, \mathbf{x}, \mathbf{e}, \mathbf{m}_\perp$, the STUR are satisfied if $h_1(t) = h_2(t) = f(t)$.

Sketch of Proof. The operators $\mathbf{e}, \mathbf{m}_\perp$ being central with joint spectrum Σ, we perform the corresponding central decomposition of E^*. In this way we obtain, for example,

$$\int_\Sigma \sum_{k=1}^{3} \Delta_{\omega_\sigma}(t)\Delta_{\omega_\sigma}(x_i) d\mu_\omega(\sigma) \geq \int_\Sigma \sqrt{\sum_{k=1}^{3} \Delta_{\omega_\sigma}(t)^2 \Delta_{\omega_\sigma}(x_i)^2 d\mu_\omega(\sigma)} \geq$$
$$\geq \frac{1}{2}\int_\Sigma ||\omega_\sigma(h_1(t)\mathbf{e})|| d\mu_\omega(\sigma) = \frac{1}{2}\int_\Sigma ||\mathbf{e}(\sigma)|| \cdot |\omega_\sigma(h_1(t))| d\mu_\omega(\sigma) \geq$$
$$\geq \frac{1}{2}|\int_\Sigma \omega_\sigma(h_1(t)) d\mu_\omega(\sigma)| = \frac{1}{2}|\omega(h_1(t))|.$$

1) It is not difficult to see that $\mathbf{m}_0 \neq 0$ entails that $\Delta_\omega A \geq const. + \frac{1}{2}\omega(h_2(t))$. Choosing $\mathbf{m}_0 = \mathbf{e}$ we then recover the the DFR model in the flat limit. In this case, however, we have a spatial characteristic length independent from time, that is on the expansion of spacetime. Should we consider this an interesting physical consequence of noncommutativity?

2) We can overcome this "problem" by adding a new generator and considering comm. rel. of the form

$$[t, \mathbf{x}] = if(t)\mathbf{e}, \quad [\mathbf{x}, \mathbf{x}] = if(t)(\mathbf{e} - f'(t)\mathbf{m}_\perp S)$$
$$[t, S] = 0, \quad [\mathbf{x}, S] = -i\mathbf{m}_\perp.$$

Defining the new central quantity $\mathbf{n} = \mathbf{e} \times \mathbf{m}_\perp$ (the symbol '$\times$' indicates here the exterior product), they take the form

$$[t, \mathbf{e} \cdot \mathbf{x}] = if(t), \quad [\mathbf{m}_\perp \cdot \mathbf{x}, S] = -iI,$$

all other commutators being zero. Now $\Delta_\omega A \geq \frac{1}{2}\left(\omega(h_2(t)) + \omega(h_2' h_2(t)S)\right)$ but we can restrict ourselves to states such that $\omega(h_2' h_2(t)S) = 0$.

Existence of covariant representations

To start with, we observe that a solution is obtained by taking $H = L^2(R^3)$ (with Lebesgue measure $d\xi_1 d\xi_2 d\xi_3$) and defining

$$t := F(\xi_1), \quad \mathbf{e}\cdot\mathbf{x} := i\frac{\partial}{\partial \xi_1},$$

$$\mathbf{m}\cdot\mathbf{x} := i\frac{\partial}{\partial \xi_2}, \quad y := i\frac{\partial}{\partial \xi_3}, \quad S := \xi_2,$$

where a function of (ξ_1, ξ_2, ξ_3) is understood as the corresponding multiplication operator. The function Φ is given by

$$-\xi = \int_{F(0)}^{F(\xi)} \frac{ds}{f(s)} = \tilde{G}\left(F(\xi)\right) \;\rightarrow\; F(\xi) = \tilde{G}^{-1}(-\xi),$$

because $\tilde{G}$ is invertible as a function of Φ (from strict positivity of the function f). Notice that $\mathrm{Ran}(F) = \mathrm{sp}(t) = (a, +\infty)$. We will see that this forbids realisations in terms of selfadjoint operators.

Given a realization through operators $t, \mathbf{x}, S, \mathbf{e}, \mathbf{m}$, and $(\mathbf{a}, R) \in R^3 \rtimes SO(3) =: G$, a new realization on the same Hilbert space is obtained by

$$t^{(\mathbf{a},R)} := t, \quad \mathbf{x}^{(\mathbf{a},R)} := R\mathbf{x} + \mathbf{a}I. \quad S^{(\mathbf{a},R)} := S,$$
$$\mathbf{e}^{(\mathbf{a},R)} := R\mathbf{e} + \mathbf{a}I, \quad \mathbf{m}^{(\mathbf{a},R)} := R\mathbf{m} + \mathbf{a}I.$$

Therefore, we will say that a realization is $R^3 \rtimes SO(3)$-covariant if there is a unitary strongly continuous representation U of $R^3 \rtimes SO(3)$ on H such that

$$U(\mathbf{a}, R)XU(\mathbf{a}, R)^* = X^{(\mathbf{a},R)},$$

for $X = \{t, \mathbf{x}, S, \mathbf{e}, \mathbf{m}\}$. A covariant realisation on the Hilbert space given by $\int_G^{\oplus} L^2(R^2)\, d\mathbf{a}dR \cong L^2(R^2) \otimes L^2(G)$ is then obtained by a direct integral construction:

$$X := \int_G^{\oplus} X^{(\mathbf{a},R)}\, d\mathbf{a}dR,$$

where $X = \{t, \mathbf{x}, S, \mathbf{e}, \mathbf{m}\}$ and $U(\mathbf{a}, R) = I \otimes \lambda(\mathbf{a}, R)$, with λ the left regular representation of G.

The C^*-algebra of the model

- According to noncommutative geometry, C^*-algebras describe *topological* noncommutative spaces, but the topological space underlying the class of spacetimes under consideration is always R^4.

However, the coordinate change $t \to t' = \int^{t'} f^{-1}$ allows us to write

$$[t', \mathbf{e} \cdot \mathbf{x}] = iI, \quad [\mathbf{m} \cdot \mathbf{x}, S] = -iI.$$

Since $\mathrm{sp}(t')$ will be bounded from below in most interesting physical situations, Schroedinger realisations in terms of selfadjoint operators are ruled out. This leads us to the Heisenberg **semigroup**, as opposed to the group He.

Proposition. The C^*-algebra of Friedmann noncomm. spacetime is the groupoid C^*-algebra $C^*(\mathfrak{H})$, where $\mathfrak{H}$ is the groupoid

$$\mathfrak{H} := \{(-\infty, +\infty] \times He \mid [0, +\infty] = \{(x, a, b, c) \in [0, \infty] \times He \, : \, x - b \in [0, \infty]\},$$

with sets of composable elements $\mathfrak{H}^2$ and of units $\mathfrak{H}^0$, product, inverse, domain and range maps d and r given by

$$\mathfrak{H}^2 := \{((x, a, b, c), (x', a', b', c')) \in \mathfrak{H} \times \mathfrak{H} \, : \, x' = x - b\},$$
$$(x, a, b, c)(x - b, a', b', c') := (x, a + a', b + b', c + c' + ab'),$$
$$(x, a, b, c)^{-1} := (x - b, -a, -b, ab - c), \qquad \mathfrak{H}^0 := [0, \infty] \times \{(0, 0, 0)\},$$
$$d(x, a, b, c) := (x, 0, 0, 0), \qquad r(x, a, b, c) := (x - b, 0, 0, 0).$$

Quantum fields and n.c. Friedmann equations

We do not really know how to make sense of "n.c. Einstein equations", but for homogeneous isotropic spacetimes these reduce (with respect to the universal time t and $T_{\mu\mu} = (\rho, P, P, P)$) to

$$H = 8\pi\rho, \qquad 3(H' + H^2) = -4\pi(\rho + 3P)$$

Conservation of energy implies we can solve for $-R = 8\pi T$ and consider

$$H = 8\pi I \otimes \Omega(T_{00})$$

where Ω is a suitable (adiabatic) state. But now the energy-momentum tensor explicitly depends on $a(t)$ through the commutation relations. Adding the equation

$$[x_\mu, x_\nu] = iL_P^2 Q_{\mu\nu}(a(t)),$$

we obtain a closed system of equations to solve for a. But how to define a quantum field?

Problem: we cannot use the naive definition for Minkowski case.
Reason: lack of time-translation invariance $\to$ no natural time Fourier transform.
The preceding discussion leads us to the following prescription: consider the commutative version of the $x(\xi)_\mu$ as a coordinate transformation and define $\hat{\Phi}$ as the (ordinary, commutative) Fourier transform of the ordinary field Φ **with respect to the ξ's** and take

$$\Phi(t, \mathbf{x}) = \int_{R^4} d^4k \, e^{ikq} \otimes \hat{\Phi}(k).$$

where the **q** are the DFR operators. This gives of course no creation/destruction operators but a bona fide object affiliated to $E \otimes F$.

Example: The two-point function of the free, scalar, conformally coupled field in suitable pure, homogeneous, quasi-free states is ($\bar{S}_k, S_k$ are known functions)

$$\Omega((t, \mathbf{x}); (t', \mathbf{x}')) = \frac{1}{8\pi^3} \int_{R^3} \frac{\bar{S}_{|\mathbf{k}|}(t)}{a(t)} \frac{S_{|\mathbf{k}|}(t')}{a(t')} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} d\mathbf{k}$$

We thus define (the q's are operators on the left hand side of the tensor product)

$$\Phi(t, \boldsymbol{x}) = \frac{1}{4\pi^2} \int_{R^4} d^4k e^{ikq} \otimes \left[\int_{R^4} d^4\xi' e^{-ik\xi'} \left(\int_{R^3} d^3\boldsymbol{k}' a(\boldsymbol{k}') \frac{S_{|\boldsymbol{k}'|}(t(\xi'))}{a(t(\xi'))} e^{-i\boldsymbol{k}'\cdot\boldsymbol{x}(\xi')} + \text{h.c.} \right) \right]$$

Remark. Outrageously preliminary calculations in a suitably chosen KMS state (we switch back to universal time) give $a(t) \simeq e^{ct}$ for small times!!

Short bibliography

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