Polyakov loop fluctuations in terms of Dirac eigenmodes

Speaker: Takahiro Doi (Kyoto University)

in collaboration with
Krzysztof Redlich (Wroclaw University & EMMI & Duke Univ.)
Chihiro Sasaki (Wroclaw University)
Hideo Suganuma (Kyoto University)

This talk is continued from the talk of Hideo Suganuma (09:30 – 10:00 today)

references

"Polyakov loop fluctuations in Dirac eigenmode expansion,"
TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

See also:
"Relation between Confinement and Chiral Symmetry Breaking in Temporally Odd-number Lattice QCD,"
TMD, H. Suganuma, T. Iritani, Phys. Rev. D90, 094505 (2014).

"Analytical relation between confinement and chiral symmetry breaking
in terms of the Polyakov loop and Dirac eigenmodes,"
H. Suganuma, TMD, T. Iritani, Prog. Theor. Exp. Phys. 2016, 013B06 (2016).

CPOD 2016, June 3, 2016, Wroclaw, Poland

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- Introduction
  - Quark confinement, Polyakov loop and its fluctuations
  - Chiral symmetry breaking and Dirac eigenmode
  - Deconfinement transition and chiral restoration at finite temperature

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  Analytical part

  - Dirac spectrum representation of the Polyakov loop

    already presented in the talk of Hideo Suganuma (09:30 – 10:00 today)
  - Dirac spectrum representation of the Polyakov loop fluctuations

  Numerical part

  - Numerical analysis for each Dirac-mode contribution
    to the Polyakov loop fluctuations

- Recent progress
  - The relation between Polyakov loop and overlap-Dirac modes

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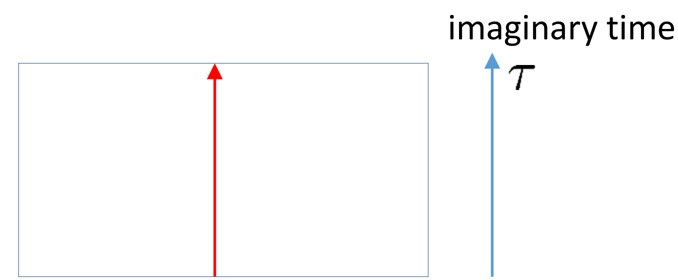
Introduction – Quark confinement

Confinement : colored state cannot be observed (quark, gluon, ・・・)
only color-singlet states can be observed (meson, baryon, ・・・)

Polyakov loop : order parameter for quark deconfinement phase transition

$$L_P(\boldsymbol{x}) = \mathrm{tr}\mathrm{T}e^{ig\int_0^\beta d\tau A_4(\boldsymbol{x},\tau)}$$ in continuum theory

$$= \mathrm{tr}\prod_{s_4=1}^{N_\tau} U_4(\boldsymbol{s}, s_4)$$ in lattice theory



Finite temperature :
(anti) periodic boundary condition for time direction

$$\langle L_P\rangle = \frac{1}{V}\sum_{\boldsymbol{x}}\langle L_P(\boldsymbol{x})\rangle$$: Polyakov loop

$$= e^{-\beta F_q}\begin{cases} = 0 & (F_q = \infty,\ \mathrm{confinement\ phase}) \\ \neq 0 & (F_q : \mathrm{finite},\ \mathrm{deconfinement\ phase}) \end{cases}$$

F_q :free energy of the system with a single static quark

Polyakov loop fluctuations

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$
- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$

- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

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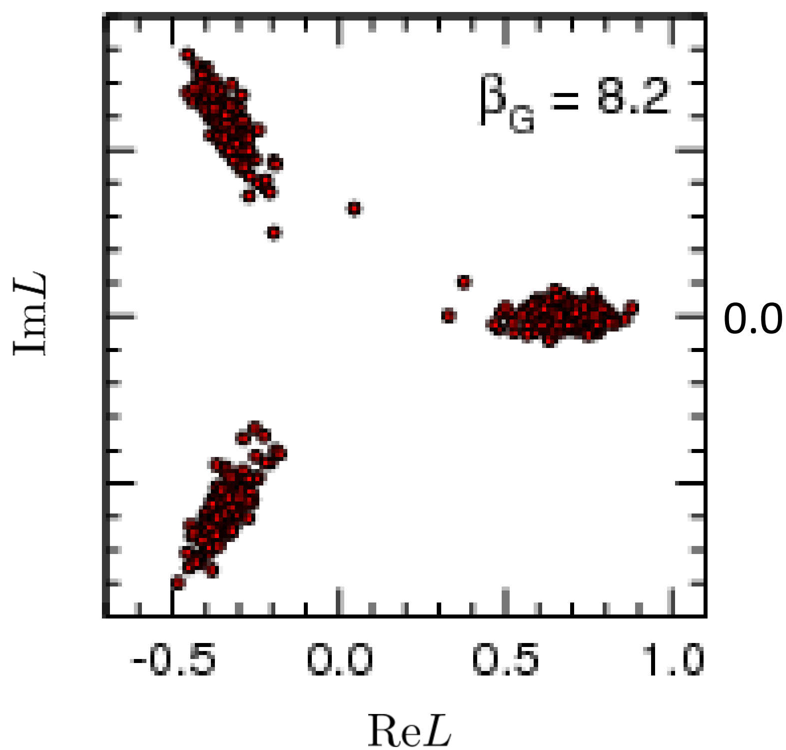
- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

T : temperature

$N_\sigma, \; N_\tau$: spatial and temporal lattice size

Scattered plot of the Polyakov loops



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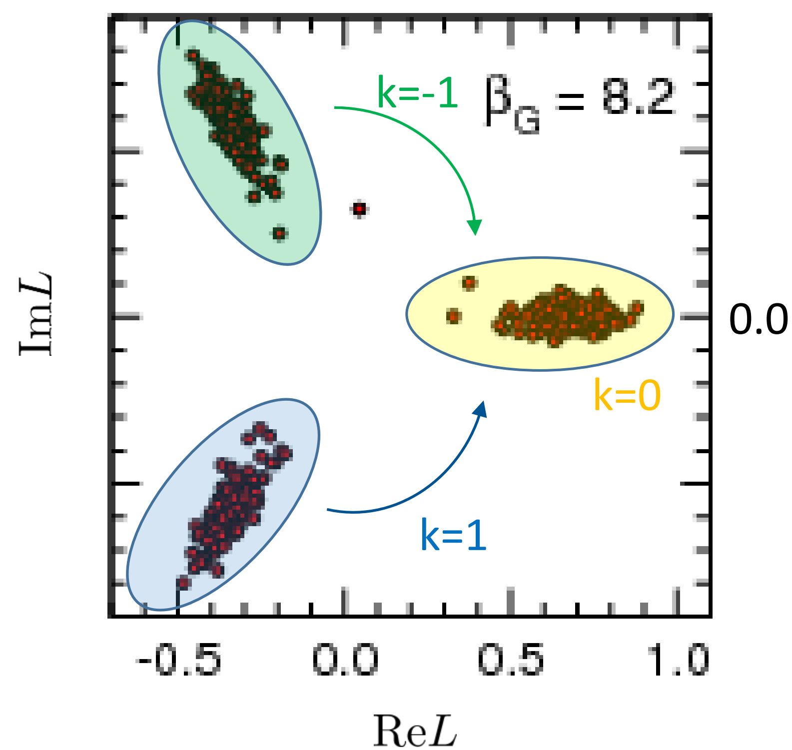

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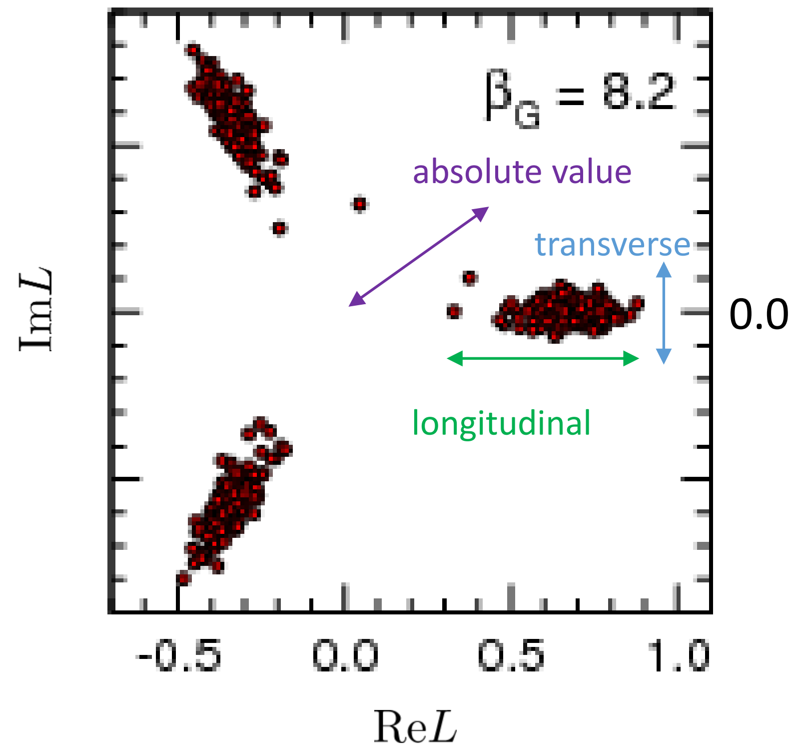

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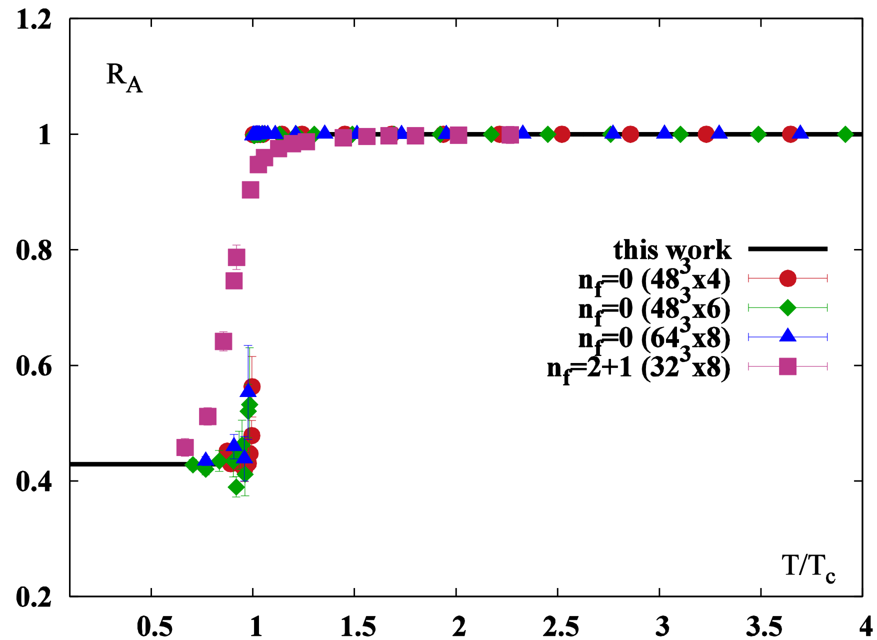
In particular,
R_A is a sensitive probe
for deconfinement transition



※nf=0: quenched level
nf=2+1: (2+1)flavor full QCD
(near physical point)

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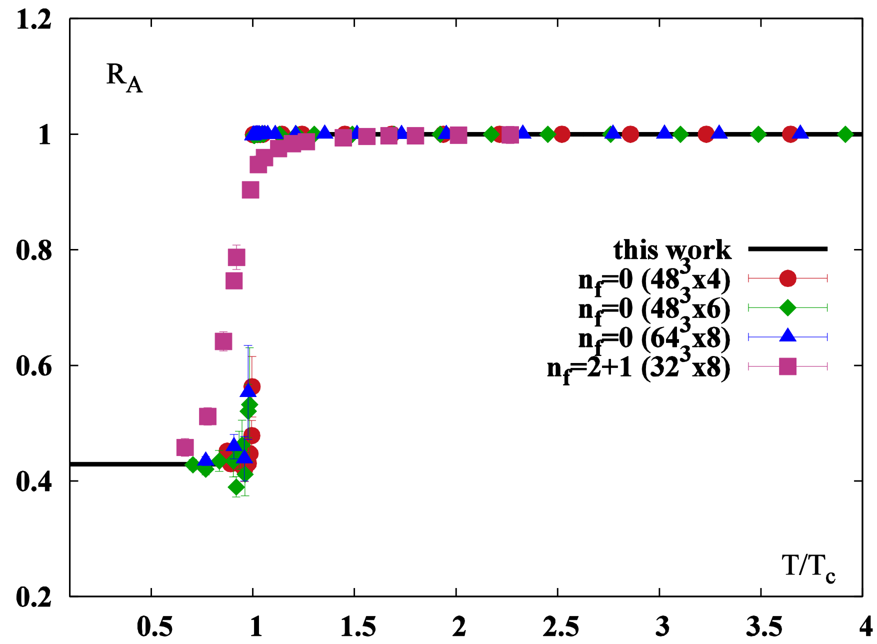


R_A is a good probe for deconfinement transition even if considering light dynamical quarks.

Introduction – Chiral Symmetry Breaking

- Chiral symmetry breaking : chiral symmetry is spontaneously broken

$$\boxed{\mathrm{SU(N)_L} \times \mathrm{SU(N)_R} \underset{\text{CSB}}{\rightarrow} \mathrm{SU(N)_V}}$$

for example $\mathrm{SU(2)}$

- u, d quarks get dynamical mass(constituent mass)
- Pions appear as NG bosons

- Chiral condensate : order parameter for chiral phase transition

$$\langle \bar{q}q \rangle \begin{cases} \neq 0 & \text{(chiral broken phase)} \\ = 0 & \text{(chiral restored phase)} \end{cases}$$

- Banks-Casher relation

$$\langle \bar{q}q \rangle = - \lim_{m \to 0} \lim_{V \to \infty} \pi \langle \rho(0) \rangle$$

$\hat{\not{D}}$:Dirac operator

$\hat{\not{D}} |n\rangle = i\lambda_n |n\rangle$:Dirac eigenvalue equation

$\rho(\lambda) = \frac{1}{V} \sum_n \delta(\lambda - \lambda_n)$:Dirac eigenvalue density

The most important point:
the low-lying Dirac modes (with small $|\lambda_n|$) are essential for chiral symmetry breaking.

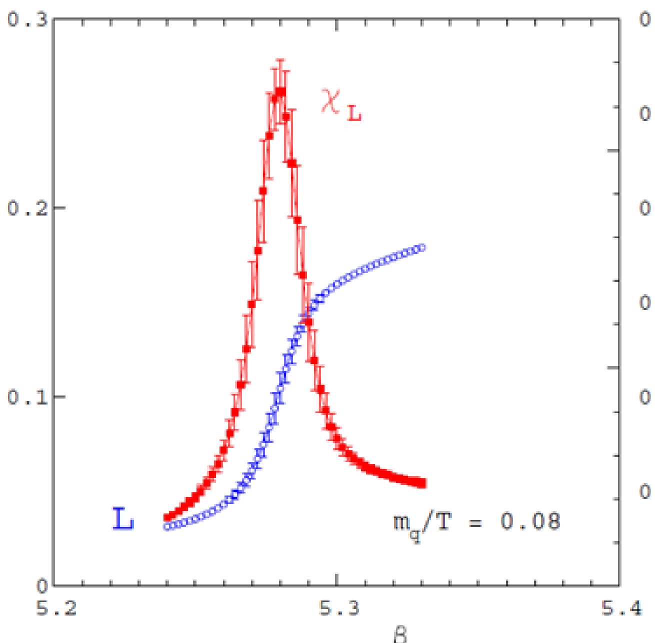
QCD phase transition at finite temperature

F. Karsch, Lect. Notes Phys. 583, 209 (2002)

$\langle L \rangle, \chi_L$: Polyakov loop and its susceptibility

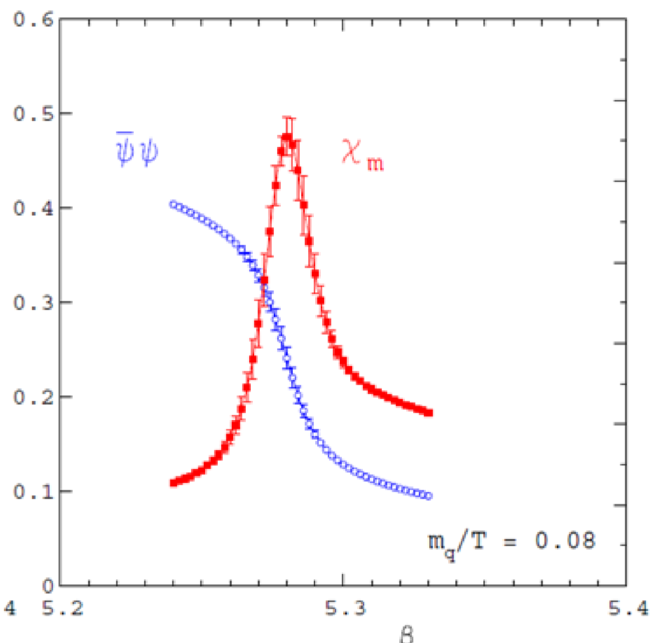
$\langle \bar{\psi}\psi \rangle, \chi_m$: chiral condensate and its susceptibility

deconfinement transition



Low T ↔ High T

chiral transition



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- $\mu = 0$
- two flavor QCD with light quarks

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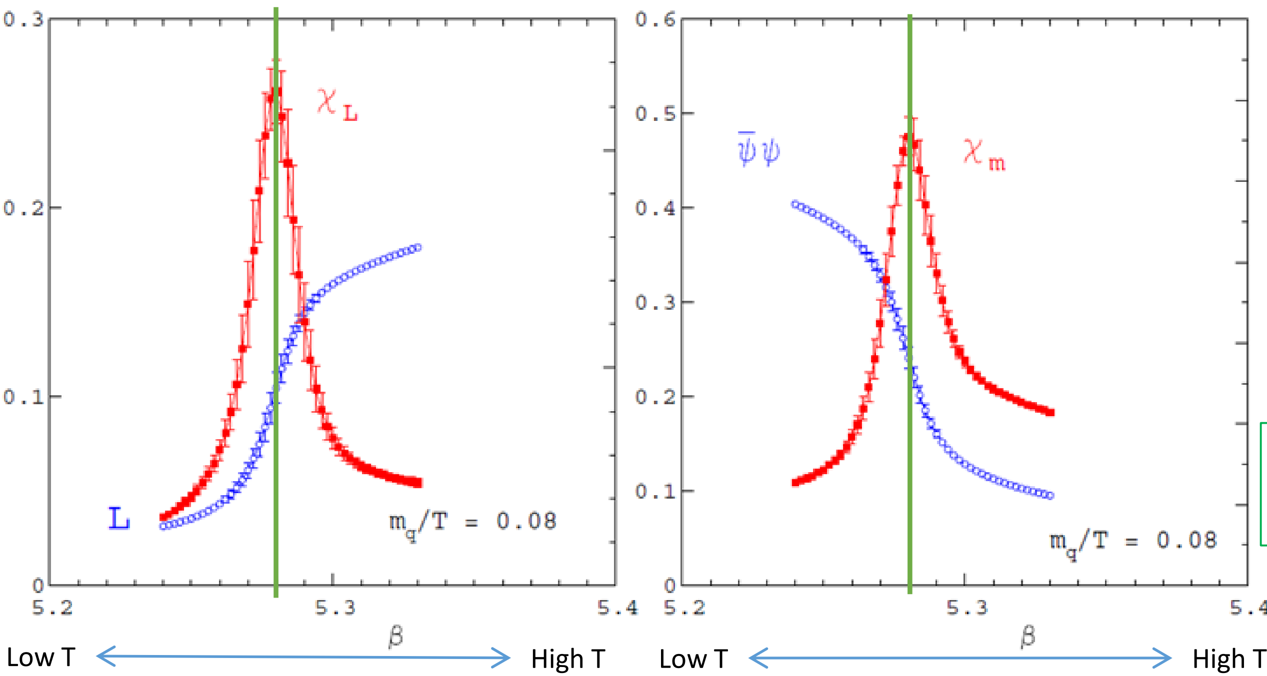

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We define critical temperature as the peak of susceptibility

$\Longrightarrow$ These two phenomena are strongly correlated(?)

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Our strategy

Our strategy to study relation between confinement and chiral symmetry breaking :

anatomy of Polyakov loop in terms of Dirac mode

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Polyakov loop L_P : an order parameter of **deconfinement** transition.

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anatomy of Polyakov loop in terms of Dirac mode

Polyakov loop L_P : an order parameter of **deconfinement** transition.

Dirac eigenmode: **low-lying Dirac modes** (with small eigenvalue $|\lambda_n| \sim 0$)

$\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$

are essential modes for **chiral symmetry breaking**.

(recall Banks-Casher relation: $\langle \bar{q}q \rangle = -\lim_{m\to 0}\lim_{V\to\infty} \pi\langle\rho(0)\rangle$)

Our strategy

Our strategy to study relation between confinement and chiral symmetry breaking :

anatomy of $\boxed{\text{Polyakov loop}}$ in terms of $\boxed{\text{Dirac mode}}$

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General discussion:

In anatomy of the Polyakov loop,
if the contribution to the Polyakov loop from the low-lying Dirac modes is very small,
we can state that the important modes for chiral symmetry breaking
are not important for confinement.

Our strategy

Our strategy to study relation between confinement and chiral symmetry breaking :

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General discussion:

In anatomy of the Polyakov loop,
if the contribution to the Polyakov loop from the low-lying Dirac modes is very small,
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We can analytically show that this situation is valid.

~next page

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

TMD, H. Suganuma, T. Iritani, Phys. Rev. D 90, 094505 (2014).
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$$L = -\frac{(2ai)^{N_\tau - 1}}{12V} \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle$$

on temporally odd number lattice: N_τ is odd

notation:

- Polyakov loop : L
- link variable operator : $\langle s | \hat{U}_\mu | s' \rangle = U_\mu(s) \delta_{s+\hat{\mu}, s'}$

  with anti p.b.c. for time direction: $\langle N_\tau, \mathbf{x} | \hat{U}_4 | 1, \mathbf{x} \rangle = -U_4(N_\tau, \mathbf{x})$
- Dirac eigenmode : $\hat{\not{D}} | n \rangle = i\lambda_n | n \rangle$

  Dirac operator : $\hat{\not{D}} = \frac{1}{2a} \sum_\mu \gamma_\mu (\hat{U}_\mu - \hat{U}_{-\mu})$ $\qquad \sum_n | n \rangle \langle n | = 1$
- The Dirac-matrix element of the link variable operator: $\langle n | \hat{U}_4 | n \rangle$

  $$|\langle n | \hat{U}_4 | n \rangle| < 1$$

properties :

- This formula is valid in full QCD and at the quenched level.
- This formula exactly holds for each gauge-configuration {U} and for arbitrary fermionic kernel K[U]

$$Z = \int \mathcal{D}\bar{q} \mathcal{D}q \mathcal{D}U \mathrm{e}^{-S_{\mathrm{G}}[U] + \bar{q}K[U]q} = \int \mathcal{D}U \mathrm{e}^{-S_{\mathrm{G}}[U]} \mathrm{det} K[U]$$

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

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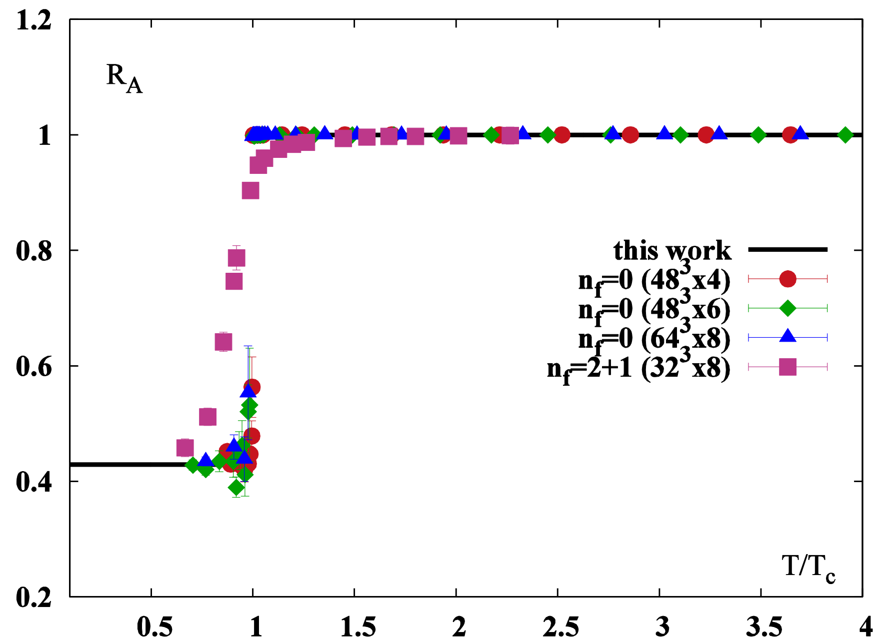

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Polyakov loop : L

Dirac eigenmode : $\hat{\not{D}} |n\rangle = i\lambda_n |n\rangle$

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$$\left\{ \begin{array}{ll} \text{Polyakov loop :} & L \\ \text{Dirac eigenmode :} & \hat{\not{D}} |n\rangle = i\lambda_n |n\rangle \\ \text{link variable operator :} & \langle s | \hat{U}_\mu | s' \rangle = U_\mu(s) \delta_{s+\hat{\mu}, s'} \\ & \langle N_\tau, \mathbf{x} | \hat{U}_4 | 1, \mathbf{x} \rangle = -U_4(N_\tau, \mathbf{x}) \end{array} \right\}$$

combine

Dirac spectrum representation of the Polyakov loop fluctuations

For example,

$$L_L = -\frac{(2ai)^{N_\tau - 1}}{12V} \sum_n \lambda_n^{N_\tau - 1} \mathrm{Re}\left(e^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle \right)$$

and...

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

In particular, the ratio R_A can be represented using Dirac modes:

$$R_A = \frac{\left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle \right|^2 \right\rangle - \left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle \right| \right\rangle^2}{\left\langle \left(\sum_n \lambda_n^{N_\tau - 1} \mathrm{Re} \left(\mathrm{e}^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle \right) \right)^2 \right\rangle - \left\langle \sum_n \lambda_n^{N_\tau - 1} \mathrm{Re} \left(\mathrm{e}^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle \right) \right\rangle^2}$$

Note 1: The ratio R_A is a good “order parameter” for deconfinement transition.

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

In particular, the ratio R_A can be represented using Dirac modes:

$$R_A = \frac{\left\langle \left| \sum_n \boxed{\lambda_n^{N_\tau - 1}} \langle n|\hat{U}_4|n\rangle \right|^2 \right\rangle - \left\langle \left| \sum_n \boxed{\lambda_n^{N_\tau - 1}} \langle n|\hat{U}_4|n\rangle \right| \right\rangle^2}{\left\langle \left(\sum_n \boxed{\lambda_n^{N_\tau - 1}} \mathrm{Re}\left(\mathrm{e}^{2\pi k i/3} \langle n|\hat{U}_4|n\rangle \right) \right)^2 \right\rangle - \left\langle \sum_n \boxed{\lambda_n^{N_\tau - 1}} \mathrm{Re}\left(\mathrm{e}^{2\pi k i/3} \langle n|\hat{U}_4|n\rangle \right) \right\rangle^2}$$

Note 1: The ratio R_A is a good "order parameter" for deconfinement transition.

Note 2: Since the damping factor $\boxed{\lambda_n^{N_\tau - 1}}$ is very small with small $|\lambda_n| \simeq 0$, low-lying Dirac modes (with small $|\lambda_n| \simeq 0$) are not important for R_A, while these modes are important modes for chiral symmetry breaking.

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⇨ Thus, the essential modes for chiral symmetry breaking in QCD are not important to quantify the Polyakov loop fluctuations, which are sensitive observables to confinement properties in QCD.

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Note 2: Since the damping factor $\boxed{\lambda_n^{N_\tau - 1}}$ is very small with small $|\lambda_n| \simeq 0$, low-lying Dirac modes (with small $|\lambda_n| \simeq 0$) are not important for R_A, while these modes are important modes for chiral symmetry breaking.

Thus, the essential modes for chiral symmetry breaking in QCD are not important to quantify the Polyakov loop fluctuations, which are sensitive observables to confinement properties in QCD.

This result suggests that there is no direct, one-to-one correspondence between confinement and chiral symmetry breaking in QCD.

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  - Chiral symmetry breaking and Dirac eigenmode
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  Analytical part

  - Dirac spectrum representation of the Polyakov loop

    already presented in the talk of Hideo Suganuma (09:30 – 10:00 today)
  - Dirac spectrum representation of the Polyakov loop fluctuations

  Numerical part

  - Numerical analysis for each Dirac-mode contribution to the Polyakov loop fluctuations
- Recent progress
  - The relation between Polyakov loop and overlap-Dirac modes

Dirac spectrum representation of the Polyakov loop fluctuations

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

$$L = -\frac{(2ai)^{N_\tau - 1}}{12V} \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle$$

Definition of the Polyakov loop fluctuations

- Polyakov loop: $L \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$
- Z3 rotated Polyakov loop: $\tilde{L} = L e^{2\pi k i/3}$
- longitudinal Polyakov loop: $L_L \equiv \mathrm{Re}(\tilde{L})$
- Transverse Polyakov loop: $L_T \equiv \mathrm{Im}(\tilde{L})$
- Polyakov loop susceptibilities:

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} [\langle |L|^2 \rangle - \langle |L| \rangle^2],$$

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2],$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2]$$

- Ratios of Polyakov loop susceptibilities:

$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

Λ-dependent Polyakov loop fluctuations

Infrared cutoff of the Dirac eigenvalue: Λ

- Λ-dependent (IR-cut) Z3 rotated Polyakov loop:

$$\tilde{L}_\Lambda = -\frac{(2ai)^{N_\tau - 1}}{12V} \sum_{|\lambda_n| > \Lambda} \lambda_n^{N_\tau - 1} e^{2\pi k i/3} \langle n | \hat{U}_4 | n \rangle$$

- Λ-dependent Polyakov loop susceptibilities:

$$(\chi)_\Lambda = \frac{1}{T^3} \frac{N_\sigma^3}{N_\tau^3} [\langle Y_\Lambda^2 \rangle - \langle Y_\Lambda \rangle^2],$$

$$Y \equiv |L|, \; L_L, \; L_T$$

- Λ-dependent ratios of susceptibilities:

$$(R_A)_\Lambda = \frac{(\chi_A)_\Lambda}{(\chi_L)_\Lambda}$$

Introduction of the Infrared cutoff for Dirac modes

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

Define Λ-dependent (IR-cut) susceptibilities:

$$(\chi)_\Lambda = \frac{1}{T^3}\frac{N_\sigma^3}{N_\tau^3}[\langle Y_\Lambda^2\rangle - \langle Y_\Lambda\rangle^2], \quad Y \equiv |L|,\ L_L,\ L_T$$

where, for example, $(L_L)_\Lambda = C_\tau \sum_{|\lambda_n|>\Lambda} \lambda_n^{N_\tau - 1} \mathrm{Re}\left(\mathrm{e}^{2\pi k i/3}(n|\hat{U}_4|n)\right)$

Define Λ-dependent (IR-cut) ratio of susceptibilities:

$$(R_A)_\Lambda = \frac{(\chi_A)_\Lambda}{(\chi_L)_\Lambda}$$

Define Λ-dependent (IR-cut) chiral condensate:

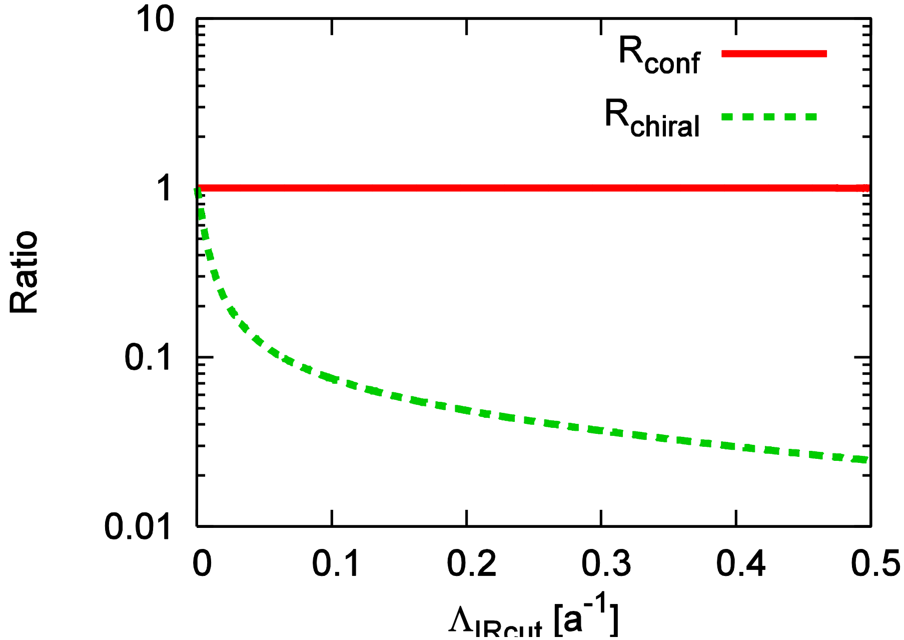
$$\langle\bar{\psi}\psi\rangle_\Lambda = -\frac{1}{V}\sum_{|\lambda_n|\geq\Lambda}\frac{2m}{\lambda_n^2 + m^2}$$

Define the ratios, which indicate the influence of removing the low-lying Dirac modes:

$$R_{\mathrm{conf}} = \frac{(R_A)_\Lambda}{R_A}, \qquad R_{\mathrm{chiral}} = \frac{\langle\bar{\psi}\psi\rangle_\Lambda}{\langle\bar{\psi}\psi\rangle}$$

Numerical analysis

$$R_{\rm conf} = \frac{(R_A)_\Lambda}{R_A}, \qquad R_{\rm chiral} = \frac{\langle\bar\psi\psi\rangle_\Lambda}{\langle\bar\psi\psi\rangle}$$



lattice setup:

- quenched SU(3) lattice QCD
- standard plaquette action
- gauge coupling: $\beta = \frac{2N_c}{g^2} = 5.6$
- lattice size: $N_\sigma^3 \times N_\tau = 10^3 \times 5$

  ⇔ lattice spacing : $a \simeq 0.25$ fm
- periodic boundary condition
  for link-variables and Dirac operator

- $R_{\rm chiral}$ is strongly reduced by removing the low-lying Dirac modes.
- $R_{\rm conf}$ is almost unchanged.

It is also numerically confirmed that
low-lying Dirac modes are important for chiral symmetry breaking
and not important for quark confinement.

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Fermion-doubling problem and chiral symmetry on the lattice

Naive Dirac operator (So far)

$$\hat{\not{D}} = \frac{1}{2a} \sum_{\mu} \gamma_\mu (\hat{U}_\mu - \hat{U}_{-\mu})$$

- include fermion doubler
- exact chiral symmetry

Wilson-Dirac operator (Example to avoid the fermion-doubling problem)

$$D_{\mathrm{W}} = D_\mu \gamma_\mu - \frac{ar}{2} D_\mu D_\mu$$

- free from fermion-doubling problem
- explicit breaking of chiral symmetry e.g.) Nielsen-Ninomiya (1981)

Overlap-Dirac operator H. Neuberger (1998)

- free from fermion-doubling problem
- exact chiral symmetry on the lattice

Recent progress: Overlap-Dirac operator

in preparation

Overlap-Dirac operator H. Neuberger (1998)

$$D_{\mathrm{ov}} = 1 + \frac{D_{\mathrm{W}}(-M_0)}{\sqrt{D_{\mathrm{W}}(-M_0)^\dagger D_{\mathrm{W}}(-M_0)}} \quad \text{(lattice unit)}$$

- Wilson-Dirac operator with negative mass ($-M_0$):

$$D_{\mathrm{W}}(-M_0) = D_\mu \gamma_\mu - \frac{ar}{2} D_\mu D_\mu - M_0$$

- The overlap-Dirac operator satisfies the Ginsparg-Wilson relation:

$$\{\gamma_5, D\} = D\gamma_5 D$$

⇒ the overlap-Dirac operator solves the fermion doubling problem with the exact chiral symmetry on the lattice.

- The overlap-Dirac operator is non-local operator.

⇒ It is very difficult to directly derive the analytical relation between the Polyakov loop and the overlap-Dirac modes.

⇒ However, we can discuss the relation between the Polyakov loop and the overlap-Dirac modes. (next page~)

(The domain-wall-fermion formalism is also free from the doubling problem with the exact chiral symmetry on the lattice. We made discussion within the formalism and it was shown by the previous Hideo Suganuma's talk, in preparation)

Polyakov loop and overlap-Dirac modes

Strategy

We discuss the relation btw the Polyakov loop and overlap-Dirac modes
by showing the following facts:

① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.
② The low-lying eigenmodes of overlap-Dirac and Wilson-Dirac operators correspond.
③ The low-lying Wilson-Dirac modes have negligible contribution to the Polyakov loop.

$$D_{\rm ov}|n\rangle = \Lambda_n^{\rm ov}|n\rangle \qquad D_{\rm W}|n\rangle = \Lambda_n^{\rm W}|n\rangle$$

Low-lying overlap-Dirac modes $|\Lambda_n^{\rm ov}| \sim 0$

② correspond

Low-lying Wilson-Dirac modes $|\Lambda_n^{\rm W}| \sim 0$

①

Essential modes for chiral symmetry breaking

③

Negligible contribution to the Polyakov loop

Polyakov loop and overlap-Dirac modes

① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.
② The low-lying eigenmodes of Wilson-Dirac and overlap-Dirac operators correspond.
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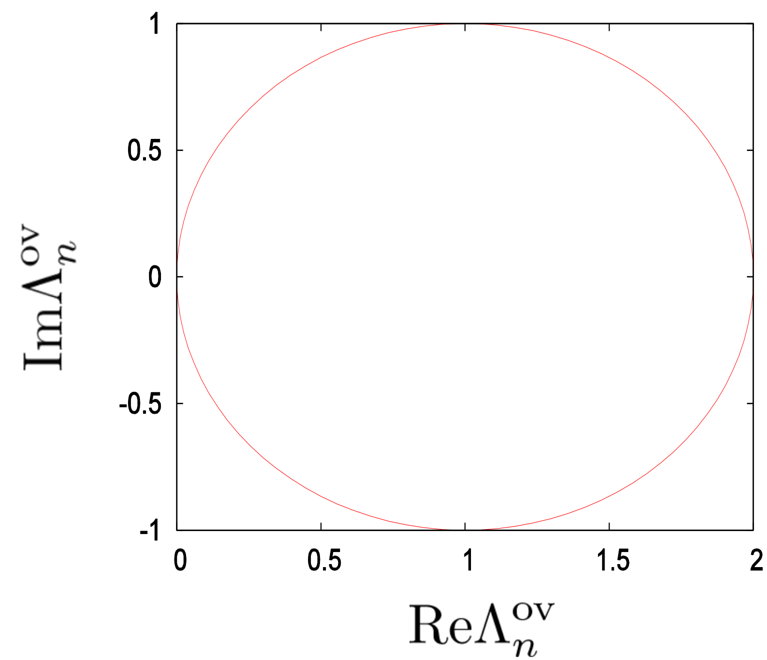
① The low-lying overlap-Dirac modes are essential modes for chiral symmetry breaking.

The chiral condensate in terms the overlap-Dirac eigenvalues:

$$\langle \bar{q}q \rangle \sim \sum_n \frac{1}{\Lambda_n^{\rm ov} + m}$$

$$D_{\rm ov}|n\rangle = \Lambda_n^{\rm ov}|n\rangle$$



The low-lying overlap-Dirac modes $(|\Lambda_n^{\rm ov}| \sim 0)$ have the dominant contribution to the chiral condensate $\langle \bar{q}q \rangle$.

(Recall the Banks-Casher relation)

$$\langle \bar{q}q \rangle = -\lim_{m\to 0}\lim_{V\to\infty} \pi\langle \rho(0) \rangle$$

Polyakov loop and overlap-Dirac modes

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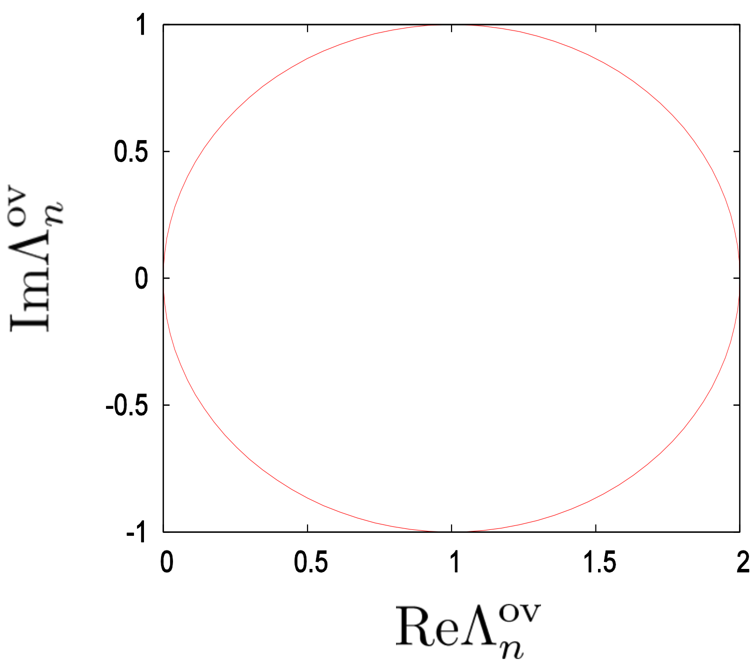
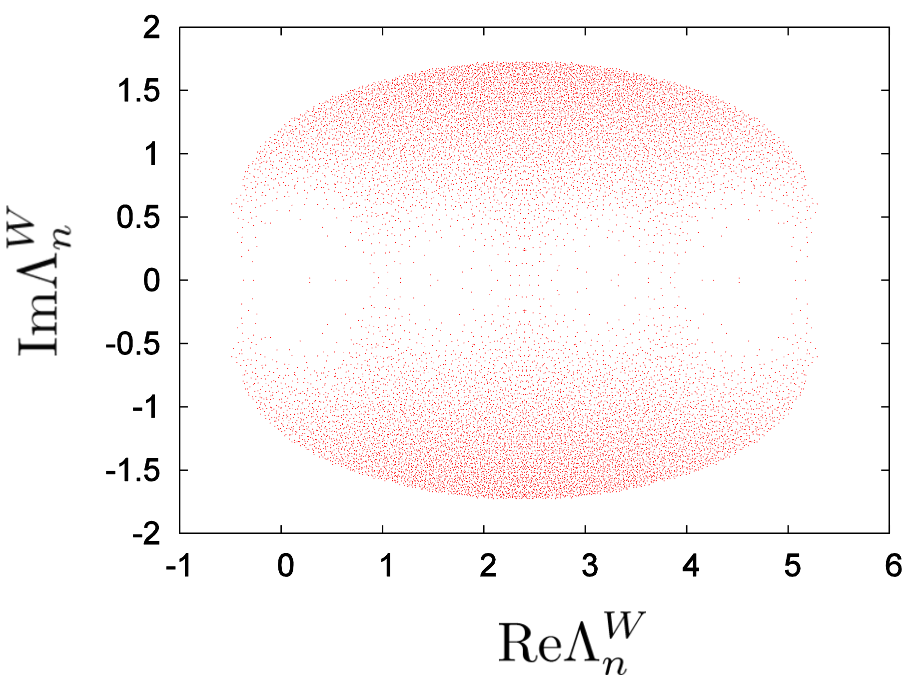
$$D_{\mathrm{ov}}|n\rangle = \Lambda_n^{\mathrm{ov}}|n\rangle$$

$$D_{\mathrm{W}}|n\rangle = \Lambda_n^{\mathrm{W}}|n\rangle$$

$$\boxed{\Lambda_n^{\mathrm{ov}} = 1 + \frac{\Lambda_n^{\mathrm{W}}}{|\Lambda_n^{\mathrm{W}}|}}$$

$$D_{\mathrm{ov}} = \frac{1}{a}\left[1 + \frac{D_{\mathrm{W}}(-M_0)}{\sqrt{D_{\mathrm{W}}(-M_0)^\dagger D_{\mathrm{W}}(-M_0)}}\right]$$





Polyakov loop and overlap-Dirac modes

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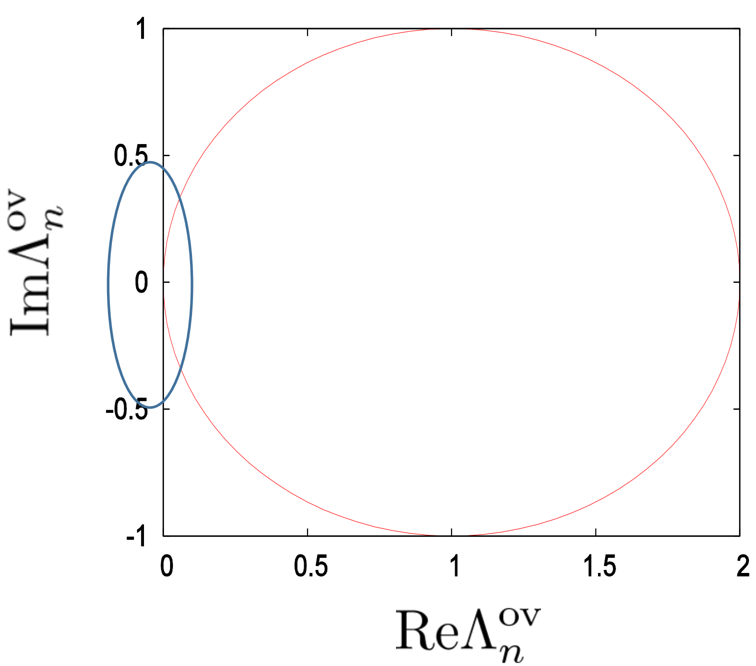
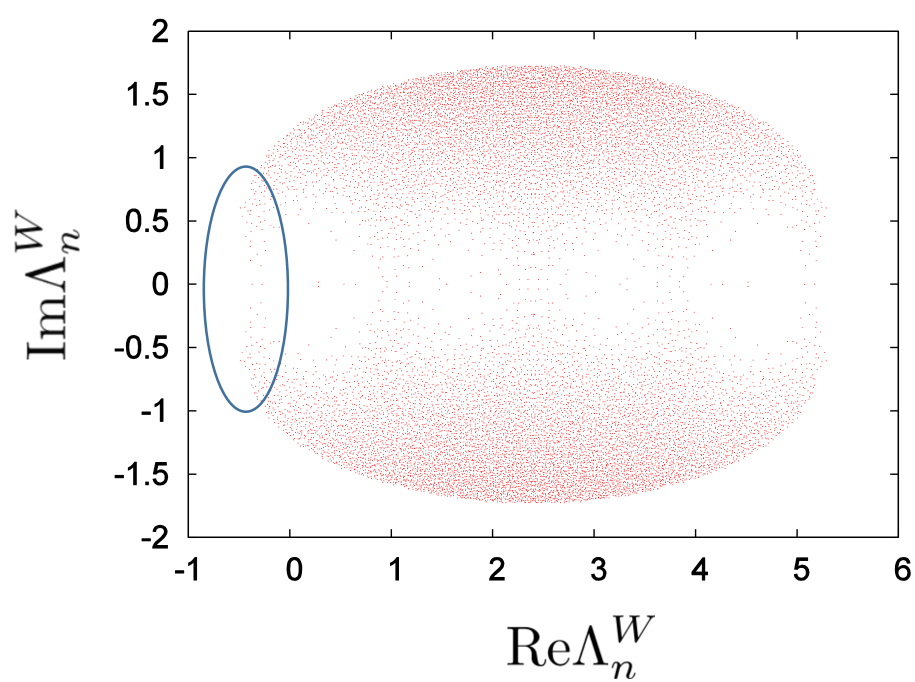
$$D_{\text{ov}}|n\rangle = \Lambda_n^{\text{ov}}|n\rangle$$
$$D_{\text{W}}|n\rangle = \Lambda_n^{\text{W}}|n\rangle$$

$$\boxed{\Lambda_n^{\text{ov}} = 1 + \frac{\Lambda_n^{\text{W}}}{|\Lambda_n^{\text{W}}|}}$$

$$D_{\text{ov}} = \frac{1}{a}\left[1 + \frac{D_{\text{W}}(-M_0)}{\sqrt{D_{\text{W}}(-M_0)^\dagger D_{\text{W}}(-M_0)}}\right]$$





Polyakov loop and overlap-Dirac modes

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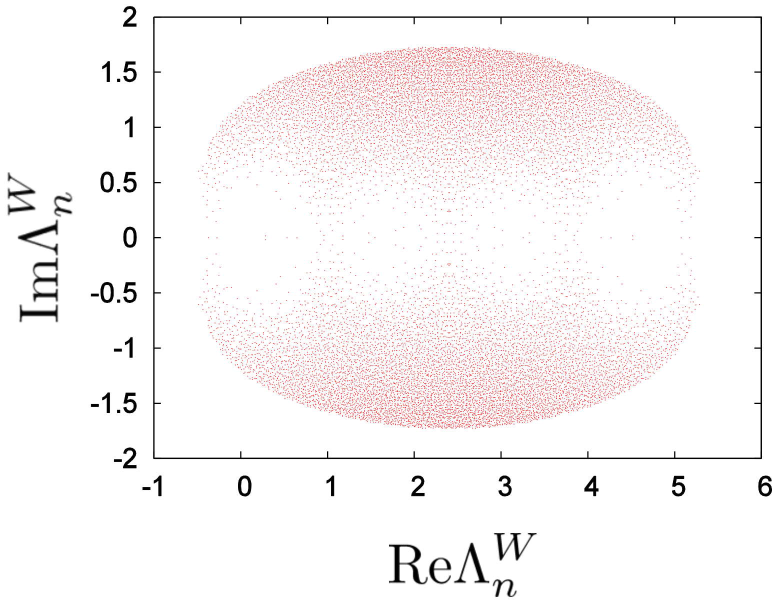
The analytical relation btw the Polyakov loop and the Wilson-Dirac modes

$$\boxed{L = \frac{2a^N}{3V} \sum_n (\Lambda_n^{\mathrm{W}})^N \langle n | \hat{U}_4^{N+1} | n \rangle} \quad (N_\tau = 2N+1)$$

$$D_{\mathrm{W}} |n\rangle = \Lambda_n^{\mathrm{W}} |n\rangle$$



Due to the damping factor $(\Lambda_n^{\mathrm{W}})^N$,
the low-lying Wilson-Dirac modes (with $|\Lambda_n^{\mathrm{W}}| \sim 0$)
have negligible contribution to the Polyakov loop.

- The derivation of the analytical relation was shown in Hideo Suganuma's previous talk. See our paper: TMD, H. Suganuma, T. Iritani, PRD90, 094505 (2014).
- These low-lying modes also have negligible contribution to Polyakov loop fluctuations and Wilson loop.

Polyakov loop and overlap-Dirac modes

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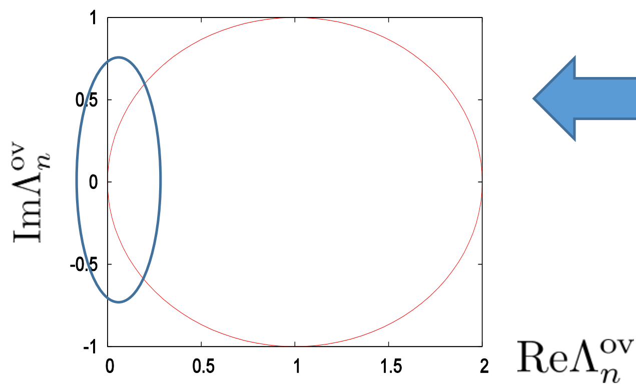
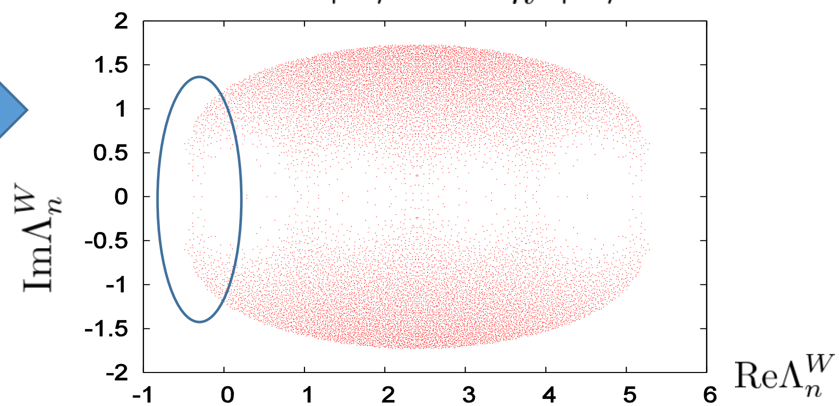
$$D_{\mathrm{ov}}|n\rangle = \Lambda_n^{\mathrm{ov}}|n\rangle$$



② correspond

$$D_{\mathrm{W}}|n\rangle = \Lambda_n^{\mathrm{W}}|n\rangle$$



① $|\Lambda_n^{\mathrm{ov}}| \sim 0$

Essential modes for chiral symmetry breaking

③ $|\Lambda_n^{\mathrm{W}}| \sim 0$

Negligible contribution to the Polyakov loop

Therefore, the presence or absence of the low-lying overlap-Dirac modes is not related to the confinement properties such as the Polyakov loop while the chiral condensate is sensitive to the density of the low-lying modes.

Summary

TMD, K. Redlich, C. Sasaki and H. Suganuma, Phys. Rev. D92, 094004 (2015).

1. We have derived the analytical relation between Polyakov loop fluctuations and Dirac eigenmodes on temporally odd-number lattice:

e.g.) $$R_A = \frac{\left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n|\hat{U}_4|n\rangle \right|^2 \right\rangle - \left\langle \left| \sum_n \lambda_n^{N_\tau - 1} \langle n|\hat{U}_4|n\rangle \right| \right\rangle^2}{\left\langle \left(\sum_n \lambda_n^{N_\tau - 1} \mathrm{Re}\left(\mathrm{e}^{2\pi k i/3} \langle n|\hat{U}_4|n\rangle \right) \right)^2 \right\rangle - \left\langle \sum_n \lambda_n^{N_\tau - 1} \mathrm{Re}\left(\mathrm{e}^{2\pi k i/3} \langle n|\hat{U}_4|n\rangle \right) \right\rangle^2}$$

N_τ : odd

Dirac eigenmode : $\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$

Link variable operator :

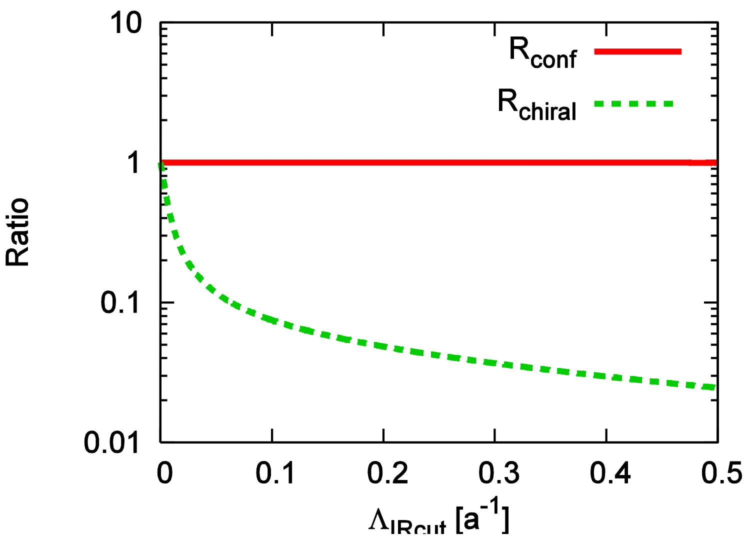
$$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$$

$$\langle N_\tau, \mathbf{x}|\hat{U}_4|1, \mathbf{x}\rangle = -U_4(N_\tau, \mathbf{x})$$

2. We have analytically and numerically confirmed that low-lying Dirac modes are not important to quantify the Polyakov loop fluctuation ratios, which are sensitive observables to confinement properties in QCD.

3. Our results suggest that there is no direct one-to-one correspondence between confinement and chiral symmetry breaking in QCD.

4. We have shown the same results within the overlap-fermion formalism, which solves the fermion-doubling problem with the exact chiral symmetry on the lattice.



Thank you very much for your attention!

Appendix

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

TMD, H. Suganuma, T. Iritani, Phys. Rev. D 90, 094505 (2014).
H. Suganuma, TMD, T. Iritani, Prog. Theor. Exp. Phys. 2016, 013B06 (2016).

$$\boxed{L = -\frac{(2ai)^{N_\tau - 1}}{12V} \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle \quad \text{on temporally odd number lattice: } N_\tau \text{ is odd}}$$

notation:

- Polyakov loop : L
- link variable operator : $\langle s | \hat{U}_\mu | s' \rangle = U_\mu(s) \delta_{s+\hat{\mu}, s'}$

  with anti p.b.c. for time direction: $\langle N_\tau, \mathbf{x} | \hat{U}_4 | 1, \mathbf{x} \rangle = -U_4(N_\tau, \mathbf{x})$
- Dirac eigenmode : $\hat{\not{D}} | n \rangle = i\lambda_n | n \rangle$

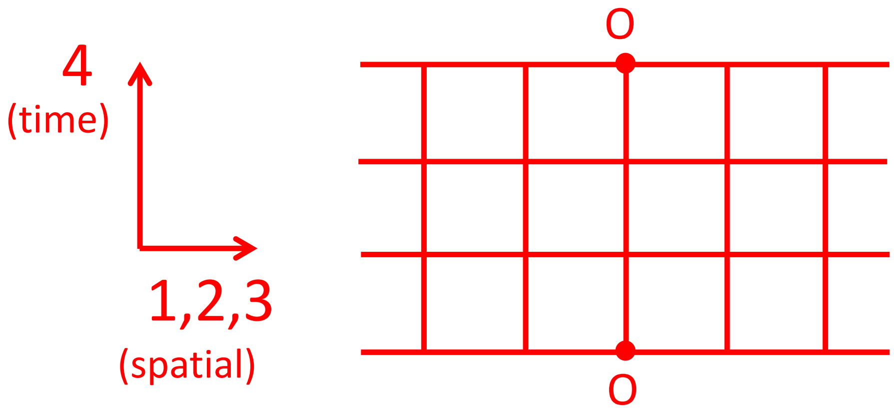
  Dirac operator : $\hat{\not{D}} = \frac{1}{2a} \sum_\mu \gamma_\mu (\hat{U}_\mu - \hat{U}_{-\mu}) \qquad \sum_n | n \rangle \langle n | = 1$

- This analytical formula is a general and mathematical identity.
  - valid in full QCD and at the quenched level.
  - holds for each gauge-configuration {U}
  - holds for arbitrary fermionic kernel K[U]

$$Z = \int \mathcal{D}\bar{q} \mathcal{D}q \mathcal{D}U \mathrm{e}^{-S_{\mathrm{G}}[U] + \bar{q} K[U] q} = \int \mathcal{D}U \mathrm{e}^{-S_{\mathrm{G}}[U]} \mathrm{det} K[U]$$

~from next page: Derivation

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice



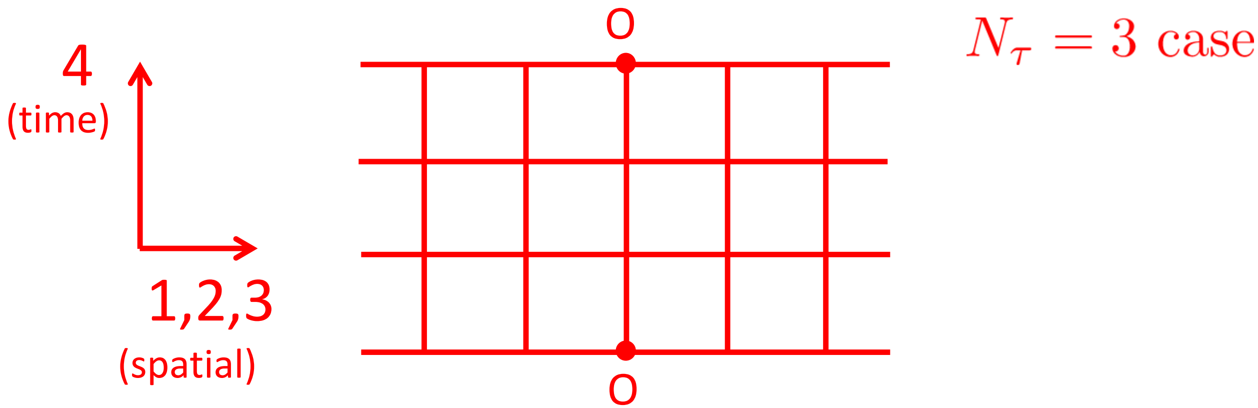
$N_\tau = 3$ case

In this study, we use

- standard square lattice
- with ordinary periodic boundary condition for gluons,
- with the odd temporal length N_τ

  (temporally odd-number lattice)

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice



In this study, we use
- standard square lattice
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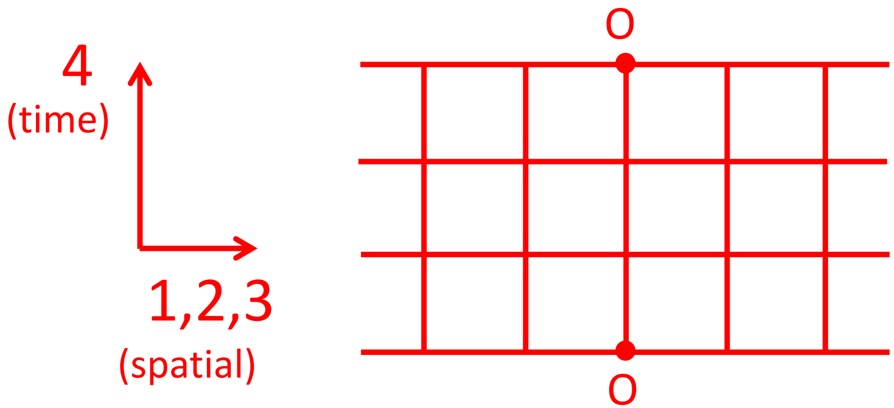
- with the odd temporal length N_τ
  (temporally odd-number lattice)

Note: in the continuum limit of $a \to 0$, $N_\tau \to \infty$,
any number of large N_τ gives the same result.
Then, it is no problem to use the odd-number lattice.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice



$N_\tau = 3$ case

In this study, we use
- standard square lattice
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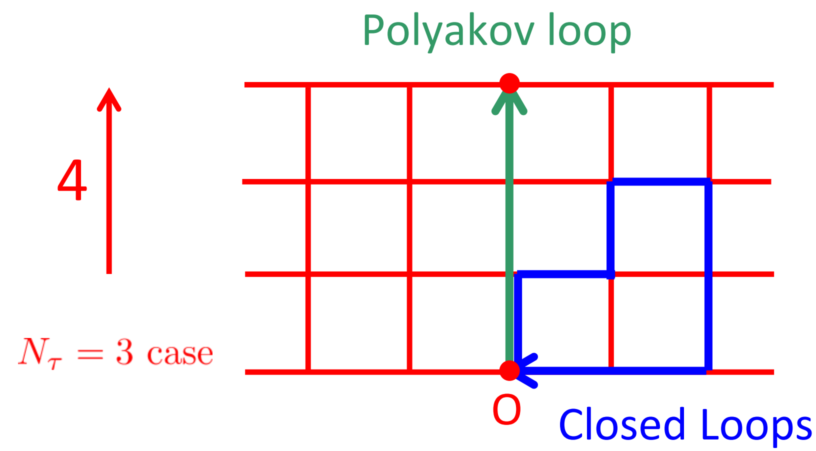
  (temporally odd-number lattice)

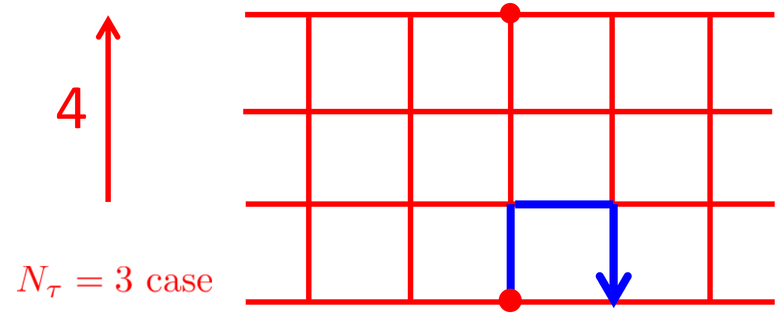
For the simple notation,
we take the lattice unit a=1 hereafter.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

In general, only gauge-invariant quantities such as Closed Loops and the Polyakov loop survive in QCD. (Elitzur's Theorem)



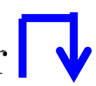
All the non-closed lines are gauge-variant and their expectation values are zero.



e.g.

$$\mathrm{Tr}\hat{U}_4\hat{U}_1\hat{U}_{-4} = \sum_s \mathrm{tr}\{U_4(s)U_1(s+\hat{4})U_4^\dagger(s+\hat{1})\}\underline{\delta_{s,s+\hat{1}}} = 0$$

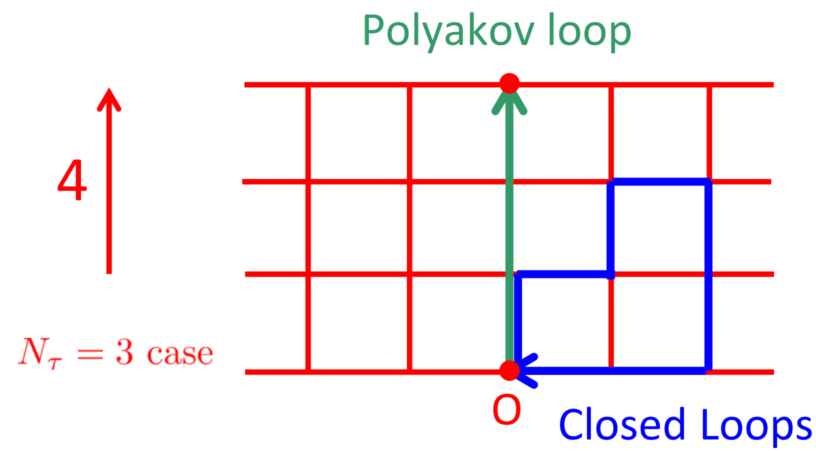
gauge-variant

(Tr  =0)

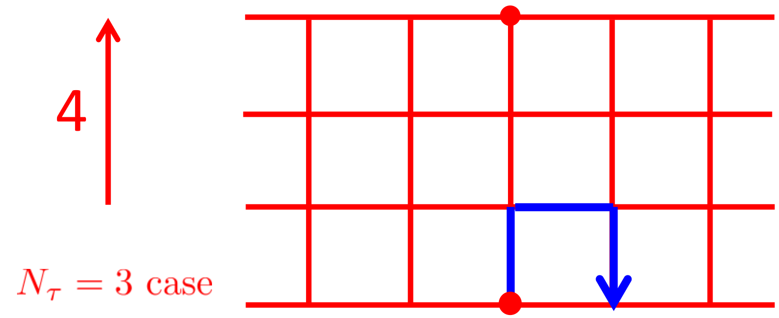
$$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$$

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

In general, only gauge-invariant quantities such as Closed Loops and the Polyakov loop survive in QCD. (Elitzur's Theorem)



All the non-closed lines are gauge-variant and their expectation values are zero.



e.g.

$$\underbrace{\mathrm{Tr}\hat{U}_4\hat{U}_1\hat{U}_{-4}}_{\text{gauge-variant}} = \sum_s \mathrm{tr}\{U_4(s)U_1(s+\hat{4})U_4^\dagger(s+\hat{1})\}\underline{\delta_{s,s+\hat{1}}} = 0$$

(Tr =0)

Key point

Note: any closed loop needs even-number link-variables on the square lattice.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

We consider the functional trace I on the temporally odd-number lattice:

$$\boxed{I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_\tau - 1})} \quad (N_\tau : \mathrm{odd})$$

definition:

$$\langle s|\hat{U}_\mu|s'\rangle = U_\mu(s)\delta_{s+\hat{\mu},s'}$$

$$\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma$$

site & color & spinor

Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_\mu \gamma_\mu (\hat{U}_\mu - \hat{U}_{-\mu})$

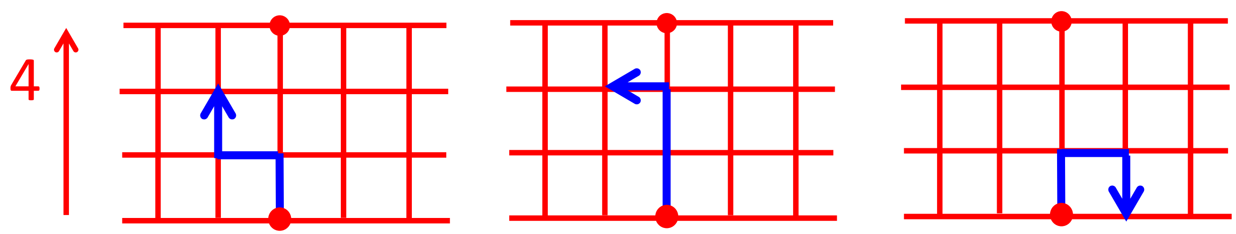
$\hat{U}_4 \hat{\not{D}}^{N_\tau - 1}$ is expressed as a sum of products of N_τ link-variable operators because the Dirac operator $\hat{\not{D}}$ includes one link-variable operator in each direction $\hat{\mu}$.

$\boxed{I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_\tau - 1})}$ includes many trajectories on the square lattice.

$N_\tau = 3$ case

length of trajectories: $\underline{N_\tau = 3}$

odd !!



An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

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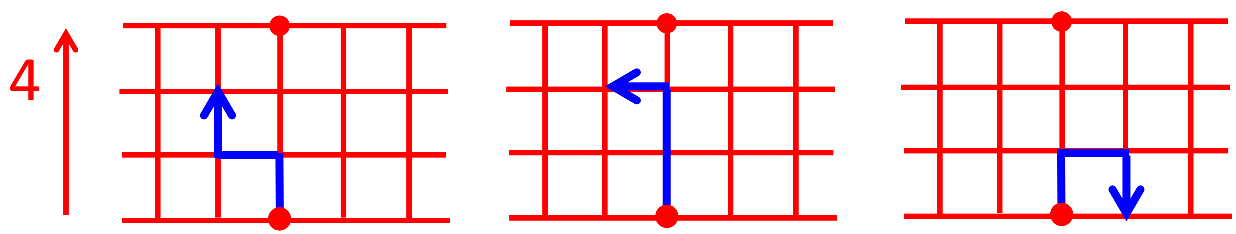


Key point

Note: any closed loop needs even-number link-variables on the square lattice.

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

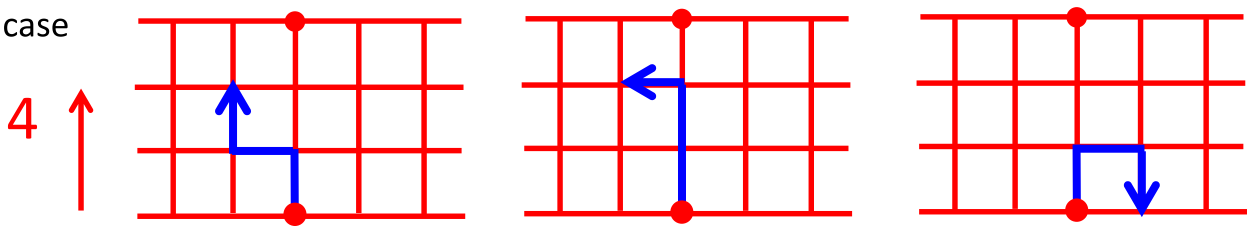
$$I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_\tau - 1}) \qquad (N_\tau : \mathrm{odd})$$

Dirac operator : $\hat{\not{D}} = \frac{1}{2}\sum_\mu \gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu})$

In this functional trace $I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \hat{\not{D}}^{N_\tau - 1})$,
it is impossible to form a closed loop on the square lattice,
because the length of the trajectories, N_τ, is odd.

Almost all trajectories are gauge-variant & give no contribution.

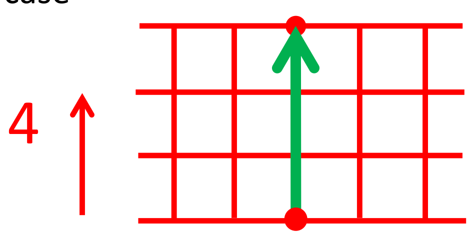
$N_\tau = 3$ case



gauge variant
(no contribution)

Only the exception is the Polyakov loop.

$N_\tau = 3$ case



gauge invariant !!

I is proportional to the Polyakov loop.

$$\boxed{I \propto L_P}$$

L_P : Polyakov loop

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

$$I = \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_\tau - 1}) \qquad (\mathrm{Tr}_{c,\gamma} \equiv \Sigma_s \mathrm{tr}_c \mathrm{tr}_\gamma)$$

$$= \mathrm{Tr}_{c,\gamma}\{\hat{U}_4(\gamma_4 \hat{D}_4)^{N_\tau - 1}\}$$ ($\because$ only gauge-invariant quantities survive)

$$= 4\mathrm{Tr}_c(\hat{U}_4 \hat{D}_4^{N_\tau - 1})$$ ($\because$ $N_\tau - 1$: even, $\gamma_4^2 = 1$ and $\mathrm{tr}_\gamma 1 = 4$)

$$= \frac{4}{2^{N_\tau - 1}} \mathrm{Tr}_c\{\hat{U}_4(\hat{U}_4 - \hat{U}_{-4})^{N_\tau - 1}\}$$

$$= \frac{4}{2^{N_\tau - 1}} \mathrm{Tr}_c\{\hat{U}_4^{N_\tau}\}$$ ($\because$ only gauge-invariant quantities survive)

$$= \frac{12V}{2^{N_\tau - 1}} L_P$$ ($\because$ $L_P = \frac{1}{3V} \mathrm{Tr}_c\{\hat{U}_4^{N_\tau}\}$: Polyakov loop)

($V = N_1 N_2 N_3 N_4$: lattice volume)

Thus, $I \equiv \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_\tau - 1})$ is proportional to the Polyakov loop.

$$\boxed{I = \mathrm{Tr}_{c,\gamma}(\hat{U}_4\, \hat{\not{D}}^{N_\tau - 1}) = \frac{12V}{(2a)^{N_\tau - 1}} L_P}$$

An analytical relation between Polyakov loop and Dirac mode on temporally odd-number lattice

On the one hand,

$$I = \frac{12V}{2^{N_\tau - 1}} L_P \qquad \cdots ①$$

On the other hand, take the Dirac modes as the basis for functional trace

$$\begin{aligned} I &= \mathrm{Tr}_{c,\gamma}(\hat{U}_4 \, \hat{\not{D}}^{N_\tau - 1}) \\ &= \sum_n \langle n | \hat{U}_4 \, \hat{\not{D}}^{N_\tau - 1} | n \rangle \\ &= i^{N_\tau - 1} \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle \qquad \cdots ② \end{aligned}$$

Dirac eigenmode

$$\hat{\not{D}} | n \rangle = i \lambda_n | n \rangle$$

$$\sum_n | n \rangle \langle n | = 1$$

from ①、②

$$\boxed{L_P = \frac{(2i)^{N_\tau - 1}}{12V} \sum_n \lambda_n^{N_\tau - 1} \langle n | \hat{U}_4 | n \rangle}$$

Note 1: this relation holds gauge-independently. (No gauge-fixing)

Note 2: this relation does not depend on lattice fermion for sea quarks.

Analytical relation between Polyakov loop and Dirac modes with twisted boundary condition

C. Gattringer, Phys. Rev. Lett. 97 (2006) 032003.

$$\boxed{L = \frac{1}{8V}\left(2\sum_{\lambda}\lambda^{N_4} - (1+i)\sum_{\lambda_+}\lambda_+^{N_4} - (1-i)\sum_{\lambda_-}\lambda_-^{N_4}\right)}$$

twisted boundary condition:

$$U_4(\mathbf{x}, N_4) \to \pm i U_4(\mathbf{x}, N_4), \qquad \forall \mathbf{x}$$

$$D(x, y) \to D_\pm(x, y)$$

λ : Eigenvalue of $D(x|y)$

$\lambda_\pm$: Eigenvalue of $D_\pm(x|y)$

$$D(x|y) = (4+m)\delta_{x,y} - \frac{1}{2}\sum_{\mu=\pm 1}^{\pm 4}[1 \mp \gamma_\mu]U_\mu(x)\delta_{x+\mu,y}$$

: Wilson Dirac operator

The twisted boundary condition is not the periodic boundary condition.

However,
the temporal periodic boundary condition is physically important
for the imaginary-time formalism at finite temperature.

(The b.c. for link-variables is p.b.c., but the b.c. for Dirac operator is twisted b.c.)

Why Polyakov loop fluctuations?

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich and C. Sasaki,
Phys. Rev. D88, 014506 (2013); Phys. Rev. D88, 074502 (2013)

Ans. 1: Avoiding ambiguities of the Polyakov loop renormalization

$$L^{\mathrm{ren}} = Z(g^2) L^{\mathrm{bare}}, \qquad L^{\mathrm{bare}} \equiv \frac{1}{N_c V} \sum_s \mathrm{tr}_c \{ \prod_{i=0}^{N_\tau - 1} U_4(s + i\hat{4}) \}$$

$Z(g^2)$: renormalization function for the Polyakov loop, which is still unknown

Avoid the ambiguity of renormalization function
by considering the ratios of Polyakov loop susceptibilities:

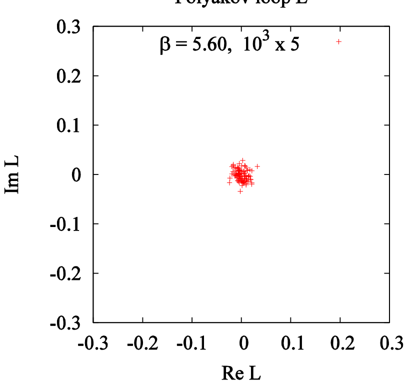
$$R_A \equiv \frac{\chi_A}{\chi_L}, \qquad R_T \equiv \frac{\chi_T}{\chi_L}$$

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L_P = \frac{(2i)^{N_4-1}}{3V}\sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

$\beta = 5.6$

lattice size : $10^3 \times 5$

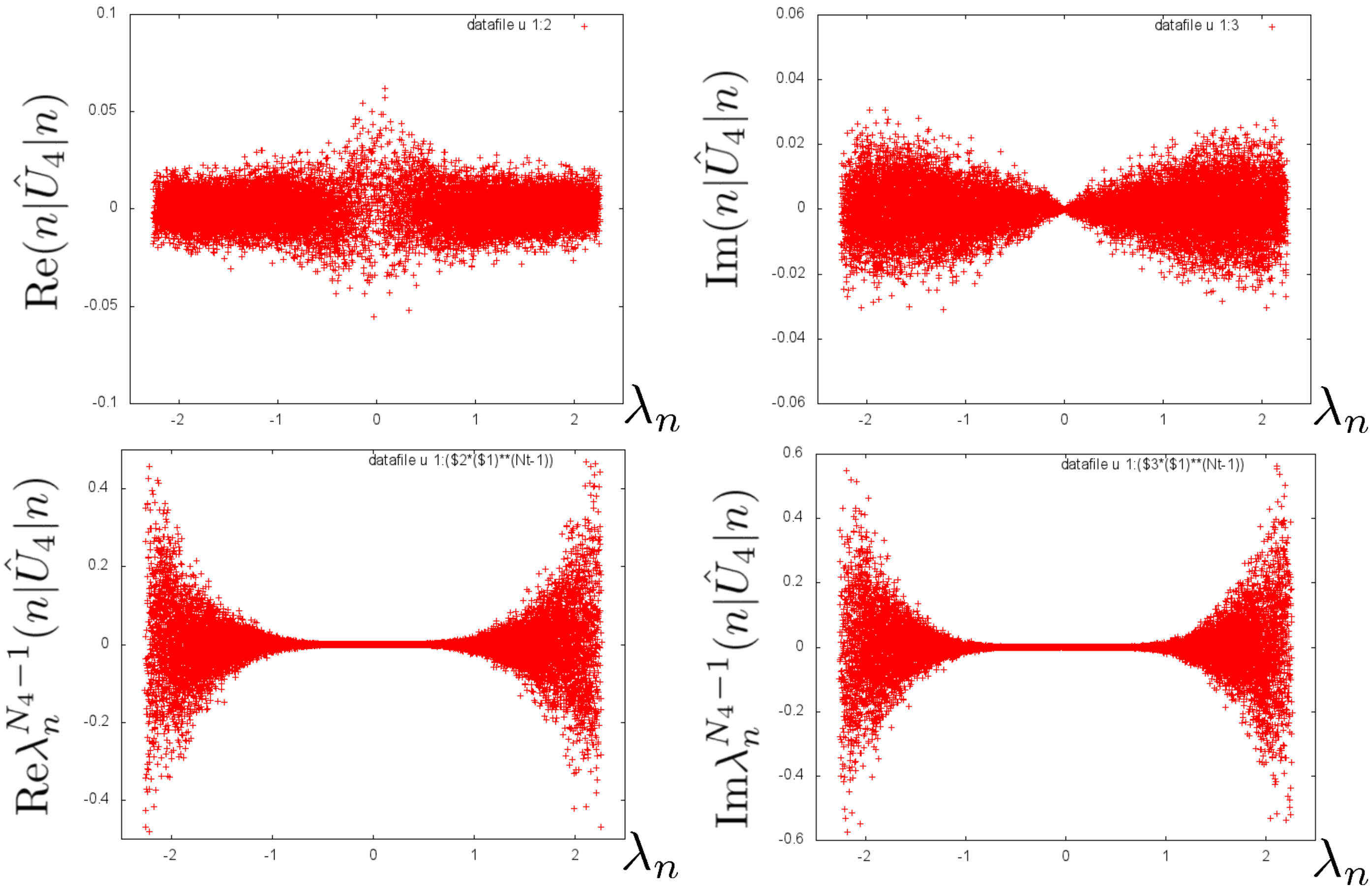


$\langle L_P \rangle = 0$

(confined phase)

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

Dirac eigenvalue: $i\lambda_n$



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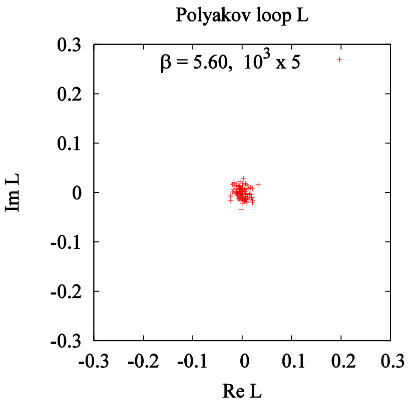

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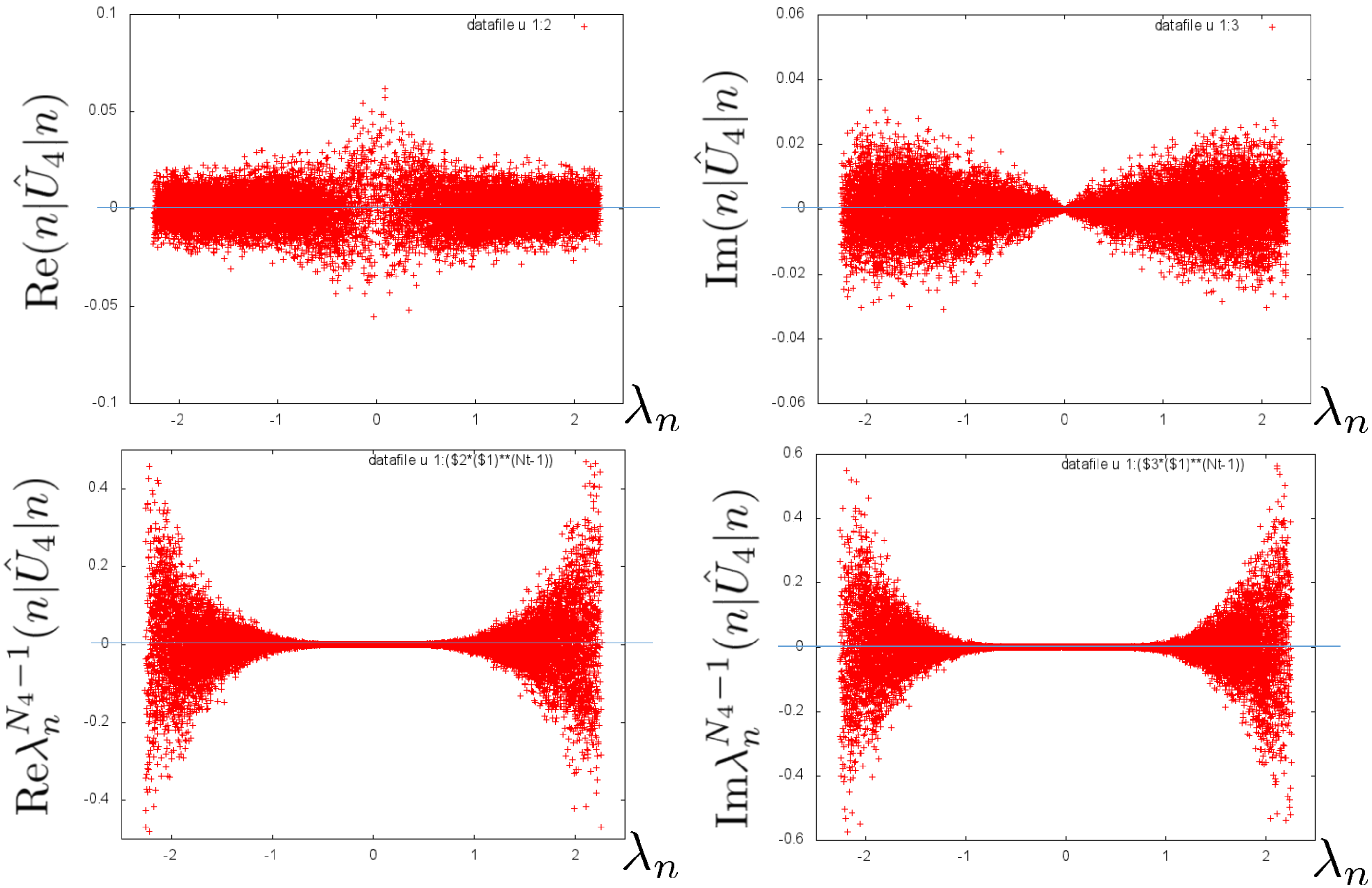


confined phase

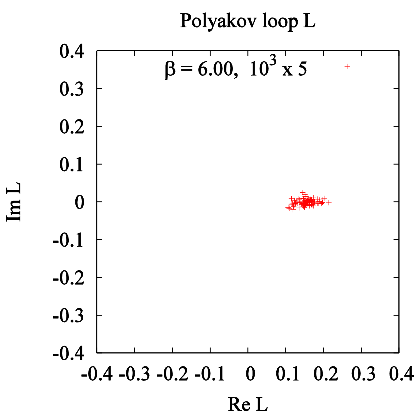
$\langle L \rangle = 0$ is due to the symmetric distribution of positive/negative value of $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

Low-lying Dirac modes have little contribution to Polyakov loop.

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L_P = \frac{(2i)^{N_4-1}}{3V} \sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

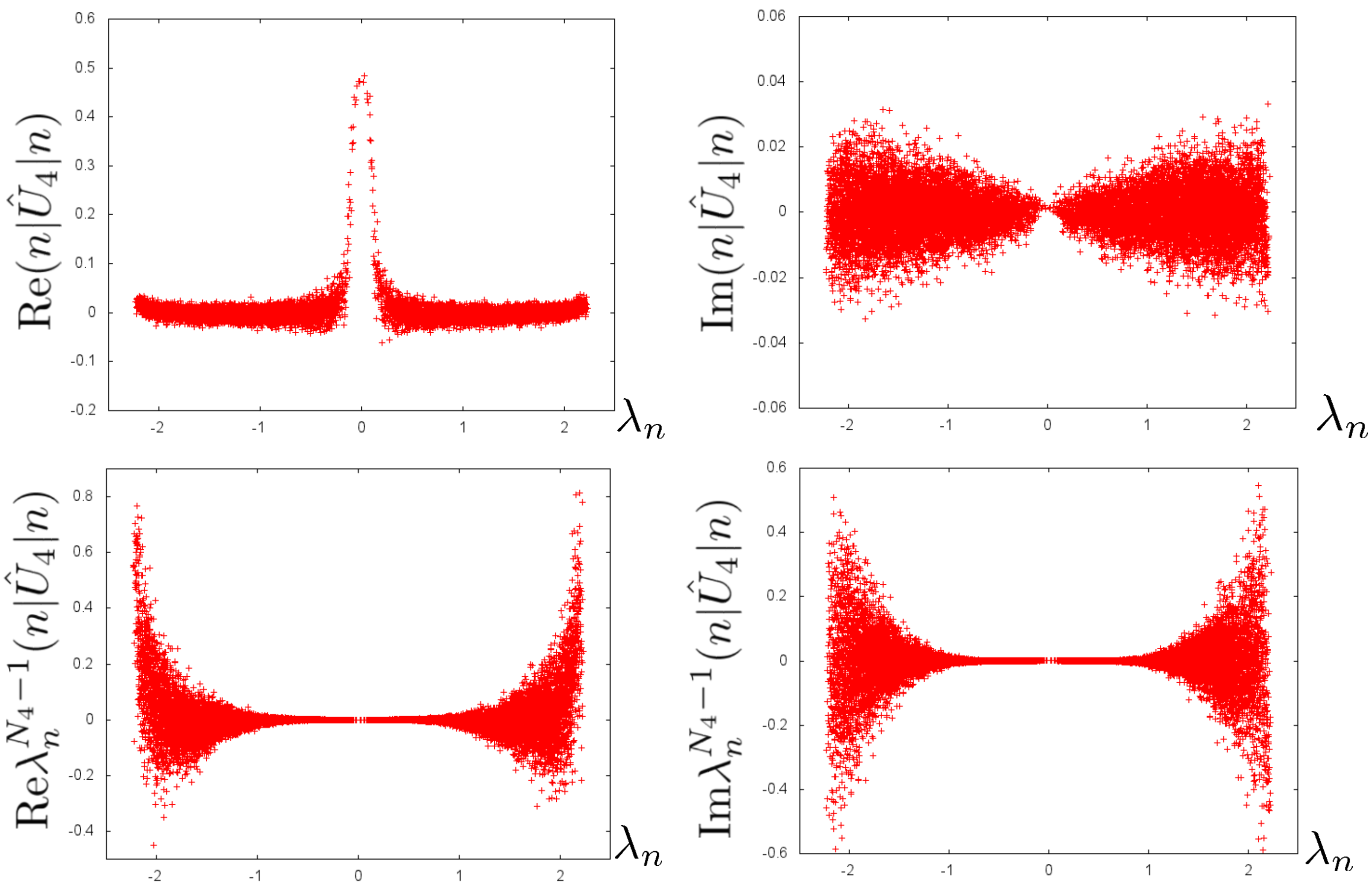
$\beta = 6.0$
lattice size : $10^3 \times 5$



$\langle L_P \rangle \neq 0$
(deconfined phase)

$$\not{D}|n\rangle = i\lambda_n|n\rangle$$

Dirac eigenvalue: $i\lambda_n$

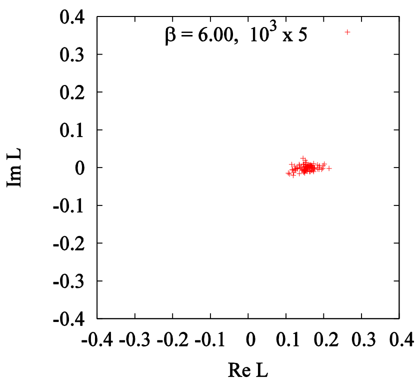


We mainly investigate the real Polyakov-loop vacuum, where the Polyakov loop is real, so only real part is different from it in confined phase.

λ_n V.S. $(n|\hat{U}_4|n)$, $\lambda_n^{N_4-1}(n|\hat{U}_4|n)$

$$L_P = \frac{(2i)^{N_4-1}}{3V}\sum_n \lambda_n^{N_4-1}(n|\hat{U}_4|n)$$

$\beta = 6.0$
lattice size : $10^3 \times 5$



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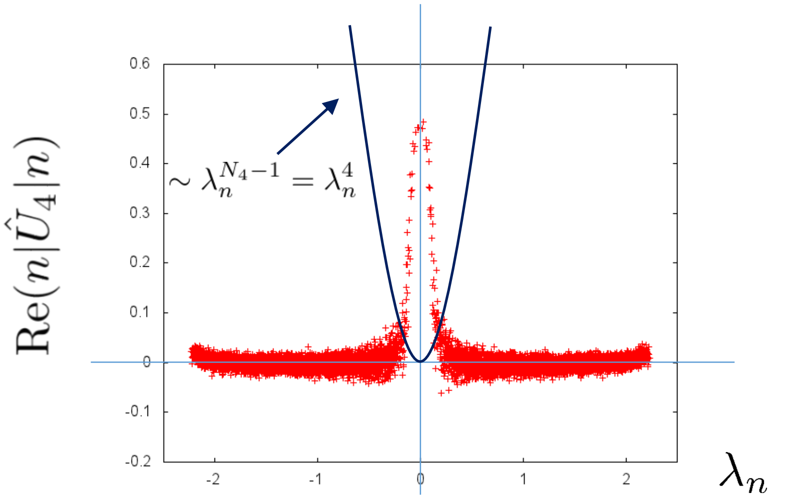

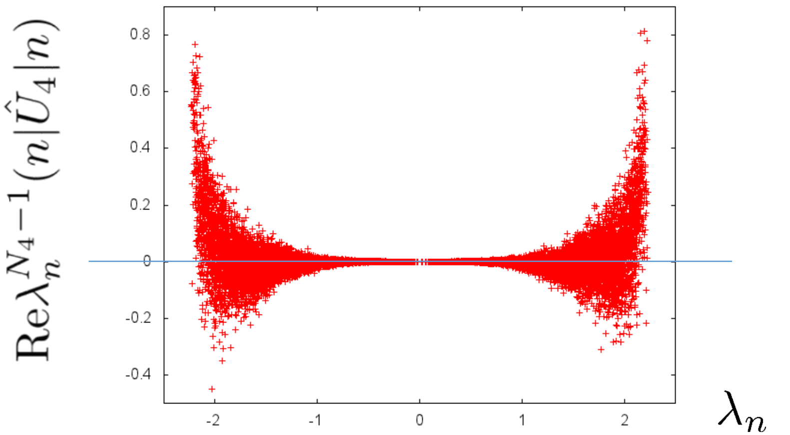

In low-lying Dirac modes region, $\mathrm{Re}(n|\hat{U}_4|n)$ has a large value,
but contribution of low-lying (IR) Dirac modes to Polyakov loop is very small
because of dumping factor $\lambda_n^{N_4-1}$