Chiral magnetic effect and chiral kinetic theory

Shi Pu
(ITP, Goethe Uni. Frankfurt am Main)

References:
• J.W. Chen, SP, Q. Wang, X.N. Wang, PRL 110 (2013) 262301
Outline

• Chiral magnetic and vortical effects

• Chiral kinetic theory

• Recent progress
  – Chiral Hall separation effect
  – Nonlinear chiral transport phenomena
  – Magneto-hydrodynamics

• Summary
Chirality of massless fermions

\[ h = \frac{\sigma \cdot p}{|p|} = \begin{cases} 
+1, & \text{right handed} \\
-1, & \text{left handed}
\end{cases} \]
If $\text{Number of } u_L = \text{Number of } u_R$
No current is observed.
Chiral Magnetic Effect

If Number of uL ≠ Number of uR
A electric current will be observed.
Chiral Magnetic Effect (CME)

\[ j^\mu = \xi_B B^\mu , \]

\[ E^\mu = F^{\mu\nu} u_\nu , \]

\[ B^\mu = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} u_\nu F_{\alpha\beta} . \]

\( u^\mu : \text{fluid velocity} \)

D.E. Kharzeev, L.D. McLerran, H.J. Warringa, NPA 803, 227
K. Fukushima, D. E. Kharzeev, H. J. Warringa, PRD78, 074033
Chiral Vortical Effect (CVE)

\[ \omega^\mu = \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta \rightarrow (0, \nabla \times \mathbf{v}) \]

\[ j^\mu = \xi \omega^\mu \]
Chiral Magnetic and Vortical Effect

\[ j^\mu = \xi_B B^\mu + \xi \omega^\mu, \]
\[ j_5^\mu = \xi_5 B^\mu + \xi_5 \omega^\mu, \]
New Transport coefficients

\[ j_\mu = \xi_B B^\mu + \xi \omega^\mu, \]
\[ j_5^\mu = \xi_5 B^\mu + \xi_5 \omega^\mu, \]

- Weakly coupling, Kubo formula
  \[(\text{Fukushima}'08, \text{Kharzeev}'11, \text{Landsteiner}'11, \text{Hou}'12, \ldots)\]
- Strong coupling, AdS/CFT duality,
  \[(\text{Erdmenger}'09, \text{Banerjee}'11, \text{Torabian}'11, \ldots)\]
Kinetic theory

- **Kinetic theory**: a microscopic dynamic theory for many-body system, to compute transport coefficients.

- **Distribution function**, e.g. Fermi-Dirac distribution $f(x, p)$
Chiral Magnetic effect (quantum effect) VS Semi-classical Boltzmann eq.

We try to study these chiral phenomena by Boltzmann equations, but we failed...

It seems that one has to modify the Boltzmann equations.

SP, J.H. Gao, Q. Wang, Phys.Rev. D83 (2011) 094017
• **Wigner function**: a quantum distribution function, ensemble average, normal ordering

\[ W(x, p) = \langle : \int \frac{d^4 y}{(2\pi)^4} \, e^{-i p y} \overline{\psi}(x + \frac{1}{2} y) \otimes \mathcal{P} U(x, y) \psi(x - \frac{1}{2} y) : > \]

Gauge link

\[ \overline{\psi}(x + \frac{1}{2} y) \quad X \quad \psi(x - \frac{1}{2} y) \]
Macroscopic quantities

**Charge current**

\[ j^\mu(x) \equiv \langle \overline{\psi}(x) \gamma^\mu \psi(x) \rangle = \int d^4p \text{Tr} \ (\gamma^\mu W), \]

**Chiral current**

\[ j_{5}^\mu(x) = \lim_{\epsilon \to 0} \langle \overline{\psi}(x + \frac{1}{2}\epsilon) \gamma^5 \gamma^\mu e^{i \int_{x-\epsilon/2}^{x+\epsilon/2} dz \cdot A(z)} \psi(x - \frac{1}{2}\epsilon) \rangle \]

\[ = \int d^4p \text{Tr} \ (\gamma^\mu \gamma^5 W) \]
Master equation from Dirac Eq.

- Massless, constant external electromagnetic fields $F_{\mu\nu}^{ext}$, neglecting particles’ interactions

\[
[\gamma^{\mu}p_{\mu} + \frac{1}{2}i \gamma^{\mu}(\partial_{\mu} - QF_{\mu\nu}^{ext}\partial_{\mu})]W = 0,
\]

Vasak, Gyulassy and Elze (’86,’87,’89)

- First order differential equation, solve it order by order
Solve the Master equation

- **Gradient expansion** to Winger function $W$ and its master equation,
  - expand all quantities at the power of derivatives
    \[ O(\partial_x^1), O(\partial_x^2), \]
  - external fields are weak
    \[ F^{\mu\nu} \sim \partial_x^\mu A^{\nu} \sim O(\partial^1), \]
• 0\textsuperscript{th} order, non-interacting ideal gas
  – classical Fermi-Dirac distribution

• input parameters:
  – finite temperature $T$,
  – chemical potential $\mu = \mu_R + \mu_L$,
  – chiral chemical potential $\mu_5 = \mu_R - \mu_L$
• It is remarkable that we obtain the chiral anomaly by Wigner function!

\[ \partial_\mu T^{\mu\nu} = F^{\nu\lambda} j_\lambda, \]

\[ \partial_\mu j_\mu = 0, \]

\[ \partial_\mu j_5^\mu = -\frac{e^2}{16\pi^2} \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}. \]
Chiral magnetic and vortical effect

\[ j^\mu = \xi_B B^\mu + \xi \omega^\mu, \]
\[ j_5^\mu = \xi_5 B^\mu + \xi_5 \omega^\mu, \]

\[ \xi = \frac{1}{\pi^2} \mu \mu_5, \]
\[ \xi_B = \frac{Q}{2\pi^2} \mu_5, \]
\[ \xi_5 = \frac{1}{6} T^2 + \frac{1}{2\pi^2} \left( \mu^2 + \mu_5^2 \right), \]
\[ \xi_{B5} = \frac{Q}{2\pi^2} \mu. \]

Consistent with other approaches!

T: temperature

Chemical potentials

\[ \mu = \mu_R + \mu_L, \]
\[ \mu_5 = \mu_R - \mu_L, \]
Spin Local Polarization Effect

Chiral current

\[ j_5^\mu \equiv j^\mu_R - j^\mu_L = \xi_5 \omega^\mu, \]

\[ \xi_5 = \frac{1}{6} T^2 + \frac{1}{2\pi^2} (\mu^2 + \mu_5^2), \]

Can be observed in both high/low energy collisions
What can we learn from these results?
3-dimensional Chiral kinetic equation

- Integral over $p_0$ and in local rest frame, we obtain the kinetic theory for chiral fermions.

$$\frac{dt}{d\tau} \partial_t f_{R/L} + \frac{dx}{d\tau} \cdot \nabla_x f_{R/L} + \frac{dp}{d\tau} \cdot \nabla_p f_{R/L} = 0,$$

$$\frac{dt}{d\tau} = 1 \pm Q \Omega \cdot B \pm 4|p|(\Omega \cdot \omega),$$

velocity

$$\frac{dx}{d\tau} = \hat{p} \pm Q(\hat{p} \cdot \Omega)B \pm Q(E \times \Omega) \pm \frac{1}{|p|} \omega,$$

force

$$\frac{dp}{d\tau} = Q(E + \hat{p} \times B) \pm Q^2(E \cdot B)\Omega$$

$$\mp Q|p|(E \cdot \omega)\Omega \pm 3Q(\Omega \cdot \omega)(p \cdot E)\hat{p},$$
3-dimensional Chiral kinetic equation

- Neglect all terms proportional to $\Omega$, it becomes the **standard** Boltzmann eq.

$$\frac{dt}{d\tau} \partial_t f_{R/L} + \frac{dx}{d\tau} \cdot \nabla_x f_{R/L} + \frac{dp}{d\tau} \cdot \nabla_p f_{R/L} = 0,$$

- $\frac{dt}{d\tau} = 1$
- $\frac{dx}{d\tau} = \hat{p}$
- $\frac{dp}{d\tau} = Q(E + \hat{p} \times B)$

$f_{R/L}$ : distribution function for right or left handed fermions
• Firstly, let us consider an adiabatic process. At each time, the system is at its eigenstate e.g. \( U \).

• Secondly, we assume the Hamiltonian is time dependent \( H=H(t) \). So do those eigenstates \( U=U(t) \), i.e. \( U(t+\Delta t)=U(t)+\Delta t \Delta U \).

• Finally, the system goes to its initial eigenstate. Then there is an additional phase factor to the wave function. It is Berry phase.

• Analogy to moving a vector in a curved space.
Berry Phase (2)

• Let us consider the Hamiltonian of Weyl fermions in momentum space.

\[ H = i \sigma \cdot \nabla \rightarrow \sigma \cdot \mathbf{p}, \quad HU(p), = EU(p), \]

\[ U(t + \Delta t) = U(t) + i\Delta t \frac{d \mathbf{p}}{dt} \cdot \mathbf{a}_p, \quad \mathbf{a}_p = -iU(t) \nabla_p U(t), \]

• If the system goes back to its initial eigenstate, then phase factor is independent on the path. So, it is physical!

\[ \Psi(p) = \exp \left( i \oint_C d\mathbf{p} \cdot \mathbf{a}_p \right) U(p) \]

\[ = \exp \left( i \iint d\mathbf{S} \cdot \Omega_p \right) U(p) \quad \Omega_p = \nabla_p \times \mathbf{a}_p, \]

Stokes’s theorem

D. Xiao, M.C. Chang, Q. Niu, Rev. Mod. Phys. 82.1959
Berry Phase (3)

• In the **absence** of external fields, the Berry phase is **decoupled** to the dynamics.

• In the **presence** of electromagnetic fields,

\[
\sqrt{\gamma} \frac{dx}{dt} = \hat{p} + eE \times \Omega + eB(\hat{p} \cdot \Omega)
\]

velocity

\[
\sqrt{\gamma} \frac{dp}{dt} = eE + \hat{p} \times eB + e^2(E \cdot B)\Omega
\]

force


\[
\Omega = \nabla_p \times a_p = \frac{p}{2|p|^3},
\]

\[
\sqrt{\gamma} = 1 + eB \cdot \Omega
\]
We used Wigner function to obtain the chiral kinetic equation with Berry phase.

\[
\frac{dt}{dT} \partial_t f_{R/L} + \frac{dx}{dT} \cdot \nabla_x f_{R/L} + \frac{dp}{dT} \cdot \nabla_p f_{R/L} = 0,
\]

\[
\frac{dt}{dT} = 1 \pm Q\Omega \cdot B \pm 4|p|(\Omega \cdot \omega),
\]

\[
\frac{dx}{dT} = \hat{p} \pm Q(\hat{p} \cdot \Omega)B \pm Q(E \times \Omega) \pm \frac{1}{|p|} \omega,
\]

\[
\frac{dp}{dT} = Q(E + \hat{p} \times B) \pm Q^2(E \cdot B)\Omega
\]

\[
\pm Q|p|(E \cdot \omega)\Omega \pm 3Q(\Omega \cdot \omega)(p \cdot E)\hat{p},
\]
Recent progress

- Chiral Hall separation effect
- Nonlinear chiral transport phenomena
- Magneto-hydrodynamics
Chiral Hall separation effect

• Assuming $E \perp B$, according to Hall effect:

\[ j_z = \sigma_H E_x B_y \]

• Charge and chirality separation in **longitudinal** direction

SP, S.Y. Wu, D.L. Yang, PRD 91 (2015) 2, 025011
Nonlinear Chiral transport effects

- An inhomogeneous chiral system
- In a large chemical potential limit
- chiral kinetic theory + relaxation time approaches

\[ \mathbf{j}_e = \sigma_E \mathbf{E} + \sigma_E \mu_5 \nabla \mu_5 \times \mathbf{E}, \]

\[ (\mu^2 + \mu_5^2) \frac{\sigma_E \mu_5}{\sigma_E} = \frac{\hbar c}{2} \quad \text{Independent on the interactions!} \]

• Magneto-hydrodynamics:
  – Relativistic hydrodynamics + Maxwell’s eq.

• Anomalous:
  – Chiral magnetic effect + other chiral transport effects
Magneto-hydrodynamics

- **1D Bjorken + ideal Magneto-hydrodynamics:**
  - analytic solution
  - with Magnetization effects
  SP, V. Roy, L. Rezzolla, D.H. Rischke, PRD 93 (2016), 074022

- **2+1 D Bjorken + ideal Magneto-hydrodynamics**
  - analytic solution:
    SP, D.L Yang, PRD93 (2016), 054042
  - Simulations:
    V. Roy, SP, L. Rezzolla, D.H. Rischke, in preparation
Summary

• We obtain the chiral magnetic and vortical effect, chiral anomaly by Wigner function.

• We derive the chiral kinetic equation (modified Boltzmann equation) related to Berry phase.

• We also made some progresses in
  • Chiral Hall separation effect,
  • nonlinear chiral transport effect
  • magneto-hydrodynamics.
Thank you!
Anomalous fluid dynamics

• We do not have those chiral transport terms in a normal fluid.

• Son and Suroń (‘09) pointed out these terms are crucial to cancel the production of negative entropy in an anomalous fluid.

\[ \partial_\mu T^{\mu\nu} = QF^{\nu\rho} j_\rho, \]
\[ \partial_\mu j^\mu = 0, \quad \partial_\mu j_5^\mu = -\frac{Q^2}{2\pi^2} E_\rho B^\rho, \]
Non-abelian Berry Phase

• For massive fermions, a particle can change its spin. In classical limit, we get,

\[
\frac{dx}{dt} = \hat{p} + \frac{dp}{dt} \times s_a \Omega^a, \\
\frac{dp}{dt} = eE + \frac{dx}{dt} \times eB, \\
\frac{ds_a}{dt} = \epsilon_{abc} \left( \frac{dp}{dt} \cdot a_b \right) s_c.
\]

J.W. Chen, J.Y. Pang, SP, Q. Wang, PRD89 (2014), 094003