Interplay between Deconfinement and Chiral Properties

H. Suganuma, T.M. Doi, K. Redlich, C. Sasaki,
Y. Nakagawa, K. Matsumoto

Critical Point and Onset of Deconfinement 2016, Wrocław, Poland, 3 June 2016

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Outline

1. Confinement and Chiral Symmetry Breaking
2. Analytic Formula between Polyakov Loop and Dirac mode at any T
3. Formula of Polyakov Loop Fluctuation with Dirac mode near T_c
4. Analytic Formula between Wilson Loop and Dirac mode at T=0
5. Formula for Wilson, Clover, Domain-Wall fermions

References:

[1] *H. S., T. M. Doi, T. Iritani, PTEP 2016 (2016) 013B06, Analytical Formulae of the Polyakov and Wilson Loops with Dirac Eigenmodes in Lattice QCD.*

[2] *T. M. Doi, H. S., T. Iritani, PRD90 (2014) 094505, Relation between Confinement and Chiral Symmetry Breaking in Temporally Odd-Number Lattice QCD.*

[3] *T. M. Doi, K. Redlich, C. Sasaki, H. S., PRD92 (2015) 094004, Polyakov Loop Fluctuations in the Dirac Eigenmode Expansion*.

Critical Point and Onset of Deconfinement 2016, Wrocław, Poland, 3 June 2016

Introduction : Confinement and Chiral Symmetry Breaking

Color Confinement and Chiral Symmetry Breaking (CSB)
are most important phenomena of Nonperturbative QCD

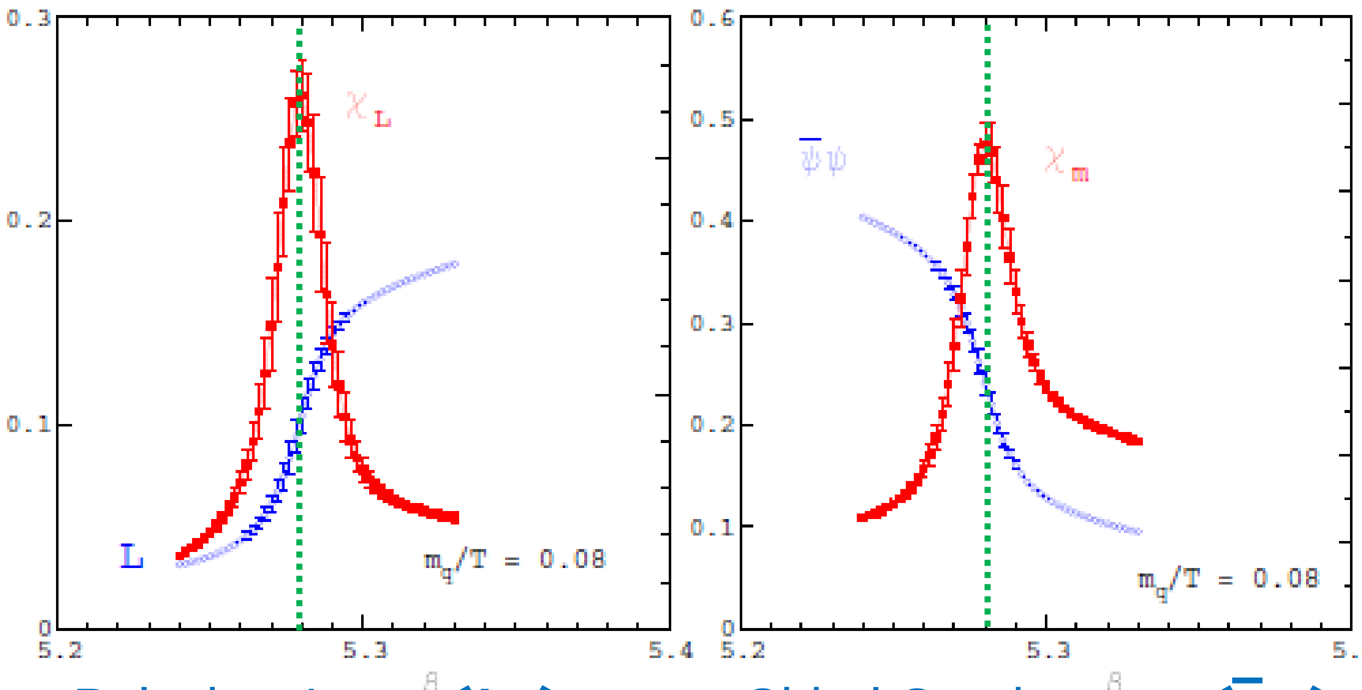
The relation between
Confinement and CSB
is not yet known
directly from QCD.

Order parameter of Confinement: Polyakov loop $\langle L_P \rangle \propto e^{-E_q/T}$

Order parameter of CSB: Chiral Condensate $\langle \bar{q}q \rangle$

Correlation between Confinement and CSB is suggested by
Simultaneous Phase Transition of
Deconfinement and Chiral Restoration.

Lattice QCD results at finite temperature F. Karsch, Lect. Notes Phys. (2002)



Polyakov Loop $\langle L_P \rangle$ Chiral Condensate $\langle \bar{q}q \rangle$

Color Confinement Chiral Symmetry Breaking

Fig. 2. Deconfinement and chiral symmetry restoration in 2-flavour QCD: Shown is $\langle L \rangle$ (left), which is the order parameter for deconfinement in the pure gauge limit ($m_q \to \infty$), and $\langle \bar{\psi}\psi \rangle$ (right), which is the order parameter for chiral symmetry breaking in the chiral limit ($m_q \to 0$). Also shown are the corresponding susceptibilities as a function of the coupling $\beta = 6/g^2$.

More on correlation between Confinement and Chiral Sym Breaking

Also, similar Coincidence between Deconfinement and Chiral Restoration is found in Finite-Size lattice QCD.

In fact, Simultaneous Phase Transitions occur according to the Box Size.

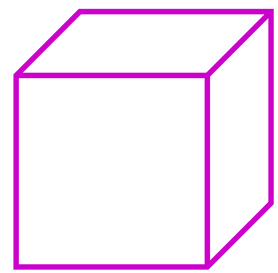
Large Volume Lattice

Confinement

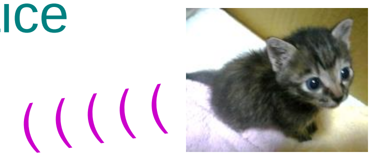
Chiral Sym. Breaking

simultaneous
Phase Transitions

Small Volume Lattice



Freedom!



Deconfinement

Chiral Restoration

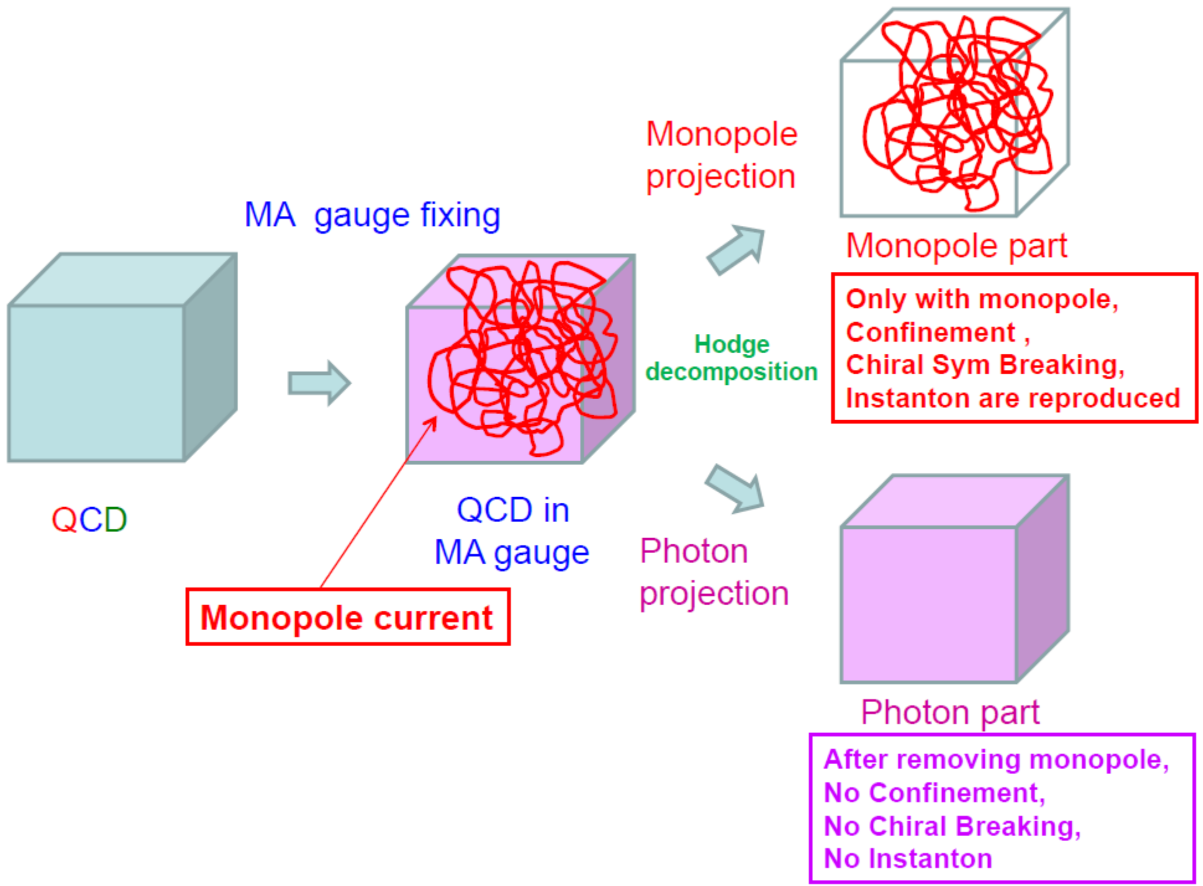
Of course, Finite-Temperature Phase transition is also a kind of Finite-Size effect of Euclidean Lattice in temporal direction.

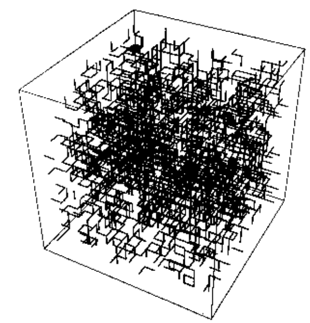
More on correlation between Confinement and Chiral Sym Breaking

The close relation between Confinement and CSB has been indicated in terms of Monopoles appearing in Maximally Abelian Gauge in QCD.

By removing the Monopoles from the QCD vacuum, the confinement property and CSB are simultaneously lost.

[e.g. Dual GL theory: H.S., S.Sasaki and H.Toki, NPB (1995),
LQCD : O.Miyamura, PLB (1995), R.Woloshyn, PRD(1995).]





Monopole world line in MA gauge In Lattice QCD

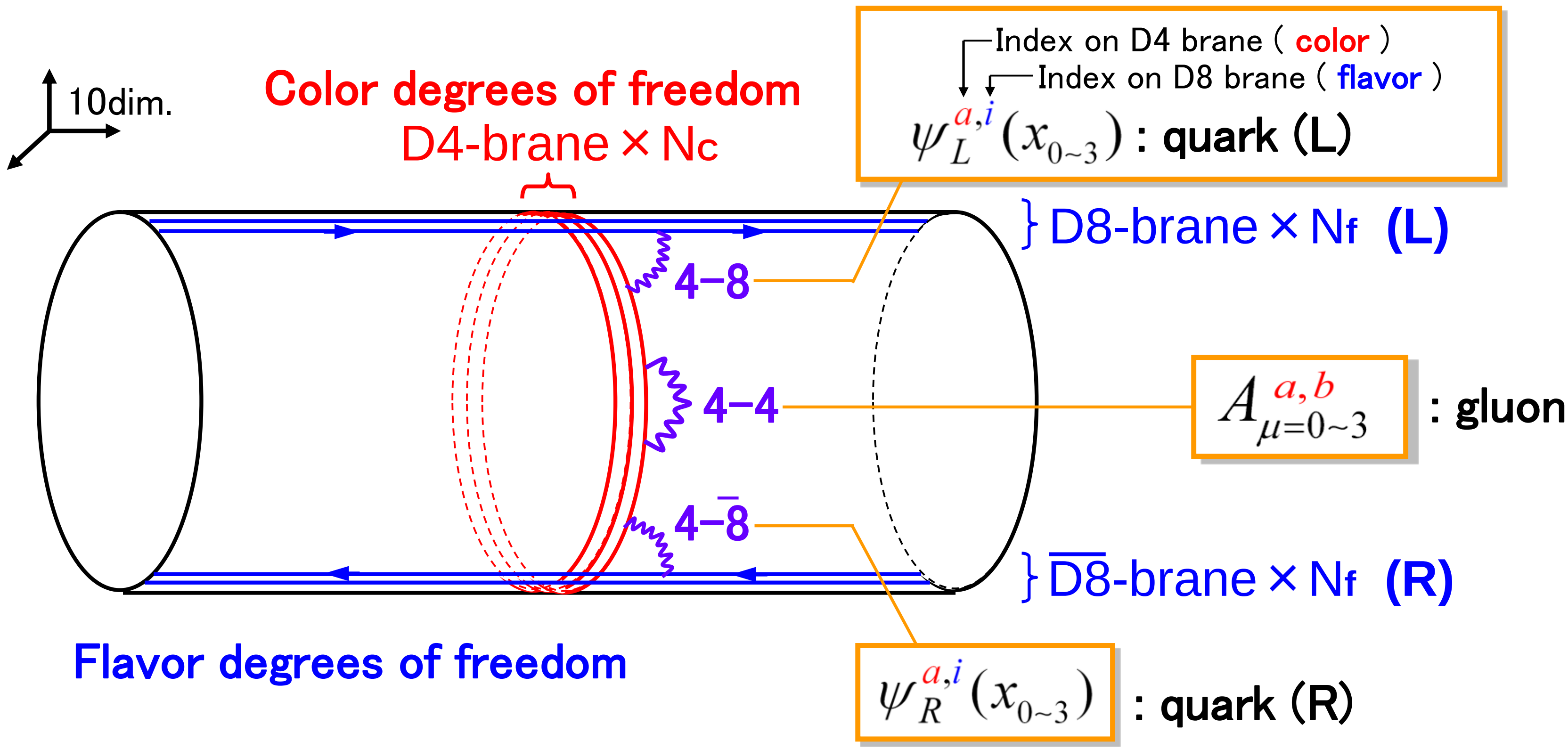
O. Miyamura

A Geometrical explanation of simultaneous Confinement and CSB in Holographic QCD (the Sakai–Sugimoto model) at $T = 0$

T. Sakai and S. Sugimoto, Prog. Theor. Phys. 113 (2005) 843.

K. Nawa, H.S. and T. Kojo, PRD75 (2007) 086003.

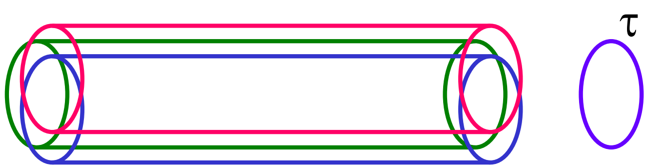
Using D4/D8/$\overline{\text{D8}}$–brane, massless QCD (quarks and gluons) can be constructed.
Here, D4–brane gives color and Gluons and D8–brane gives flavor.
Left (Right) Quarks appear at the cross point between D4 and D8 (D8bar).



Holographic construction of non-SUSY Yang-Mills theory with Gauge/Gravity correspondence

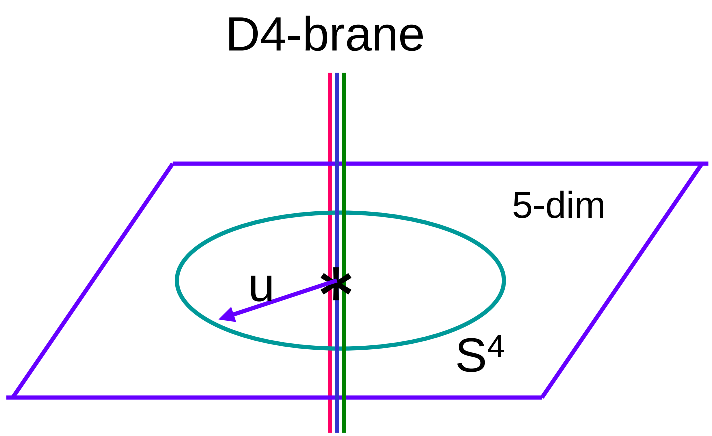
S^1-compactified N_c D4-brane → SURGA solution of D4 brane

$X_\mu = (t, x, y, z)$

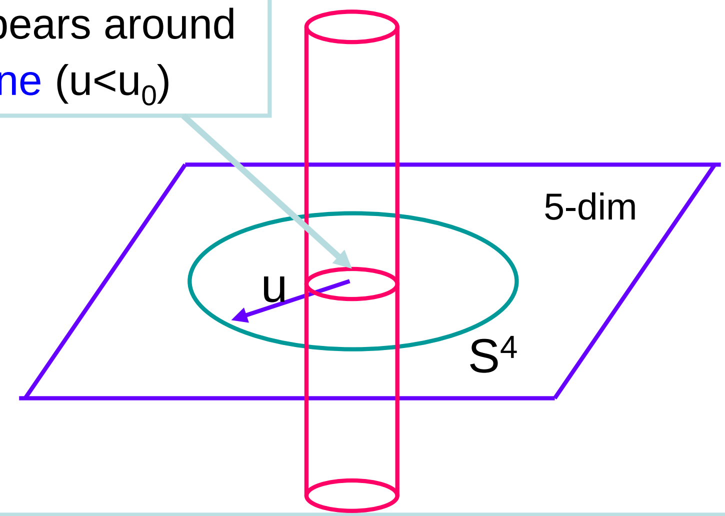


$$ds^2 = H(u)^{-1/2}(\eta_{\mu\nu}dx^\mu dx^\nu + f(u)d\tau^2) + H(u)^{1/2}(du^2 / f(u) + u^2 d\Omega_4^2)$$

$$f(u) = 1 - (u_0 / u)^3$$



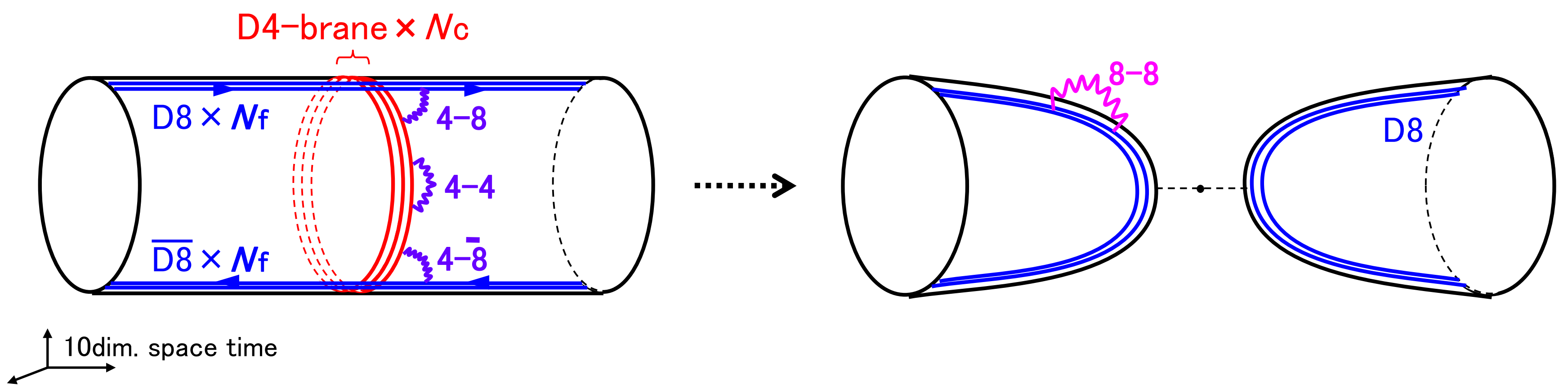
10-dim space-time Genus appears around D4-brane ($u<u_0$)



- SUSY is broken by the periodic/antiperiodic boundary condition of S^1. (Witten 98)
- D4-brane is replaced by Curved Space of Gravitational Solution.
- Around the D4-brane, 10 dim. space-time disappear, that is, D4-brane leads Genus of 10 dim. space-time. This leads to linear confinement potential for quarks.

In Large N_c limit, D4-brane is extremely massive and is replaced by Gravitational background, under assumption of AdS/CFT (gauge/gravity) correspondence.

D4 × Nc	→	Gravitational background
D8, $\overline{\text{D8}}$ × Nf	→	D8, $\overline{\text{D8}}$ × Nf



K. Nawa, H. S. and T. Kojo, PRD75 (2007) 086003.

Chiral symmetry breaking is thus realized by the geometrical connection of D8 and D8bar branes. … These considerations may suggest that chiral symmetry breaking and color confinement occur simultaneously.

In Holographic QCD, Confinement and Chiral SB may occur simultaneously due to the “Genus” in 10 dim. Space-time created by large *Nc* D4-brane.

A possible Difference between Deconfinement and Chiral Restoration in QCD

~ controversial

Other Lattice QCD calculation

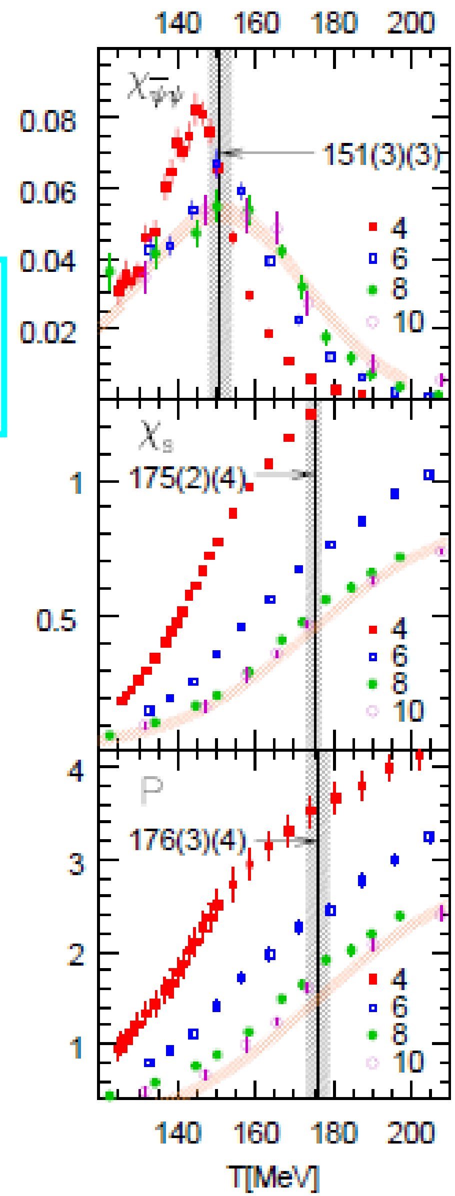
Y. Aoki, Z. Fodor, S.D. Katz,
K.K. Szabo, PLB643 (2006)

Chiral Restoration：Tc≒151MeV

Deconfinement：Td≒176MeV

For 151～176MeV,
Chiral Restored but Confined?

Don't be so serious…



Light Quark Condensate
$\langle \bar{q}q \rangle$

Chiral Symmetry

Strange quark condensate
$\langle \bar{s}s \rangle$

Polyakov Loop $\langle P \rangle$

Quark Confinemnet

A Large Difference between Deconfinement and Chiral Restoration in the case of Adjoint-Color Fermion

Lattice QCD calculation with Adjoint-Color Fermion

"Deconfinement and chiral symmetry restoration in an SU(3) gauge theory with adjoint fermions", F. Karsch, M. Lutgemeier, NPB550 (1999).

We analyze the finite temperature phase diagram of QCD with fermions in the adjoint representation. The simulations performed with four dynamical Majorana fermions show that the deconfinement and chiral phase transitions occur at two distinct temperatures. While the deconfinement transition is first order at T_d we find evidence for a continuous chiral transition at a higher temperature $\boxed{T_c \simeq 8\ T_d}$. We observe a rapid change of bulk thermodynamic observables at T_d which reflects the increase in the number of degrees of freedom. However, these show little variation at T_c, where the fermion condensate vanishes. We also analyze the potential between static fundamental and adjoint charges in all three phases and extract the corresponding screening masses above T_d.

QCD with fundamental color fermion: Coincidence of Chiral transition and deconfinement $T_c \fallingdotseq T_d$

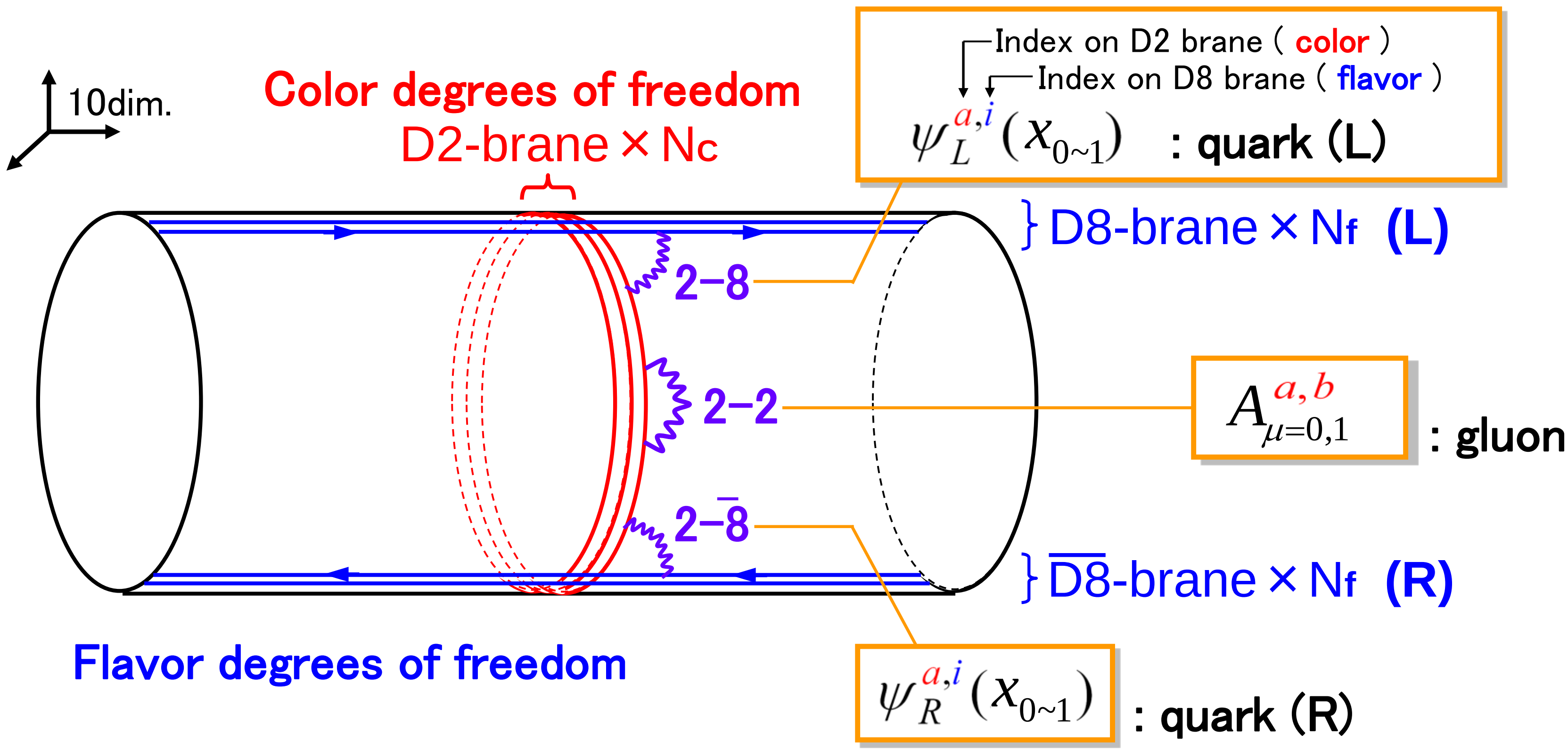
Adjoint-color fermion: Chiral transition at much higher temperarture $T_c \fallingdotseq 8T_d$

Umm… It is not ordinary QCD…

Holographic Construction of Massless 1+1 QCD at $T = 0$

Y. Nakagawa, K. Matsumoto and H.S.

Using D2/D8/$\overline{\text{D8}}$–brane system, massless 1+1 QCD (quarks and gluons) can be constructed. Here, D2–brane gives color and Gluons and D8–brane gives flavor. Left (Right) Quarks appear at the cross point between D2 and D8 (D8bar).

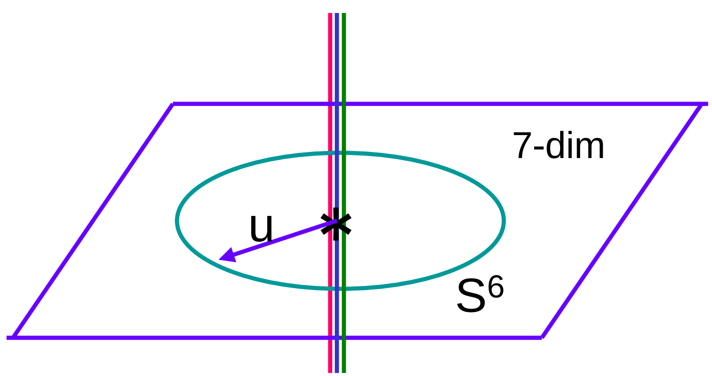


Holographic description of 1+1 Yang-Mills theory with Gauge/Gravity correspondence

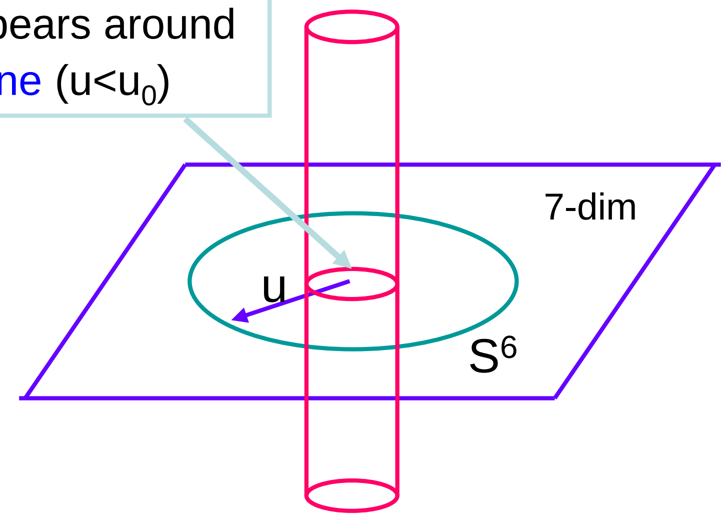
S^1-compactified N_c D2-brane → SURGA solution of D2 brane

$$ds^2 = H(u)^{-1/2}(\eta_{\mu\nu}dx^\mu dx^\nu + f(u)d\tau^2) + H(u)^{1/2}(du^2/f(u) + u^2 d\Omega_6^2)$$

$$f(u) = 1-(u_0/u)^5$$



10-dim space-time
Genus appears around
D2-brane ($u<u_0$)



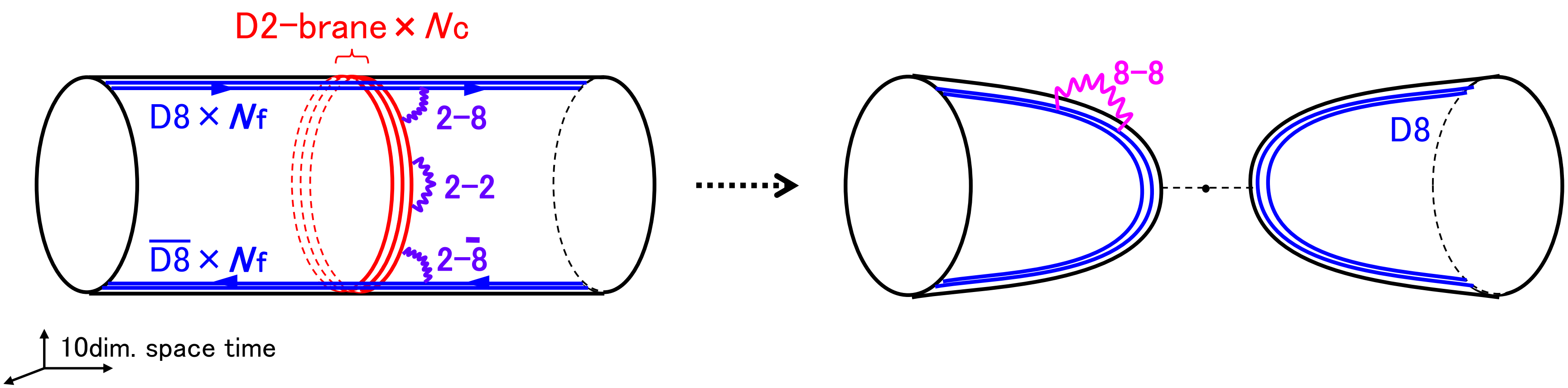
- SUSY is broken by the periodic/antiperiodic boundary condition of S^1.
- D2-brane is replaced by Curved Space of Gravitational Solution.
- Around the D2-brane, 10 dim. space-time disappear, that is, D2-brane leads Genus of 10 dim. space-time. This leads to linear confinement potential for quarks.

Y. Nakagawa, K. Matsumoto and H.S.

Consider S^1-compactified Large N_c D2-brane with N_f D8-$\overline{\text{D8}}$-brane.
This system leads to non-SYSY 1+1 QCD at $T = 0$.

D2 × Nc ⋯⋯→ Gravitational background

D8, $\overline{\text{D8}}$ × Nf ⋯⋯→ D8, $\overline{\text{D8}}$ × Nf



Like the Sakai-Sugimoto model, this holographic QCD leads to "Color Confinement". Simultaneously, Chiral Symmetry is also spontaneously broken due to the connection (topological change) of D8 branes.
However, including $1/N_c$ higher order, CSB is no more realized in 1+1 dim.

Examples with Confinement but without Chiral Symmetry Breaking

1+1 QCD ($N_f \geqq 2$)

- Confinement is realized.
- Spontaneous Chiral Symmetry Breaking does NOT occur.
  (Because of Coleman–Mermin–Wagner theorem：
  In 1+1 space–time, massless scalar leads to IR instability
  and cannot appear, so that spontaneous symmetry
  breaking accompanying massless NG bosons NEVER occurs.

$\mathcal{N} = 1$ SUSY 1+3 QCD with $N_f = N_c + 1$

- Confinement is realized.
- Spontaneous Chiral Symmetry Breaking does NOT occur.

Banks-Casher Relation

$$\Sigma \equiv \left| \langle \bar{q} q \rangle \right| = \lim_{m \to 0} \lim_{V \to \infty} \pi\, \rho(0)$$

$$\rho(\lambda) = \frac{1}{V} \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle$$ **: QCD Dirac operator eigenvalue density**

$$\hat{\not{D}} | n \rangle = i \lambda_n | n \rangle$$

Zero-eigenvalue density ρ(0) of QCD Dirac operator $\hat{\not{D}}$ gives Chiral Condensate.

⇒ The essential modes for Chiral Sym Breaking are Low-lying Dirac modes.

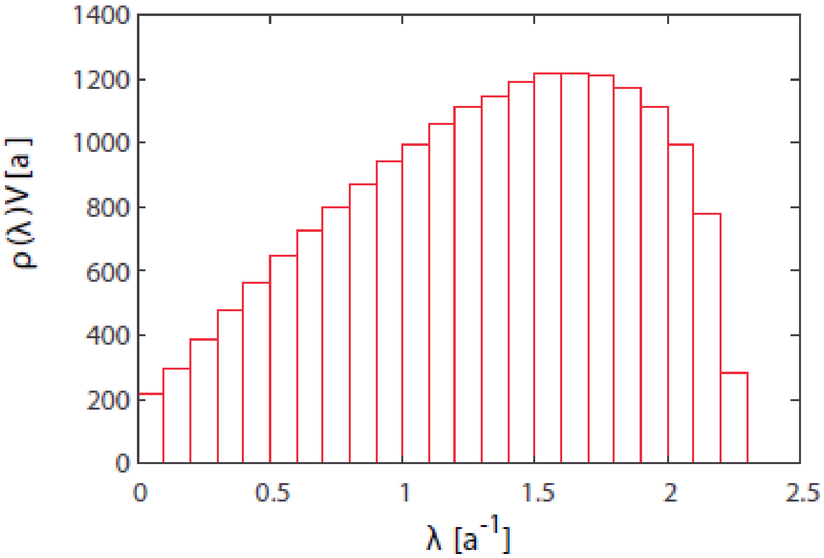
※ The non-zero spectrum is symmetric due to $\{\gamma_5, \hat{\not{D}}\} = 0$

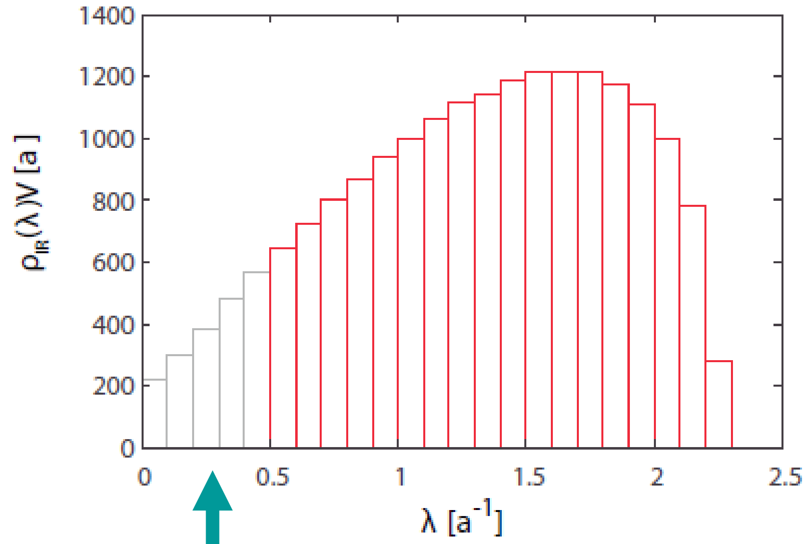
$$\because \hat{\not{D}} \psi_n = \lambda_n \psi_n \to \hat{\not{D}} (\gamma_5 \psi_n) = -\lambda_n (\gamma_5 \psi_n)$$

Let us consider Confinement in terms of Dirac modes.

Previous study: Eigen-value distribution of QCD Dirac operator

β=5.6 (a=0.25fm), 6^4





We Remove the contribution of Low-lying Dirac modes.

$$\langle \bar{q}q \rangle_{IR} \propto \sum_{\lambda_n \geq \Lambda_{IR}} \frac{2m}{\lambda_n^2 + m^2}$$

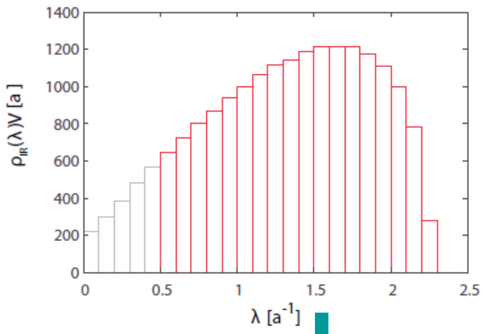
$$\boxed{\frac{\langle \bar{q}q \rangle_{IR}}{\langle \bar{q}q \rangle} \approx 0.02}$$

for m_q~5 MeV

Chiral Condensate is extremely reduced (only 2% remains) after removing the low-lying Dirac modes.

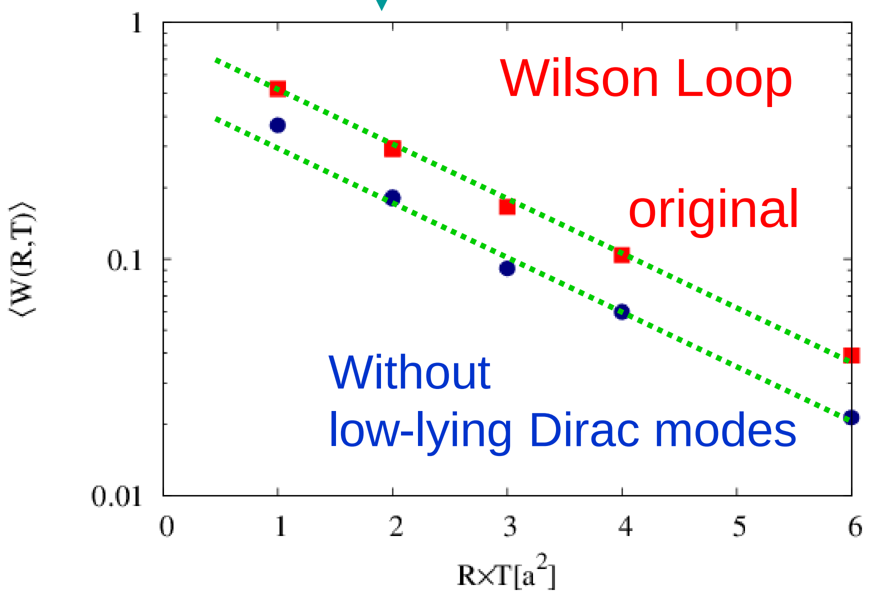
(cf. Banks-Casher relation)

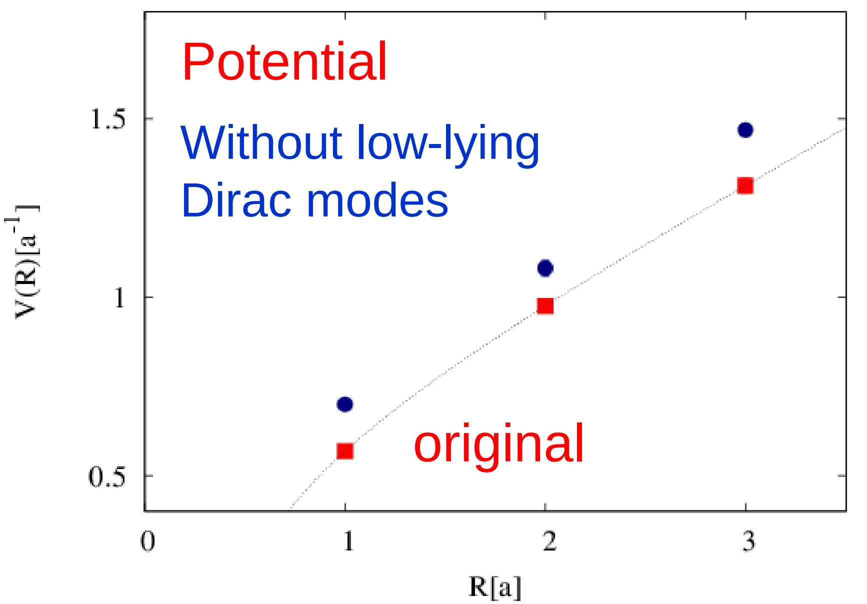
<u>Previous study</u>: Wilson Loop after removing low-lying Dirac modes (T=0)



S.Gongyo, T.Iritani, H.S., PRD86 (2012) 034510.

Lattice QCD result of
Wilson Loop and Inter-Quark Potential
before/after the removal of low-lying Dirac modes





Wilson Loop obeys the Area Law and
the confining force is almost unchanged
even after the removal of low-lying Dirac modes,
which are responsible to chiral symmetry breaking.

Previous study: *IR-Dirac-mode-cut Polyakov Loop* (finite T)

IR-Dirac-mode-cut Polyakov Loop T.Iritani, H.S., PTEP 2014 3 033B03.

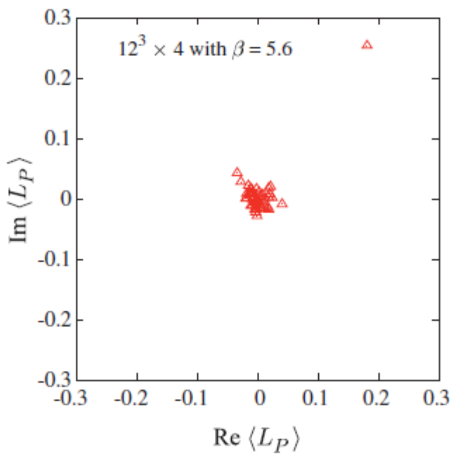
$$\langle L_P \rangle_{IR-cut} \equiv \frac{1}{3V} \mathrm{Tr}(\hat{U}_4^P)^{Nt} = \frac{1}{3V} \sum_{n_1, n_2, \ldots, n_{Nt} \in A} \mathrm{tr} \, \langle n_1 | \hat{U}_4 | n_2 \rangle \langle n_2 | \hat{U}_4 | n_3 \rangle \cdots \langle n_{Nt} | \hat{U}_4 | n_1 \rangle$$

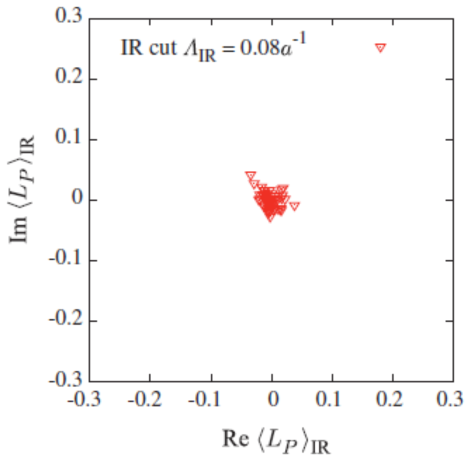
on periodic lattice

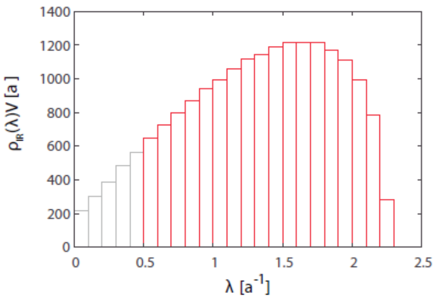
Example in confinement phase

Polyakov Loop

Without IR-Dirac modes







Dirac spectral density

Fig. 11. The Polyakov loop $\langle L_P \rangle$ (left) and the IR Dirac mode cut Polyakov loop $\langle L_P \rangle_{IR}$ (right) with $\Lambda_{IR} \simeq 0.08a^{-1}$ on a $12^3 \times 4$ lattice at $\beta = 5.6$ (confinement phase).

Even after removing the low-lying Dirac modes, Polyakov loop remains to be zero, which means confinement phase and unbroken Z_3-center symmetry.

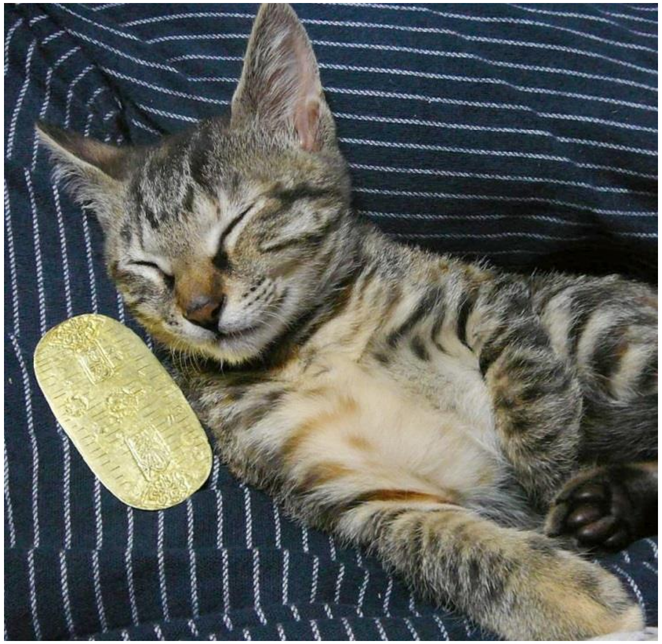
Thus, our lattice-QCD results indicated *negligible contribution of low-lying Dirac modes for confinement*, while these modes are essential for chiral symmetry breaking.

References:

[1] H.S., S. Gongyo, T. Iritani, A. Yamamoto, "Relevant Gluonic Momentum for Confinement and Gauge-Invariant Formalism with Dirac-mode Expansion". PoS (QCD-TNT-II) 044 (2011).

[2] S.Gongyo, T.Iritani, H.S., Phys. Rev. D86 (2012) 034510, "Gauge-Invariant Formalism with Dirac-mode Expansion for Confinement and Chiral Symmetry Breaking".

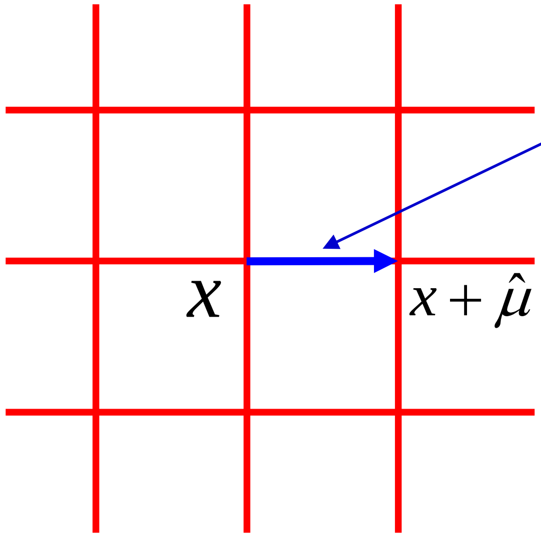
[3] T. Iritani, H.S., PTEP 2014 (2014) 033B03, "Lattice QCD Analysis of the Polyakov Loop in terms of Dirac Eigenmodes".



Main Dish !

Lattice QCD

In lattice QCD, gauge degrees of freedom is described by *Link-variable* defining on a Link on the lattice.
Gluon field is exponentiated into Link-variable.



$$U_\mu(x) \equiv e^{iagA_\mu(x)} \in SU(N_c)$$

Gauge transformation : Link-variable is sandwiched by the gauge functions on both edges of the link.

$$U_\mu(x) \to U_\mu^V(x) = V(s)U_\mu(x)V^\dagger(s+\hat{\mu})$$

2. Analytic Formula between Polyakov Loop and Dirac mode at any Temperature

Setup:

- ordinary square lattice with temporally odd number N_t ($< N_s$)
- normal periodic boundary condition for gluons

- *Elitzur Th.* : only *gauge-invariant quantities* have *nonzero* expectation value

Analytical Formula**:** Dirac spectral representation of Polyakov loop

$$L_P \propto \sum_n \underline{\lambda_n^{N_t - 1}} \langle n | \hat{U}_4 | n \rangle$$

L_P : Polyakov Loop**,** λ_n : Dirac eigenvalue**,** $|n\rangle$: Dirac eigenmode

Result: small contribution of low-lying Dirac modes to Polyakov loop

References:

[1] T. M. Doi, H. S., T. Iritani, Phys. Rev. D90 (2014) 094505.

[2] T. M. Doi, K. Redlich, C. Sasaki, H. S., Phys. Rev. D92 (2015) 094004.

[3] H. S., T. M. Doi, T. Iritani, PTEP 2016 (2016) 013B06.

Eigen-mode of Dirac operator in Lattice QCD

$$\hat{\not{D}}_{xy} = \frac{1}{2a}\sum_{\mu=1}^{4}\gamma^{\mu}[U_{\mu}(x)\delta_{y,x+\hat{\mu}} - U_{-\mu}(x)\delta_{y,x-\hat{\mu}}]$$:Lattice Dirac operator

$$U_{\mu}(x) \equiv e^{iagA_{\mu}(x)} \qquad U_{-\mu}(x) \equiv U_{\mu}^{\dagger}(x-\hat{\mu})$$

$$\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$$:Dirac eigen-state $|n\rangle$, Dirac eigen-value $\lambda_n \in \mathbf{R}$

$$\sum_{y}\hat{\not{D}}_{xy}\psi_n(y) = i\lambda_n\psi_n(x)$$:Dirac eigen-function $\psi_n(x)$

Explicit form of Dirac eigen-value equation in lattice QCD

$$\frac{1}{2a}\sum_{\mu=1}^{4}\gamma^{\mu}[U_{\mu}(x)\psi_n(x+\hat{\mu}) - U_{-\mu}(x)\psi_n(x-\hat{\mu})] = i\lambda_n\psi_n(x)$$

Gauge trans. property:
$$\begin{cases} U_{\mu}(x) \to V(x)U_{\mu}(x)V^{\dagger}(x+\hat{\mu}) \\ \psi_n(x) \to V(x)\psi_n(x) \end{cases}$$

same as quark field apart from an irrelevant global phase factor

$$\langle m|n\rangle = \int d^4x\, \psi_m^{\dagger}(x)\psi_n(x) = \delta_{mn}$$:normalization

We introduce
Link-variable operator $\boxed{\hat{U}_{\pm\mu}}$ defined by the matrix element of

$$\boxed{\langle x|\hat{U}_{\pm\mu}|y\rangle = U_{\pm\mu}(x)\delta_{x\pm\hat{\mu},y}} \qquad U_{-\mu}(x) \equiv U^{\dagger}_{\mu}(x-\hat{\mu})$$

Using link-variable operator, many notations are quite simplified:

$$\boxed{\hat{D}_{\mu} = \frac{1}{2a}(\hat{U}_{\mu} - \hat{U}_{-\mu})}$$:covariant derivative operator

$$\boxed{\hat{\not{D}} = \frac{1}{2a}\sum_{\mu=1}^{4}\gamma^{\mu}(\hat{U}_{\mu} - \hat{U}_{-\mu})}$$:Lattice Dirac operator

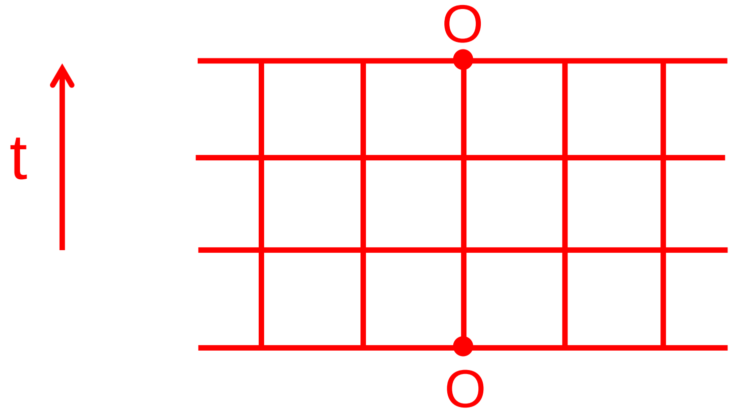
$$\boxed{\hat{\not{D}}|n\rangle = i\lambda_n|n\rangle}$$:Dirac eigenstate, Dirac eigenvalue $\lambda_n \in \mathbf{R}$

$$\boxed{\langle L_P\rangle \propto \mathrm{Tr}\hat{U}_4^{N_t} = \sum_x \langle x|\mathrm{tr}_c\hat{U}_4^{N_t}|x\rangle}$$:Polyakov loop

$\mathrm{Tr} = \sum_x \mathrm{tr}_c$:functional trace

Temporally Odd-Number Lattice

In this study, we use a standard square lattice with ordinary (nontwisted) periodic boundary condition for gluons.
But we consider temporally odd-number lattice,
where the temporal lattice size N_t is odd. ($N_t < N_s$)



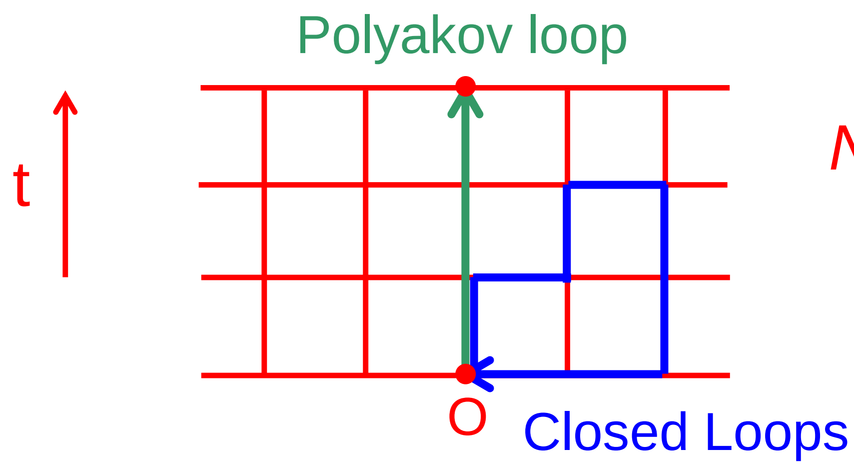
N_t =3 case

NB: in the continuum limit of $a \to 0$, $N_t \to \infty$,
any number of large N_t gives the same result.
Then, it is no problem to use the odd-number lattice.

For the simple notation,
we take the lattice unit a=1 hereafter.

Temporally Odd-Number Lattice

In general, only gauge-invariant quantities such as Closed Loops and the Polyakov loop survive in QCD. (Elitzur's Theorem)



N_t =3 case

All the non-closed lines are gauge-variant and their expectation values are zero.

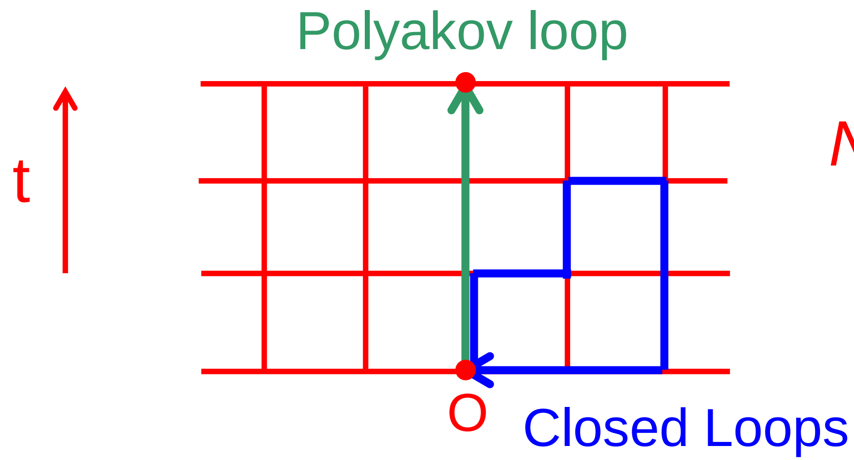
e.g. $\mathrm{Tr}\hat{U}_4\hat{U}_1\hat{U}_{-4} = \sum_x \mathrm{tr}\{U_4(x)U_1(x+\hat{4})U_4^\dagger(x+\hat{1})\} \underline{=0}$

gauge-variant

Tr =0

Temporally Odd-Number Lattice

In general, only gauge-invariant quantities such as Closed Loops and the Polyakov loop survive in QCD. (Elitzur's Theorem)

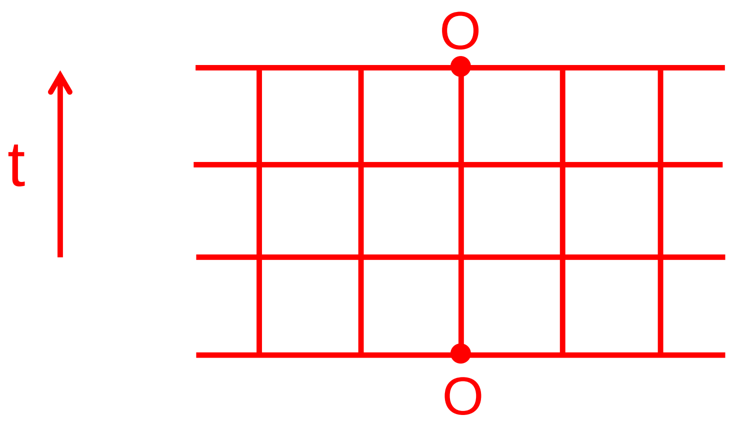


N_t =3 case

All the non-closed lines are gauge-variant and their expectation values are zero.

NB: any closed loop needs even-number link-variables on the square lattice.

Temporally Odd-Number Lattice



N_t =3 case

On the temporally odd-number lattice,
we consider the functional trace:

$$\boxed{\underline{I \equiv \mathrm{Tr}\,\hat{U}_4 \hat{\not{D}}^{N_t-1}} = \sum_x \left\langle x \,|\, \mathrm{tr}\,\hat{U}_4 \hat{\not{D}}^{N_t-1} \,|\, x \right\rangle \equiv \left\langle \mathrm{tr}\,\hat{U}_4 \hat{\not{D}}^{N_t-1} \right\rangle_{space\text{–}time}}$$

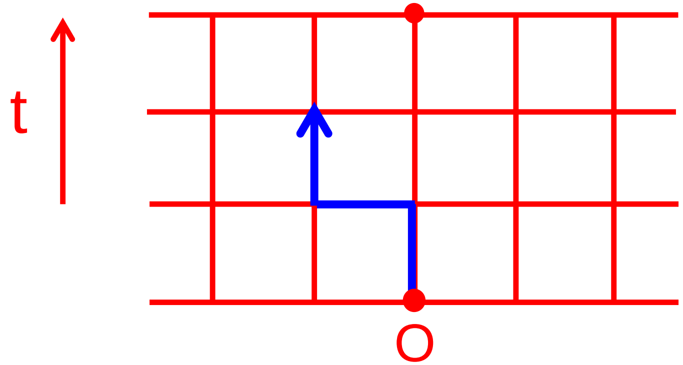
$$\mathrm{Tr} = \sum_x \mathrm{tr}_c\ \mathrm{tr}_\gamma \qquad \mathrm{tr} = \mathrm{tr}_c \mathrm{tr}_\gamma$$

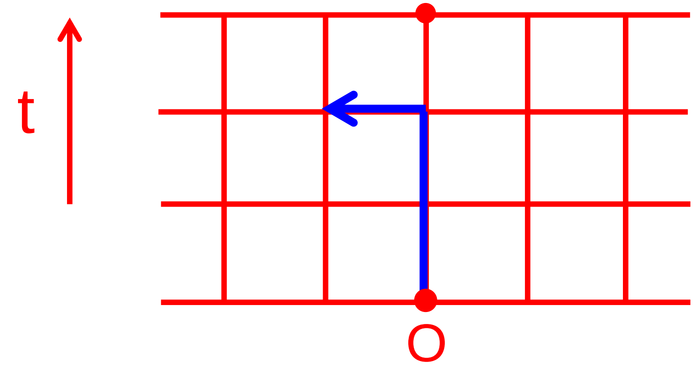
color & spinor

Property on functional trace $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} = \left\langle \mathrm{tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} \right\rangle_{space-time}$

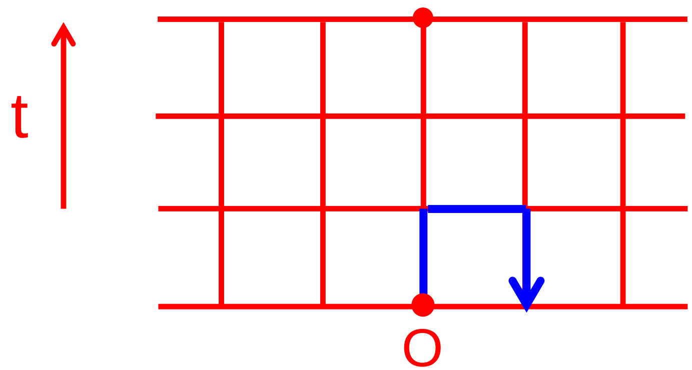
NB: $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ consists of products of N_t link-variable operators, since the Dirac operator $\hat{\not{D}} = \frac{1}{2}\sum_{\mu=1}^{4} \gamma^{\mu} (\hat{U}_{\mu} - \hat{U}_{-\mu})$ includes one link-variable operator in each direction $\pm\mu$.

$I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ includes many trajectories on the square lattice.





N_t =3 case



Any closed loop needs even-number link-variables on the square lattice.

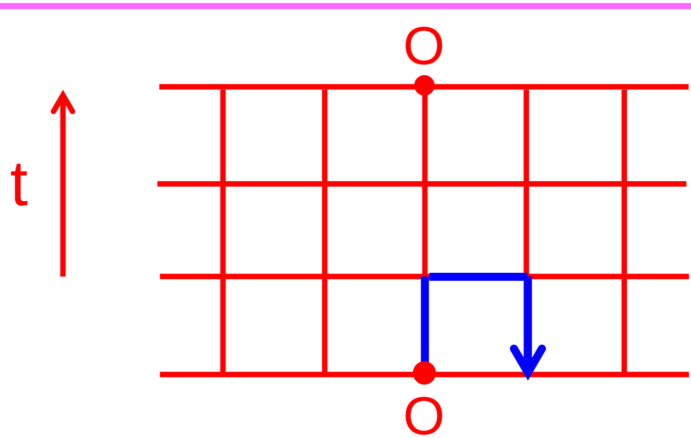
Property on functional trace $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} = \left\langle \mathrm{tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} \right\rangle_{space-time}$

NB: $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ consists of products of N_t link-variable operators, since the Dirac operator $\hat{\not{D}} = \frac{1}{2}\sum_{\mu=1}^{4}\gamma^{\mu}(\hat{U}_{\mu} - \hat{U}_{-\mu})$ includes one link-variable operator in each direction $\pm\mu$.

$I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ includes many trajectories on the square lattice.

In this functional trace $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$,
it is impossible to form a closed loop on the square lattice,
because the total number of the link-variable, N_t, is odd.
Almost all trajectories are gauge-variant & give no contribution.

N_t =3 case



Gauge-variant!

Any closed loop needs even-number link-variables on the square lattice.

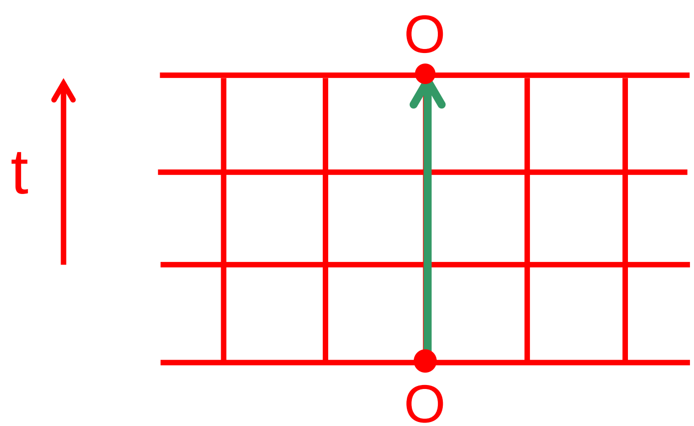
The trace on γ-matrix also gives a constraint.

Property on functional trace $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} = \left\langle \mathrm{tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} \right\rangle_{space-time}$

NB: $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ consists of products of N_t link-variable operators, since the Dirac operator $\hat{\not{D}} = \frac{1}{2}\sum_{\mu=1}^{4} \gamma^{\mu}(\hat{U}_{\mu} - \hat{U}_{-\mu})$ includes one link-variable operator in each direction $\pm\mu$.

$I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ includes many trajectories on the square lattice.

In this functional trace $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$,
it is impossible to form a closed loop on the square lattice,
because the total number of the link-variable, N_t, is odd.
Only the exception is the Polyakov loop.



N_t =3 case

Gauge-invariant!

Any closed loop needs even-number link-variables on the square lattice.

The trace on γ-matrix also gives a constraint.

Property on functional trace $\underline{I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}} = \left\langle \mathrm{tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} \right\rangle_{space\text{-}time}$

NB: $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ consists of products of N_t link-variable operators, since the Dirac operator $\hat{\not{D}} = \frac{1}{2}\sum_{\mu=1}^{4} \gamma^\mu (\hat{U}_\mu - \hat{U}_{-\mu})$ includes one link-variable operator in each direction $\pm\mu$.

$I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ includes many trajectories on the square lattice.

In this functional trace $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$,
it is impossible to form a closed loop on the square lattice,
because the total number of the link-variable, N_t, is odd.
Only the exception is the Polyakov loop.

Therefore, in this functional trace $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$,
only the Polyakov-loop ingredient can survive:

$$I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} = \mathrm{Tr}\hat{U}_4 (\gamma_4 \hat{D}_4)^{N_t-1} = \mathrm{Tr}\hat{U}_4 \hat{D}_4^{N_t-1}$$

$$\propto \mathrm{Tr}\hat{U}_4 (\hat{U}_4 - \hat{U}_{-4})^{N_t-1} = \mathrm{Tr}\hat{U}_4^{N_t} = \mathrm{Tr}\,\hat{L}_P = \left\langle \mathrm{tr}\,\hat{L}_P \right\rangle_{space\text{-}time}$$

$$I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$$

$$= \mathrm{Tr}\hat{U}_4 (\gamma_4 \hat{D}_4)^{N_t-1}$$ (∵ only gauge-invariant quantities survive)

$$= \mathrm{Tr}\hat{U}_4 \hat{D}_4^{N_t-1}$$ ($\because\ \gamma_4^{N_t-1} = 1$, NB: N_t−1 is even)

$$= \frac{1}{2^{N_t-1}} \mathrm{Tr}\hat{U}_4 (\hat{U}_4 - \hat{U}_{-4})^{N_t-1}$$

$$= \frac{1}{2^{N_t-1}} \mathrm{Tr}\hat{U}_4^{N_t}$$ (∵ only gauge-invariant quantities survive)

$$= -\frac{1}{2^{N_t-1}} \mathrm{Tr}\,\hat{L}_P$$ (∵ anti-periodicity of $\hat{D}_4$ in temporal direction)

$$= -\frac{4}{2^{N_t-1}} \left\langle \mathrm{tr}_c\ \hat{L}_P \right\rangle_{space-time}$$ ($\because\ \mathrm{tr}_\gamma 1 = 4$, $\mathrm{Tr} = \sum_{space-time} \mathrm{tr}_c\ \mathrm{tr}_\gamma$)

Thus, $I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1}$ is proportional to Polyakov loop $\left\langle \mathrm{tr}_c\ \hat{L}_P \right\rangle_{space-time}$

On one hand, we obtain

$$\boxed{I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} = -\frac{4}{2^{N_t-1}} \left\langle \mathrm{tr}_c \, \hat{L}_P \right\rangle_{space-time}}$$

On the other hand,
using the complete set of the Dirac eigen-states $|n\rangle$

$$\boxed{I \equiv \mathrm{Tr}\hat{U}_4 \hat{\not{D}}^{N_t-1} = \sum_n \left\langle n \middle| \hat{U}_4 \hat{\not{D}}^{N_t-1} \middle| n \right\rangle = i^{N_t-1} \sum_n \lambda_n{}^{N_t-1} \left\langle n \middle| \hat{U}_4 \middle| n \right\rangle}$$

$$\sum_n |n\rangle\langle n| = 1 \qquad\qquad \hat{\not{D}}|n\rangle = i\lambda_n |n\rangle$$

Combining them, we obtain the analytical relation:

$$\boxed{\left\langle \mathrm{tr}_c \, \hat{L}_P \right\rangle_{space-time} = -\frac{(2i)^{N_t-1}}{4} \sum_n \lambda_n{}^{N_t-1} \left\langle n \middle| \hat{U}_4 \middle| n \right\rangle}$$

[T.M. Doi et al., PRD90 (2014) 09405; PRD92 (2015) 094004; PTEP 2016 (2016) 013B06.]

$$\left\langle \mathrm{tr}_c\, \hat{L}_P \right\rangle_{space-time} = -\frac{(2i)^{N_t-1}}{4} \sum_n \lambda_n^{N_t-1} \left\langle n \middle| \hat{U}_4 \middle| n \right\rangle$$

Each Dirac-mode contribution specified by n can be individually calculated in actual lattice QCD simulations.

NB: the sum of RHS can be expressed with Dirac eigenvalue λ_n , Dirac eigenfunction $\psi_n(x)$, and temporal link-variable $U_4(x)$ as

$$\sum_n \lambda_n^{N_t-1} \left\langle n \middle| \hat{U}_4 \middle| n \right\rangle = \sum_n \lambda_n^{N_t-1} \sum_x \langle n | x \rangle \langle x | \hat{U}_4 | x+\hat{t} \rangle \langle x+\hat{t} | n \rangle$$

$$= \sum_n \lambda_n^{N_t-1} \sum_x \psi_n^\dagger(x) U_4(x) \psi_n(x+\hat{t})$$

Each term is manifestly Gauge Invariant.

Gauge trans. property: $\begin{cases} U_\mu(x) \to V(x) U_\mu(x) V^\dagger(x+\hat{\mu}) \\ \psi_n(x) \to V(x) \psi_n(x) \end{cases}$

Comment: There is no cancellation between chiral-pair Dirac states,

$|n\rangle$ and $\gamma_5 |n\rangle$, because $N_t - 1$ is even and $(-\lambda_n)^{N_t-1} = \lambda_n^{N_t-1}$

$$\left\langle \mathrm{tr}_c\, \hat{L}_P \right\rangle_{space\text{-}time} = -\frac{(2i)^{N_t-1}}{4} \sum_n \underline{\lambda_n^{N_t-1}} \left\langle n \middle| \hat{U}_4 \middle| n \right\rangle$$

As a remarkable fact, because of the *crucial factor* $\lambda_n^{N_t-1}$, the contribution from *small* λ_n region is *negligibly small* in this sum.

(in comparison with other terms with large λ_n)

Here, the matrix element $\langle n|\hat{U}_4|n\rangle$ is generally *nonzero* for each Dirac mode, and does *not* include explicit N_t -dependence.
→The "counter factor" $1/\lambda_n^{N_t-1}$ never arise from $\langle n|\hat{U}_4|n\rangle$.

Comments:

- Even if $\langle n|U_4|n\rangle$ behaves as δ-function δ(λ), the factor $\lambda_n^{N_t-1}$ is still crucial, because of $\lambda\delta(\lambda)= 0$. In fact, without appearance of counter factor from $\langle n|U_4|n\rangle$, the crucial factor $\lambda_n^{N_t-1}$ leads to small contribution for low-lying Dirac modes.
- *If* RHS *were not* a sum *but a product*, the small λ_n region should have given an important contribution, i.e., a critical reduction factor, to the Polyakov loop. However, in the sum, contribution from the small λ_n region is negligible.
- Even in the presence of a possible multiplicative renormalization factor for the Polyakov loop, the low-λ_n contribution is negligible in this sum, relatively in comparison with other terms.

$$\left\langle \mathrm{tr}_c\, \hat{L}_P \right\rangle_{space\text{-}time} = -\frac{(2i)^{N_t-1}}{4} \sum_n \underline{\lambda_n{}^{N_t-1}} \langle n | \hat{U}_4 | n \rangle$$

Comments:

- This relation is correct for any color number N_c and any gauge group.
- This is correct at any temperature and both confined/deconfined phases.
- This is correct regardless of presence or absence of dynamical quarks, although dynamical quark effects appear in Polyakov loop, Dirac eigenvalue distribution, and Dirac-mode matrix element.
- Then, this relation is applicable to finite-density QCD.

This relation obtained on temporally odd-number lattice is expected to hold in the continuum limit of $a \to 0$, $N_t \to \infty$, since any number of large N_t gives the same physical result.

$$\left\langle \mathrm{tr}_c\, \hat{L}_P \right\rangle_{space-time} = -\frac{(2i)^{N_t-1}}{4} \sum_n \underline{\lambda_n{}^{N_t-1}} \left\langle n \middle| \hat{U}_4 \middle| n \right\rangle$$

Lattice QCD calculation

For detail, see T.M. Doi's Talk

Using actual lattice QCD calculations at quenched level, we confirm this analytical relation in both confined and deconfined phases, and also show the negligible contribution of low-lying Dirac modes to the Polyakov loop numerically.

Conclusion

From this relation, the contribution of low-lying Dirac modes to the Polyakov loop is negligibly small, while the low-lying Dirac modes are essential for CSB.

Then, this analytical relation indicates no direct (one-to-one) correspondence between confinement and CSB in QCD.

Local Summary: Analytic Formula between Polyakov Loop and Dirac mode at any Temperature

Setup:

- ordinary square lattice with temporally odd number N_t ($< N_s$)
- normal periodic boundary condition for gluons

Analytical Formula: Dirac spectral representation of Polyakov loop

$$L_P \propto \sum_n \underline{\lambda_n^{N_t-1}} \langle n | \hat{U}_4 | n \rangle$$

L_P : Polyakov Loop, λ_n : Dirac eigenvalue, $|n\rangle$: Dirac eigenmode

Result: small contribution of low-lying Dirac modes to Polyakov loop

References:

[1] T. M. Doi, H. S., T. Iritani, Phys. Rev. D90 (2014) 094505.

[2] T. M. Doi, K. Redlich, C. Sasaki, H. S., Phys. Rev. D92 (2015) 094004.

[3] H. S., T. M. Doi, T. Iritani, PTEP 2016 (2016) 013B06.

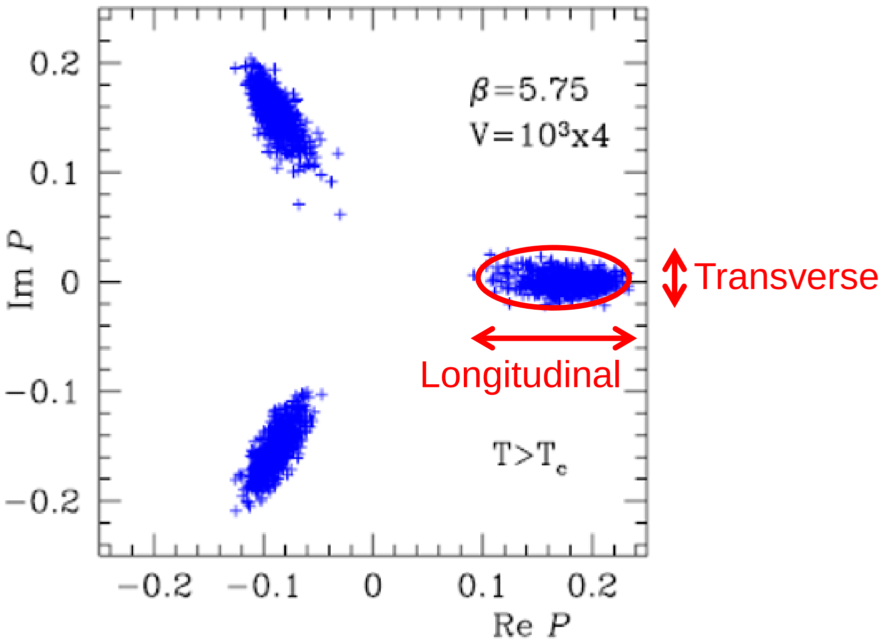
3. Analytic Formula of Polyakov Loop Fluctuation with Dirac mode near T_c

This subject will be mainly presented by T.M.Doi's Talk.

T. M. Doi, K. Redlich, C. Sasaki and H.S., PRD92 (2015) 094004,
"Polyakov Loop Fluctuations in the Dirac Eigenmode Expansion".

Polyakov loop fluctuation is paid attention for an indicator of the QCD transition.

P.M. Lo, B. Friman, O. Kaczmarek, K. Redlich, C.Sasaki, PRD88 (2013) 074502.



Scatter plot of Polyakov loop

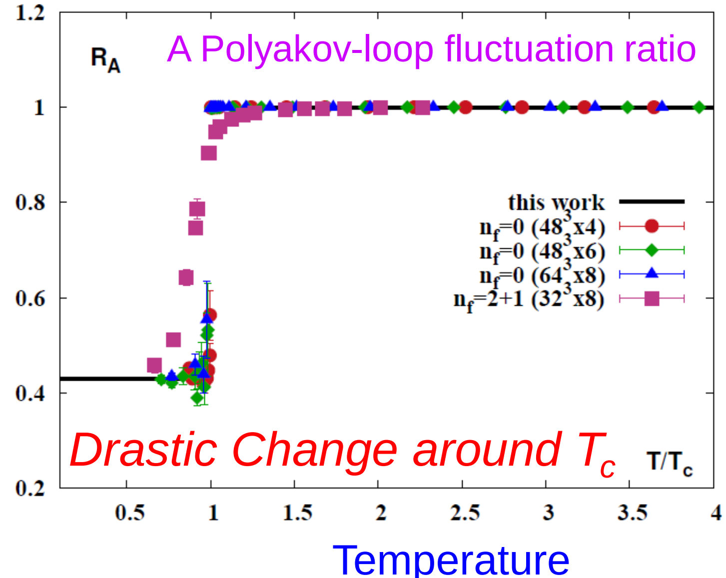


FIG. 3: The ratio of the absolute (5) to longitudinal (6) part of the Polyakov loop susceptibilities calculated within lattice gauge theory for pure gauge system and (2+1)-flavor QCD (see text). The temperature is normalized to its (pseudo) critical value for respective lattice. The line is the model result explained in the text.

3. Analytic Formula of Polyakov Loop Fluctuation with Dirac mode

We derive Dirac-mode expansion formula for Polyakov loop fluctuations, and find a *small* contribution of low-lying Dirac modes to them.

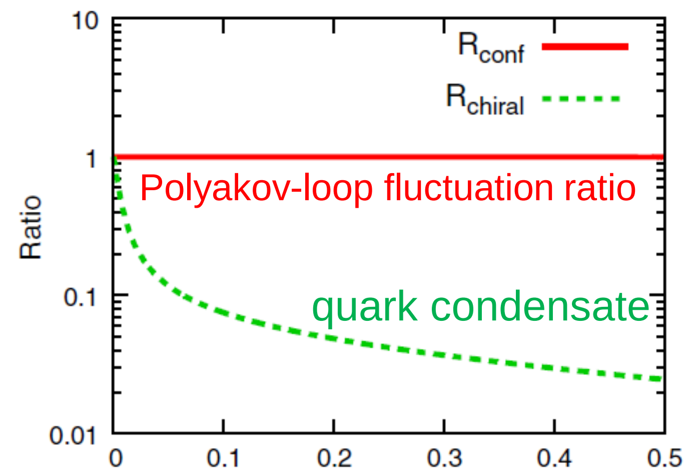
T. M. Doi, K. Redlich, C. Sasaki and H.S., PRD92 (2015) 094004,
"Polyakov Loop Fluctuations in the Dirac Eigenmode Expansion".

Dirac mode representation of Polyakov loop fluctuation:

$$R_A = \frac{\langle |\sum_n \lambda_n^{N_\tau -1} \langle n|\hat{U}_4|n\rangle|^2 \rangle - \langle |\sum_n \lambda_n^{N_\tau -1} \langle n|\hat{U}_4|n\rangle| \rangle^2}{\langle (\sum_n \lambda_n^{N_\tau -1} \mathrm{Re}(e^{2\pi k i/3} \langle n|\hat{U}_4|n\rangle))^2 \rangle - \langle \sum_n \lambda_n^{N_\tau -1} \mathrm{Re}(e^{2\pi k i/3} \langle n|\hat{U}_4|n\rangle) \rangle^2},$$

Lattice QCD result of a Polyakov-loop fluctuation ratio plotted with IR cut of Dirac modes in a confined phase.

By removing low-lying Dirac modes, quark condensate is rapidly reduced, but the Polyakov-loop fluctuation is almost unchanged.



Λ_{IRcut} [a^{-1}] IR cut of Dirac modes

FIG. 5 (color online). The numerical results for the R_{chiral} and R_{conf} ratio from Eqs. (38) and (39), respectively, as a function of an infrared cutoff Λ introduced on Dirac eigenvalues, expressed in lattice units. The Monte Carlo calculations have been performed on the $10^3 \times 5$ lattice at $\beta = 5.6$ and for the quark mass of $m = 5$ MeV.

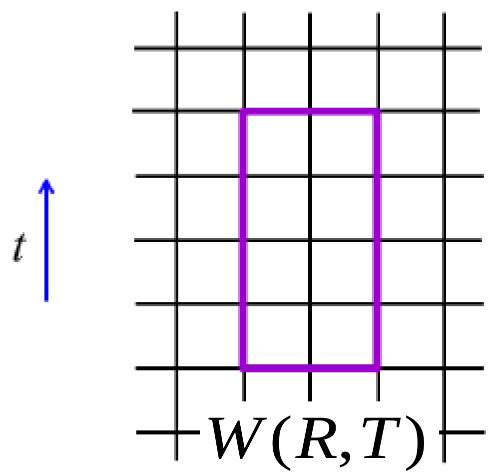
4. Analytical Relation between Wilson loop and Dirac mode at zero-temperature

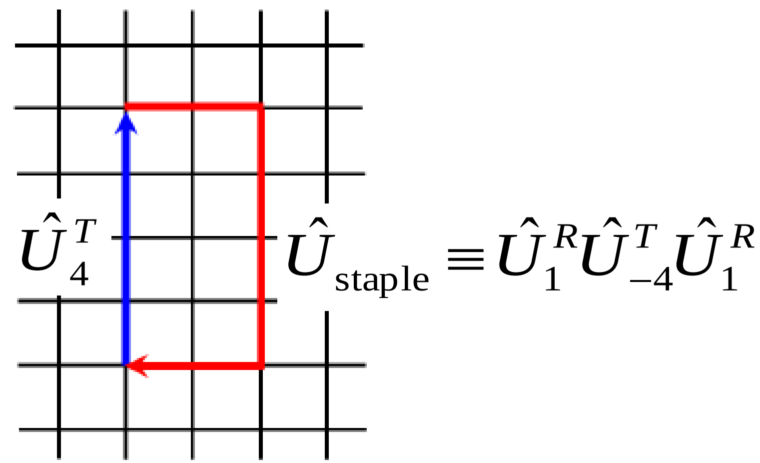
Reference: *H. S., T. M. Doi, T. Iritani, PTEP 2016 (2016) 013B06, Analytical Formulae of the Polyakov and Wilson Loops with Dirac Eigenmodes in Lattice QCD.*

- arbitrary square lattice (Even number lattice is OK.)
- *Elitzur Th.* : only *gauge-invariant quantities* have *nonzero* expectation value

Wilson loop : $W(R,T) \equiv \mathrm{Tr}\{\hat{U}_1^R \hat{U}_{-4}^T \hat{U}_1^R \hat{U}_4^T\} = \mathrm{Tr}\{\hat{U}_{\mathrm{staple}} \hat{U}_4^T\}$

Staple operator : $\hat{U}_{\mathrm{staple}} \equiv \hat{U}_1^R \hat{U}_{-4}^T \hat{U}_1^R$





We consider $\boxed{J \equiv \mathrm{Tr}\{\hat{U}_{\mathrm{staple}} \hat{\not{D}}^T\}}$ (T : even)

$$J \equiv \mathrm{Tr}\,\hat{U}_{\mathrm{staple}} \hat{\not{D}}^{T}$$

$$= \mathrm{Tr}\,\hat{U}_{\mathrm{staple}} (\gamma_4 \hat{D}_4)^{T}$$ (∵ only gauge-invariant quantities survive)

$$= \mathrm{Tr}\,\hat{U}_{\mathrm{staple}} \hat{D}_4^{T}$$ ($\because\ \gamma_4^{N_t - 1} = 1$, T is even)

$$= \frac{1}{2^T} \mathrm{Tr}\,\hat{U}_{\mathrm{staple}} (\hat{U}_4 - \hat{U}_{-4})^{T}$$

$$= \frac{1}{2^T} \mathrm{Tr}\,\hat{U}_{\mathrm{staple}} \hat{U}_4^{T}$$ (∵ only gauge-invariant quantities survive)

($\because\ \mathrm{tr}_\gamma 1 = 4,\ \mathrm{Tr} = \sum_x \mathrm{tr}_c\ \mathrm{tr}_\gamma$)

$$= \frac{4}{2^T} W$$ ($\because\ W = \mathrm{Tr}\{\hat{U}_{\mathrm{staple}} \hat{U}_4^{T}\}$)

Thus, $J \equiv \mathrm{Tr}\{\hat{U}_{\mathrm{staple}} \hat{\not{D}}^{T}\}$ is proportional to Wilson loop W

On one hand, we obtain for even T

$$\boxed{J \equiv \mathrm{Tr}\hat{U}_{\mathrm{staple}}\hat{\not{D}}^T = \frac{4}{2^T} W}$$

On the other hand, using the complete set of the Dirac eigen-states $|n\rangle$

$$\boxed{J \equiv \mathrm{Tr}\hat{U}_{\mathrm{staple}}\hat{\not{D}}^T = \sum_n \langle n|\hat{U}_{\mathrm{staple}}\hat{\not{D}}^T|n\rangle = (-)^{T/2}\sum_n \lambda_n{}^T \langle n|\hat{U}_{\mathrm{staple}}|n\rangle}$$

$$\sum_n |n\rangle\langle n| = 1 \qquad \hat{\not{D}}|n\rangle = i\lambda_n|n\rangle$$

Combining them, we obtain a relation for even T:

$$\boxed{W = (-)^{T/2} 2^{T-2} \sum_n \lambda_n{}^T \langle n|\hat{U}_{\mathrm{staple}}|n\rangle}$$

$$\rightarrow \quad \boxed{V(R) = -\lim_{T\to\infty}\frac{1}{T}\ln W = -\lim_{T\to\infty}\frac{1}{T}\ln\left|\sum_n (2\lambda_n)^T \langle n|\hat{U}_{\mathrm{staple}}|n\rangle\right|}$$

$$\rightarrow \quad \boxed{\sigma = -\lim_{R,T\to\infty}\frac{1}{RT}\ln W = -\lim_{R,T\to\infty}\frac{1}{RT}\ln\left|\sum_n (2\lambda_n)^T \langle n|\hat{U}_{\mathrm{staple}}|n\rangle\right|}$$

$$W = (-)^{T/2} 2^{T-2} \sum_n \lambda_n^{\ T} \left\langle n \,|\, \hat{U}_{\text{staple}} \,|\, n \right\rangle$$

$$\sigma = -\lim_{R,T\to\infty} \frac{1}{RT} \ln W = -\lim_{R,T\to\infty} \frac{1}{RT} \ln\left| \sum_n (2\lambda_n)^T \left\langle n \,|\, \hat{U}_{\text{staple}} \,|\, n \right\rangle \right|$$

Because of the factor $\lambda_n^{\ T}$ in the sum,
low-lying Dirac-mode contribution is to be small
for the Wilson loop *W*, the inter-quark potential V(R) and
the string tension σ, unless the extra counter factor $1/\lambda_n^{\ T}$
appears from the matrix element $\left\langle n \,|\, \hat{U}_{\text{staple}} \,|\, n \right\rangle$.

Thus, the string tension σ, or the confining force,
is expected to be unchanged by the removal of
low-lying Dirac-mode contribution.

Local Summary: Analytical Relation between Wilson loop and Dirac mode

Setup:

- Arbtitary square lattice

Analytic formla of Wilson loop with Dirac modes

$$W(R,T) \propto \sum_n \underline{\lambda_n{}^T} \left\langle n \,|\, \hat{U}_{\text{staple}} \,|\, n \right\rangle$$

Wilson loop : $W(R,T) \equiv \mathrm{Tr}\{\hat{U}_1^R \hat{U}_{-4}^T \hat{U}_1^R \hat{U}_4^T\} = \mathrm{Tr}\{\hat{U}_{\text{staple}} \hat{U}_4^T\}$

λ_n : Dirac eigenvalue, $|n\rangle$: Dirac eigenmode

Result: small contribution of low-lying Dirac modes to the string tension or the quark confining force

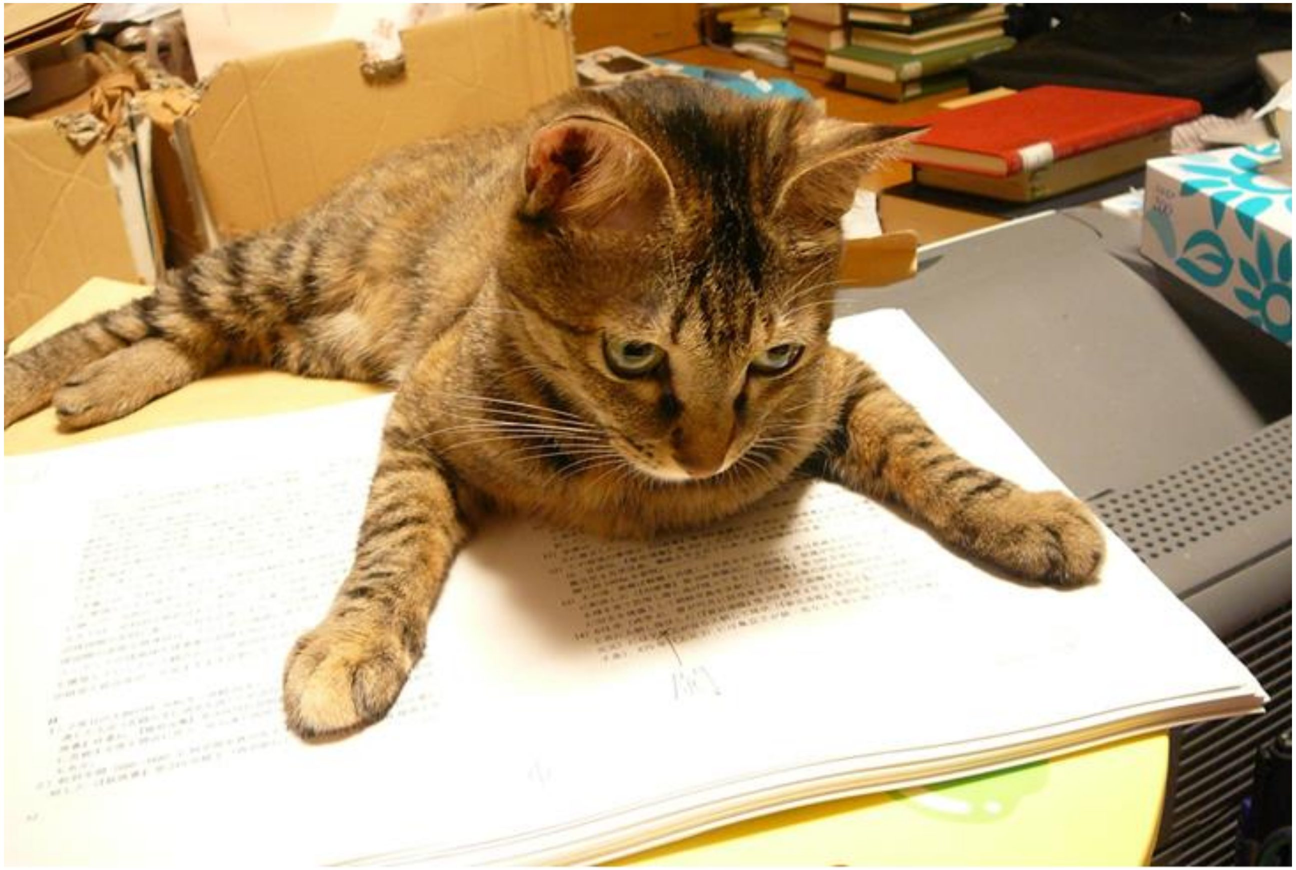
The above formulae are *mathematically valid*.

Because, we have just used the Elitzur theorem and the identity of completeness: $\sum_n |n\rangle\langle n| = 1$

However, one may wonder the doubler in the use of simple Lattice Dirac operator: $\hat{\not{D}} = \frac{1}{2a}\sum_{\mu=1}^{4}\gamma^{\mu}(\hat{U}_{\mu} - \hat{U}_{-\mu})$

5. Formula for Wilson, Clover, Domain-Wall fermions

Let us remove the doublers !



With T. M. Doi

5-1. Formula for Wilson and Clover fermions

Clover fermion: O(a)-improved Wilson fermion

Setup:

- ordinary square lattice with $N_t = 4l+1$ ($< N_s$)
- normal periodic boundary condition for gluons

- *Elitzur Th.* : only *gauge-invariant quantities* have *nonzero* expectation value

Analytic Formula**:** Dirac spectral representation of Polyakov loop

$$L_P \propto \sum_n \underline{\lambda_n^{2l}} \langle n | \hat{U}_4^{2l+1} | n \rangle$$

L_P : Polyakov Loop**,** λ_n : Dirac eigenvalue**,** $|n\rangle$: Dirac eigenmode

Result: small contribution of low-lying Dirac modes to Polyakov loop

Wilson and Clover fermions

Wilson Fermion Kernel in lattice formalism:

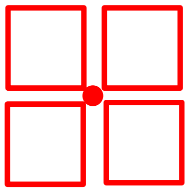
$$\boxed{K = \frac{1}{2a}\gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu}) + \frac{r}{2a}\sum_{\mu=1}^{4}(\hat{U}_\mu + \hat{U}_{-\mu} - 2) + m}$$

Clover Fermion Kernel in lattice formalism:

$$\boxed{K = \frac{1}{2a}\gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu}) + \frac{r}{2a}\sum_{\mu=1}^{4}(\hat{U}_\mu + \hat{U}_{-\mu} - 2) + ar\frac{1}{2}g\sigma_{\mu\nu}G_{\mu\nu} + m}$$

Clover fermion: O(a)-improved Wilson fermion

Here, Field Strength is defined as Clover-type:



$$G_{\mu\nu} = \frac{1}{8}(P_{\mu\nu} - P^\dagger_{\mu\nu})$$

$$P_{\mu\nu}(s) = \langle s|\hat{U}_\mu\hat{U}_\nu\hat{U}_{-\mu}\hat{U}_{-\nu} + \hat{U}_\nu\hat{U}_{-\mu}\hat{U}_{-\nu}\hat{U}_\mu + \hat{U}_{-\mu}\hat{U}_{-\nu}\hat{U}_\mu\hat{U}_\nu + \hat{U}_{-\nu}\hat{U}_\mu\hat{U}_\nu\hat{U}_{-\mu}|s\rangle$$

Details

Derivation of O(a)-improved Wilson fermion: Clover fermion

Consider the transformation of fermion field variables:

$$\psi = (1+\frac{c}{2}\not{D})\psi' \qquad \bar{\psi} = \bar{\psi}'(1+\frac{c}{2}\not{D}) \qquad c = O(a) \in \mathrm{R}$$

Fermion action is transformed as

$$\bar{\psi}(\not{D}+m)\psi = \bar{\psi}'(1+\frac{c}{2}\not{D})(\not{D}+m)(1+\frac{c}{2}\not{D})\psi' = \bar{\psi}'\{(1+cm)\not{D}+c\not{D}^2+m\}\psi' + O(c^2)$$
$$= \bar{\psi}''\{\not{D}+c'\not{D}^2+m'\}\psi'' + O(a^2)$$

Fermion variable measure is transformed as

$$D\psi D\bar{\psi} = D\psi' D\bar{\psi}' \det{}^{-2}(1+\frac{c}{2}\not{D}) = D\psi' D\bar{\psi}'(1+O(c^2)) = D\psi'' D\bar{\psi}''(1+cm)\{1+O(a^2)\}$$

Fermion generating functional up to O(a):

$$Z = \int D\psi D\bar{\psi} e^{-\int \bar{\psi}(\not{D}+m)\psi} = \int D\psi'' D\bar{\psi}'' e^{-\int \bar{\psi}''(\not{D}+c'\not{D}^2+m')\psi''}$$

Details

Derivation of O(a)-improved Wilson fermion: Clover fermion

Fermion generating functional up to O(a):

$$Z=\int D\psi D\overline{\psi}e^{-\int\overline{\psi}(\not{D}+m)\psi}=\int D\psi' D\overline{\psi}' e^{-\int\overline{\psi}''(\not{D}+c'\not{D}^2+m')\psi''}$$

Clover fermion action: O(a)-improved Wilson fermion

$$\boxed{\begin{aligned}S&=\int d^4x\,\overline{\psi}(\not{D}+c\not{D}^2+m)\psi\\&=\int d^4x\,\overline{\psi}(\not{D}+c\{D^2+\frac{1}{2}g\sigma_{\mu\nu}G_{\mu\nu}\}+m)\psi\end{aligned}}$$

$$c=O(a)\in\mathrm{R}$$

cf. Wilson fermion removes doublers but includes O(a) error:

$$S_W=\int d^4x\,\overline{\psi}(\not{D}+cD^2+m)\psi$$

Details

Lattice Clover fermion: O(a)-improved Wilson fermion

Fermion Kernel of Clover fermion:

$$\boxed{K = \not{D} + c\{D^2 + \frac{1}{2} g\sigma_{\mu\nu} G_{\mu\nu}\} + m} \qquad c = O(a) \in \mathrm{R}$$

Clover Fermion Kernel in lattice formalism:

$$\boxed{K = \frac{1}{2a}\gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu}) + \frac{r}{2a}\sum_{\mu=1}^{4}(\hat{U}_\mu + \hat{U}_{-\mu} - 2) + ar\frac{1}{2} g\sigma_{\mu\nu} G_{\mu\nu} + m}$$

Dirac term Wilson term Clover term

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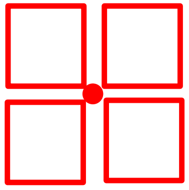

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$$P_{\mu\nu}(s) = \left\langle s \middle| \hat{U}_\mu \hat{U}_\nu \hat{U}_{-\mu} \hat{U}_{-\nu} + \hat{U}_\nu \hat{U}_{-\mu} \hat{U}_{-\nu} \hat{U}_\mu + \hat{U}_{-\mu} \hat{U}_{-\nu} \hat{U}_\mu \hat{U}_\nu + \hat{U}_{-\nu} \hat{U}_\mu \hat{U}_\nu \hat{U}_{-\mu} \middle| s \right\rangle$$

Formula for Wilson and Clover fermions

WIlson (or Clover) Fermion Kernel in lattice formalism:

$$K = \frac{1}{2a}\gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu}) + \frac{r}{2a}\sum_{\mu=1}^{4}(\hat{U}_\mu + \hat{U}_{-\mu} - 2) + \left(ar\frac{1}{2}g\sigma_{\mu\nu}G_{\mu\nu}\right) + m$$

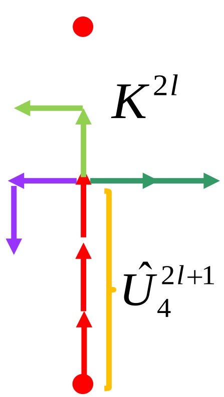
On square lattice with $N_t = 4l + 1$

Consider $J \equiv \mathrm{Tr}\{\hat{U}_4^{2l+1}K^{2l}\}$

cf. Appendix of T. M. Doi, H. S., T. Iritani, PRD90 (2014) 094505.

The fermion kernel *K* includes many terms. However, to form *Loop*, you have to select only U_4 term in all 2*l* *K* in the Trace *J*.

eg $N_t = 5\ (l = 1)$ case



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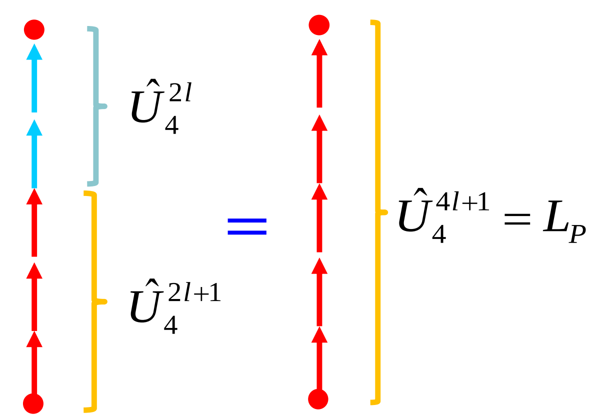

The Only Loop in

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Then

$$J = \mathrm{Tr}\left[\hat{U}_4^{2l+1}\left(\frac{1}{2a}\gamma_4\hat{U}_4 + \frac{r}{2a}\hat{U}_4\right)^{2l}\right] = \frac{1}{(2a)^{2l}}\mathrm{Tr}\left[\hat{U}_4^{4l+1}(\gamma_4 + r)^{2l}\right]$$

NB: At this stage, non-commutable nature disappears.

By taking Tr, all the odd γ_4 terms vanishes:

$$J = \frac{1}{(2a)^{2l}}[(1+r)^{2l} + (1-r)^{2l}]\mathrm{Tr}(\hat{U}_4^{4l+1}) \propto L_P$$

Thus, *J* is found to be proportional to *Polyakov loop*.

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On one hand, *J* is proportional to *Polyakov loop*.

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On the other hand, using the eigenmode of $K|n\rangle = i\lambda_n |n\rangle$

$$J = \sum_n \langle n|\hat{U}_4^{2l+1} K^{2l}|n\rangle = \sum_n (i\lambda_n)^{2l} \langle n|\hat{U}_4^{2l+1}|n\rangle$$

Then

$$\boxed{L_P \propto \sum_n \lambda_n^{2l} \langle n \,|\, \hat{U}_4^{2l+1} \,|\, n\rangle}$$

→ Similar arguments are applicable.

Local Summary: Formula for Wilson and Clover fermions

Clover fermion: O(a)-improved Wilson fermion

Setup:

- ordinary square lattice with $N_t = 4l+1$ ($< N_s$)
- normal periodic boundary condition for gluons

Analytic Formula: Dirac spectral representation of Polyakov loop

$$L_P \propto \sum_n \underline{\lambda_n^{2l}} \langle n | \hat{U}_4^{2l+1} | n \rangle$$

L_P : Polyakov Loop, λ_n : Dirac eigenvalue, $|n\rangle$: Dirac eigenmode

Result: small contribution of low-lying Dirac modes to Polyakov loop

5-2. Formula for Domain-Wall fermion

Domain-Wall (DW) fermion: lattice fermion with “exact” chiral symmetry

Setup:

- ordinary square lattice with $N_t = 4l+1$ ($< N_s$)
- normal periodic boundary condition for gluons

- *Elitzur Th.* : only *gauge-invariant quantities* have *nonzero* expectation value

Analytic Formula**:** Dirac spectral representation of Polyakov loop

$$L_P \propto \sum_\nu \Lambda_\nu^{2l} \left\langle \nu \,|\, \hat{U}_4^{2l+1} \,|\, \nu \right\rangle = \sum_\nu \underline{\lambda_{n_\nu}^{2l}} \left\langle n_\nu \,|\, \hat{U}_4^{2l+1} \,|\, n_\nu \right\rangle + O(M_0^{-2})$$

L_P: Polyakov Loop, λ : Dirac eigenvalue, $|n\rangle$: Dirac eigenmode

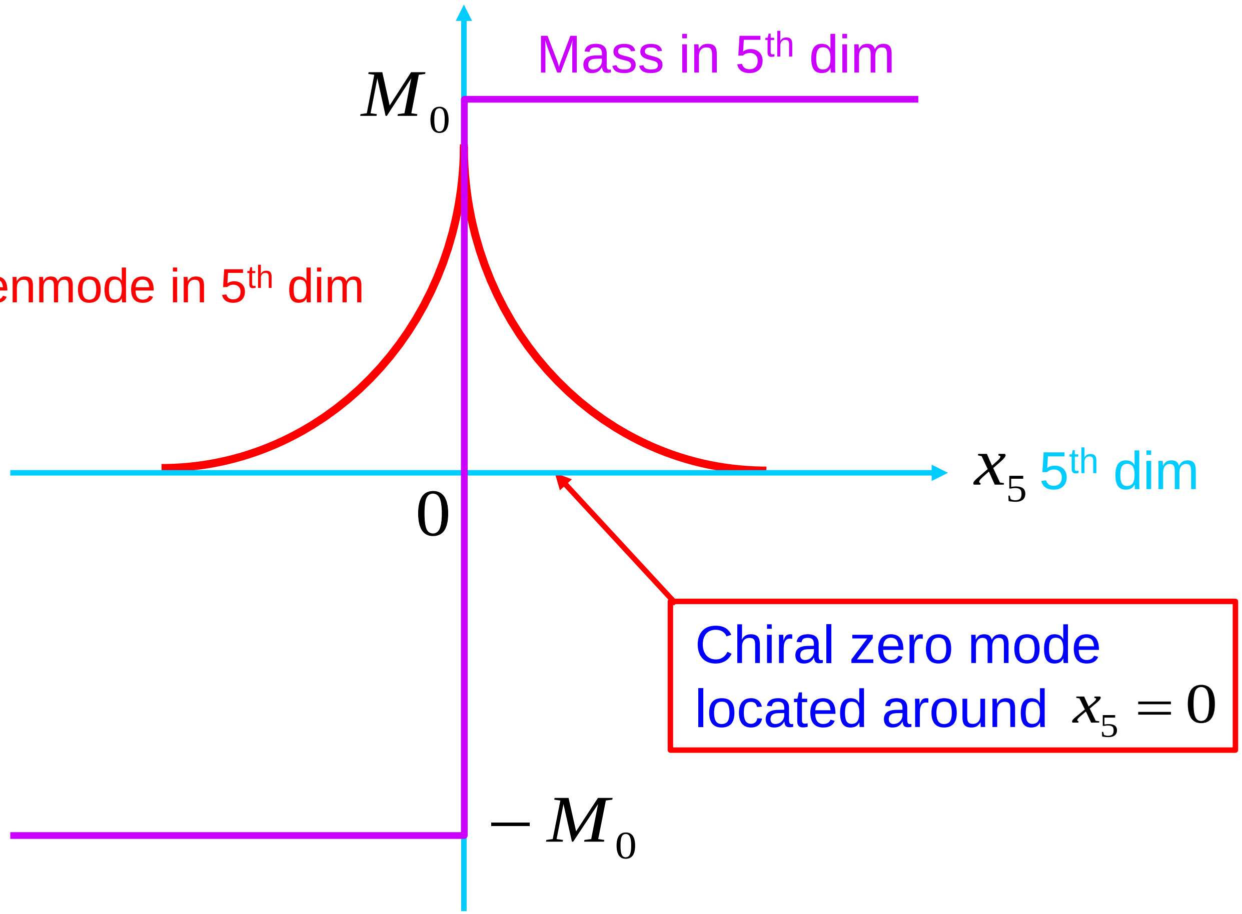
M_0: large DW mass of O(1/a)

Result: small contribution of low-lying Dirac modes to Polyakov loop

Domain-Wall (DW) fermion: lattice fermion with "exact" chiral symmetry

By introducing 5th dimension, "exact" chiral symmetry is realized.

Mass term in 5th dim: $\boxed{M(x_5) = M_0\,\mathrm{sgn}(x_5)}$ $M_0 \approx a^{-1}$: *very large*



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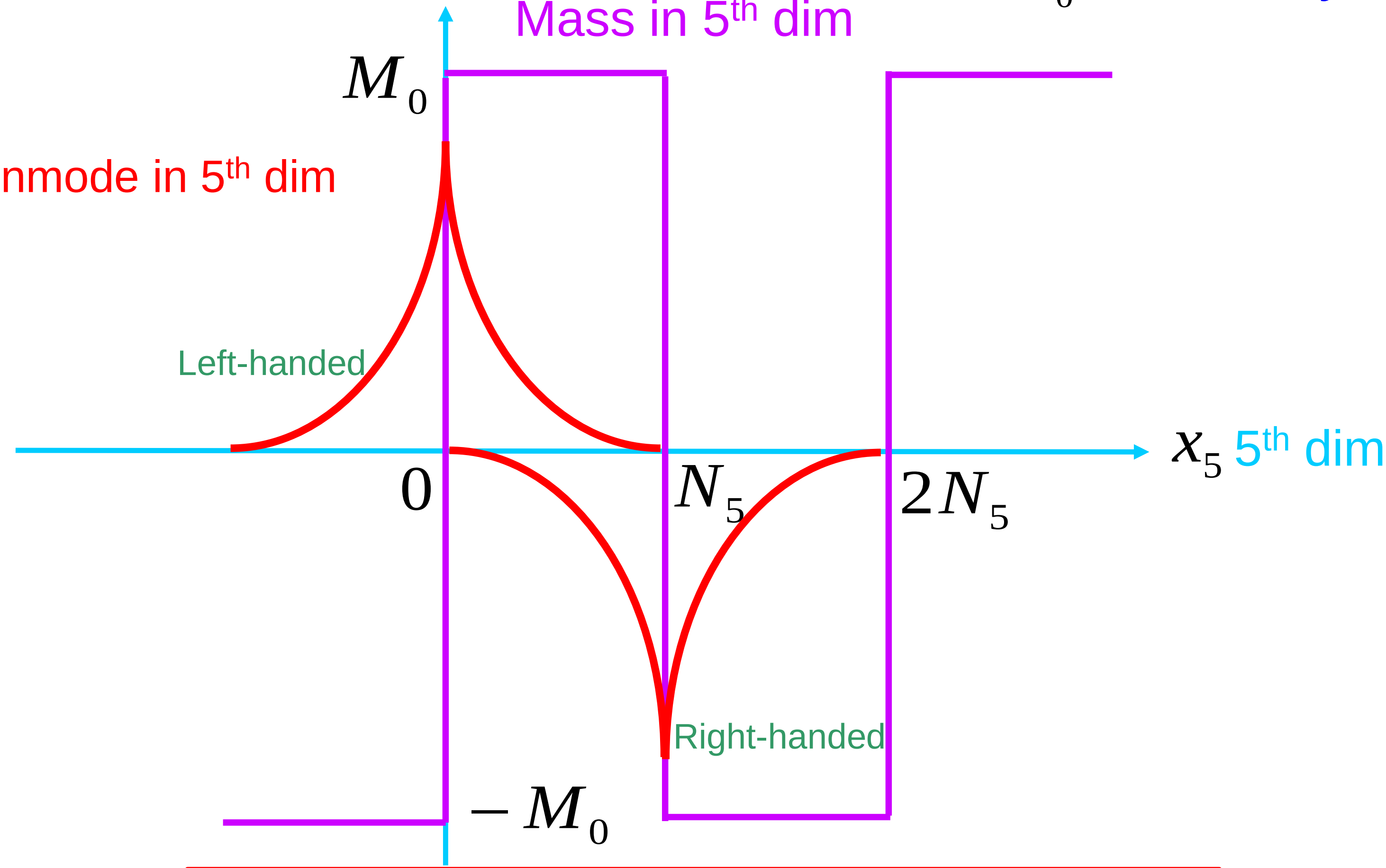

Chiral zero mode localized around $x_5 = 0, N_5$

Domain-Wall (DW) fermion: lattice fermion with "exact" chiral symmetry

Fermion Kernel of DW fermion:

$$\boxed{K = \not{D} + c\{D^2 + \frac{1}{2} g\sigma_{\mu\nu}G_{\mu\nu}\} + m + \underline{\gamma_5\partial_5 + M(x_5)}} \qquad c = O(a) \in \mathrm{R}$$

Only kinetic and mass terms in 5th dim.

DW Fermion Kernel in lattice formalism:

$$\boxed{K = \frac{1}{2a}\gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu}) + \frac{r}{2a}\sum_{\mu=1}^{4}(\hat{U}_\mu + \hat{U}_{-\mu} - 2) + ar\frac{1}{2} g\sigma_{\mu\nu}G_{\mu\nu} + m + \underline{\gamma_5\hat{\partial}_5 + M(x_5)}}$$

Dirac term Wilson term Clover term 5th-dim term

In 5th dim, No coupling
with Gluons (Link-variables)

5th dim has only kinetic and mass terms, so that it is solvable in 5th direction. The extra degrees of freedom in 5th dim. is integrated out in the generating functional. The UV divergence can be removed by Pauli-Villars regularization.

Formula for Domain-Wall (DW) fermion

DW Fermion 5-dim Kernel in lattice formalism:

$$K = \frac{1}{2a}\gamma_\mu(\hat{U}_\mu - \hat{U}_{-\mu}) + \frac{r}{2a}\sum_{\mu=1}^{4}(\hat{U}_\mu + \hat{U}_{-\mu} - 2) + ar\frac{1}{2}g\sigma_{\mu\nu}G_{\mu\nu} + m + \gamma_5\hat{\partial}_5 + M(x_5)$$

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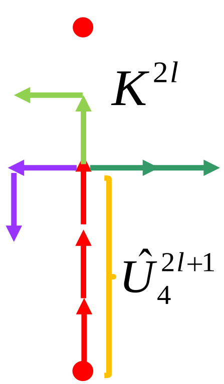

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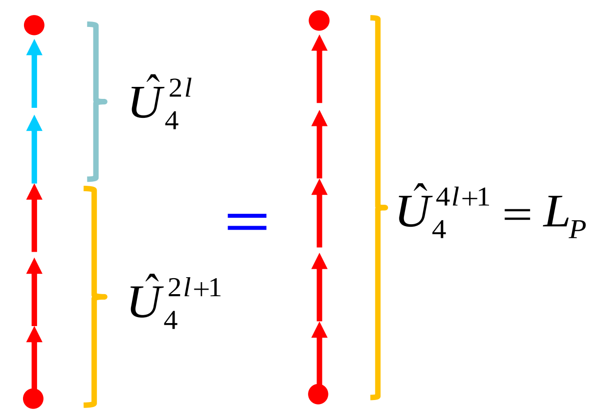

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Then $$\boxed{L_P \propto \sum_n \Lambda_n^{2l} \langle n \,|\, \hat{U}_4^{2l+1} \,|\, n \rangle}$$

5-dim kernel eigenvalue Λ ≒ 4-dim physical Dirac eigenvalue λ

$$\lambda_{n_\nu} = \Lambda_\nu + O(M_0^{-2})$$

$$\boxed{L_P \propto \sum_\nu \Lambda_\nu^{2l} \langle \nu \,|\, \hat{U}_4^{2l+1} \,|\, \nu \rangle = \sum_\nu \lambda_{n_\nu}^{2l} \langle \nu \,|\, \hat{U}_4^{2l+1} \,|\, \nu \rangle + O(M_0^{-2})}$$

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Local Summary: Formula for Domain-Wall fermion

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L_P : Polyakov Loop**,** λ : Dirac eigenvalue, $|n\rangle$: Dirac eigenmode

M_0: large DW mass of O(1/a)

Result: small contribution of low-lying Dirac modes to Polyakov loop

Discussion

It seems natural to consider *No direct one-to-one correspondence* between confinement and chiral symmetry breaking in QCD. Actually, independent of quark mass, confinement is realized. In fact, even without chiral symmetry, color confinement occurs.

How about "coincidence" of Deconfinement and Chiral Restoration temperatures ?

In general, around the (pseudo)critical point, all the physical quantities can be largely changed, according to a drastic change of order parameter.

If there is some independence between Confinement and Chiral Sym Breaking ,Various Phase Structure of QCD can be expected in Various Circumstances (T, μ, H,…) .


Thank you!