Heavy flavors and exotic hadrons

EXERCISE

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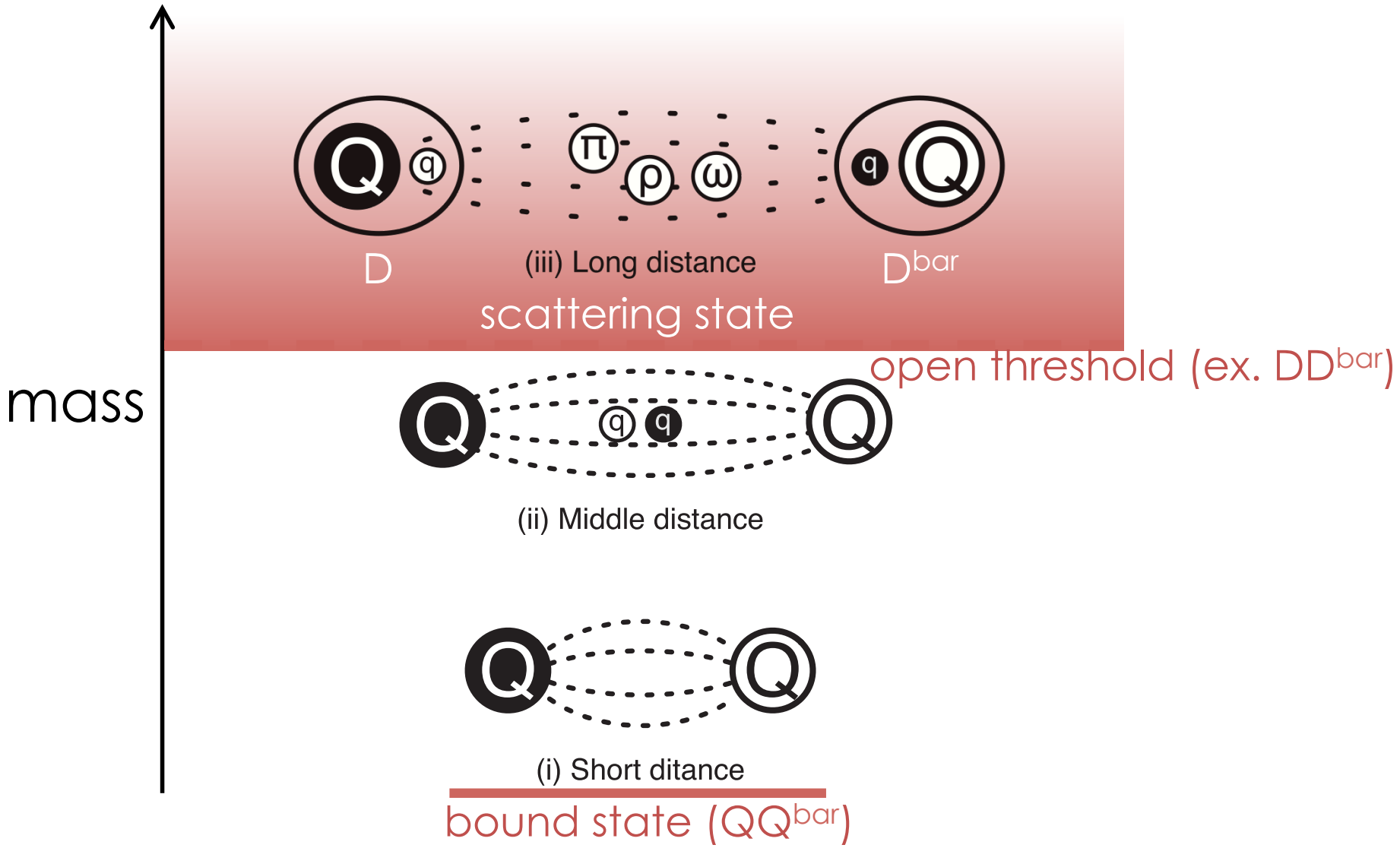
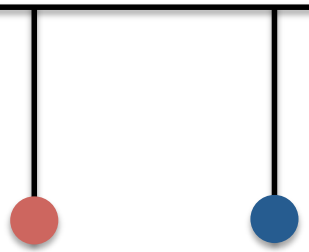
Tokyo Institute of Technology

53th Karpacz Winter School of Theoretical Physics and THOR COST Action Training School
@Karpacz, Poland 26 Feb. – 4 Mar. 2017

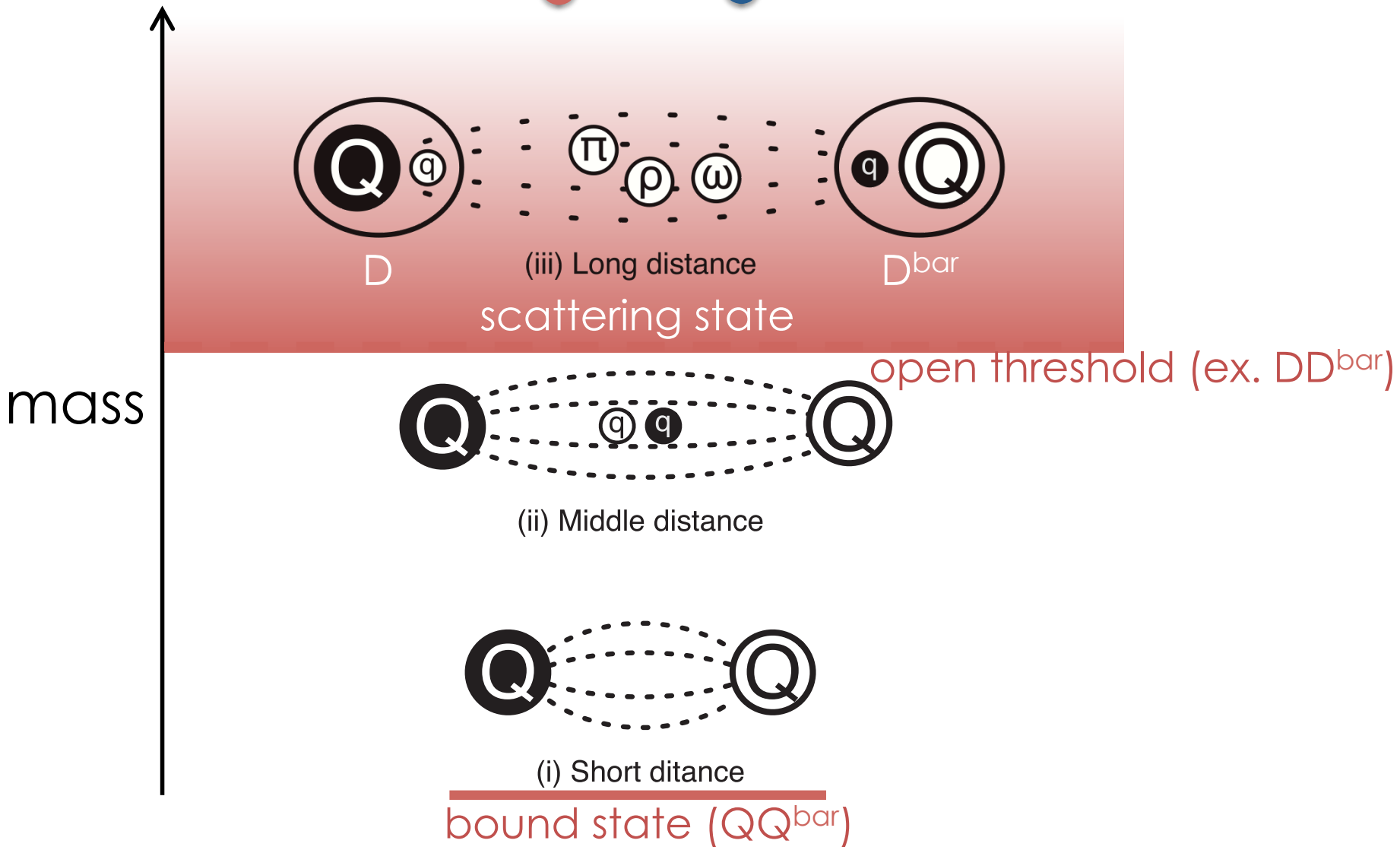
Resonance

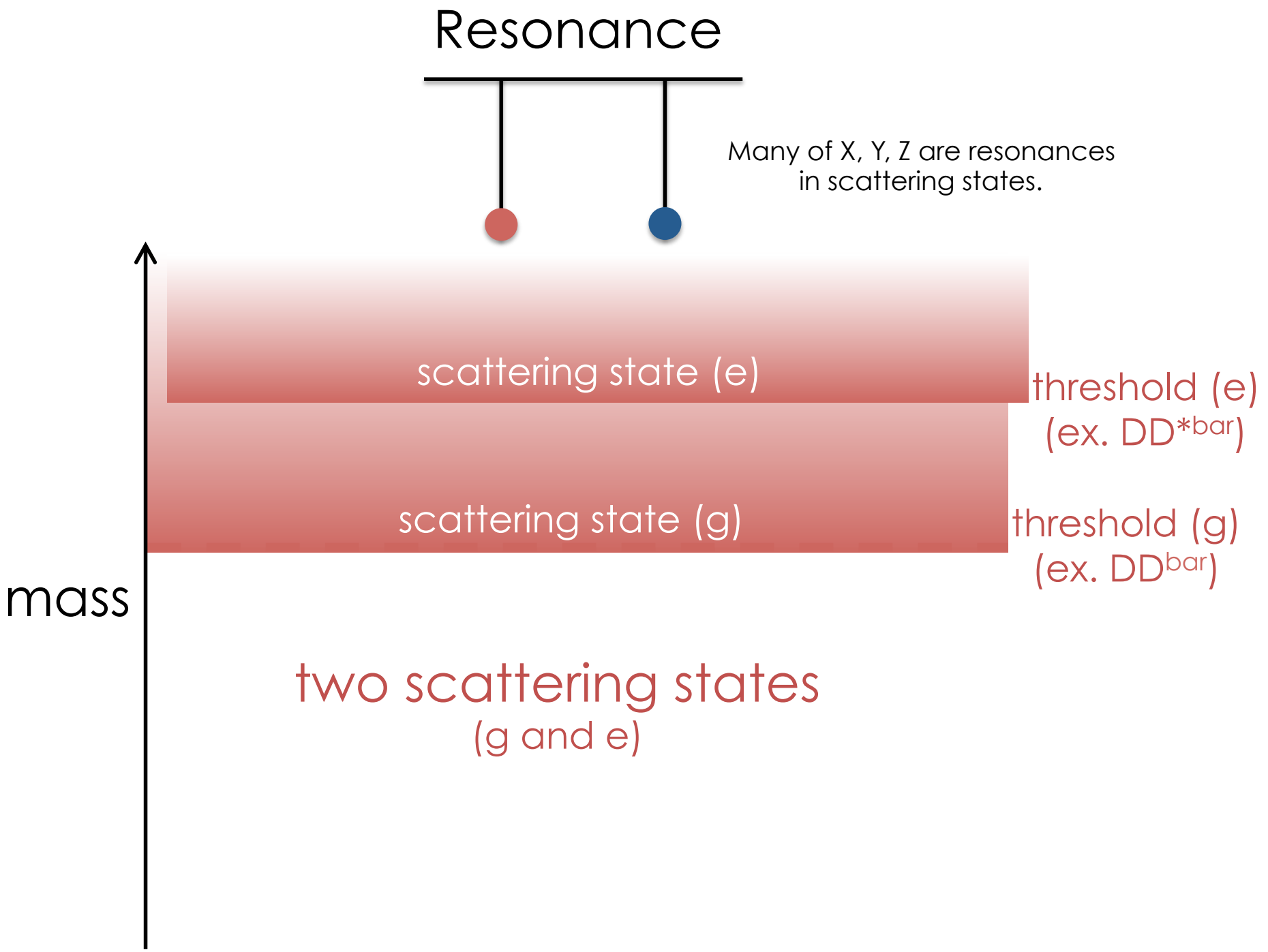

mass
Q
q
π
ρ
ω
q
Q
D
(iii) Long distance
D^bar
scattering state
open threshold (ex. DD^bar)
Q
q
q
Q
(ii) Middle distance
Q
Q
(i) Short ditance
bound state (QQ^bar)

Resonance

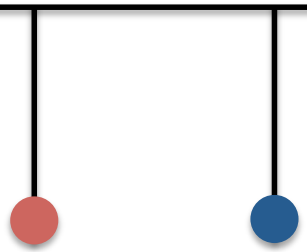


Many of X, Y, Z are resonances
in scattering states.




Resonance
Many of X, Y, Z are resonances
in scattering states.
scattering state (e)
threshold (e)
(ex. DD*bar)
scattering state (g)
threshold (g)
(ex. DDbar)
mass
two scattering states
(g and e)

Resonance

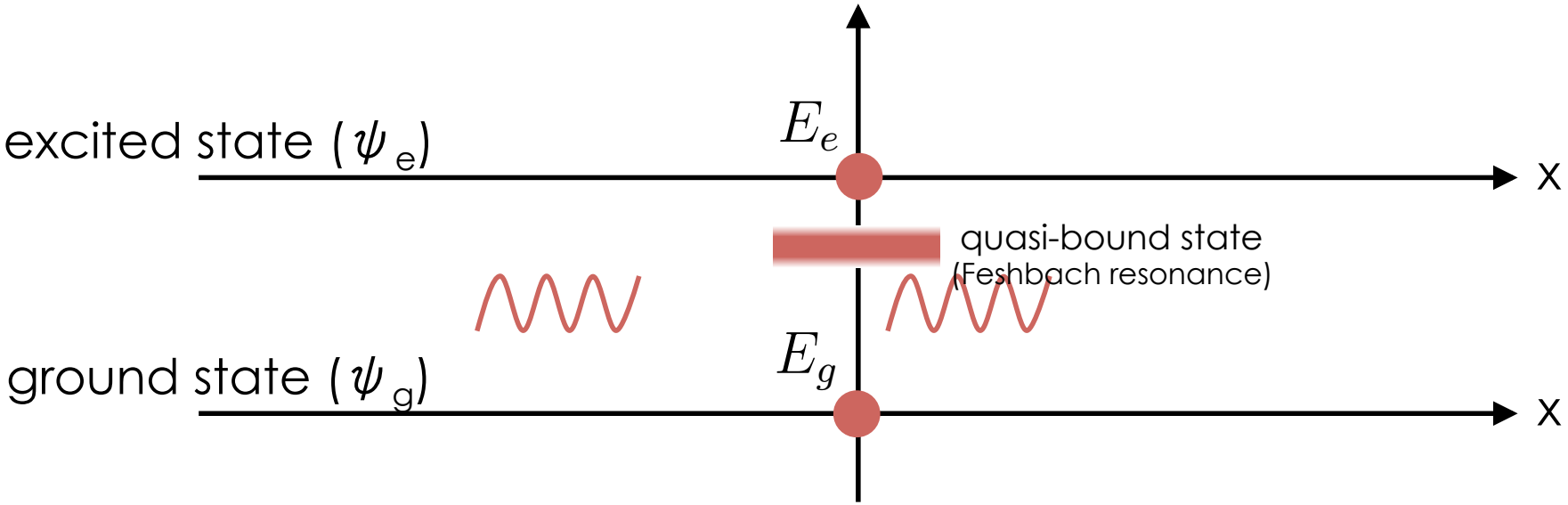


Many of X, Y, Z are resonances
in scattering states.

Purpose of this exercise:

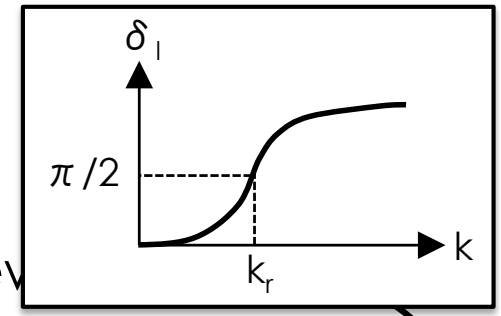
Understanding *Feshbach* resonance

two-channel problem in "one-dimensional" system

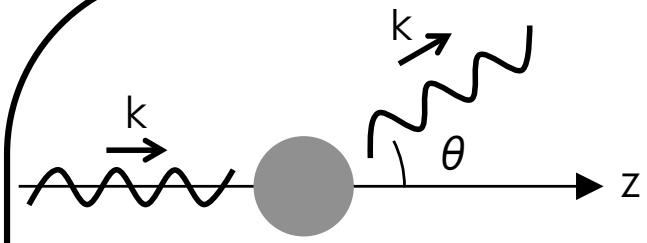


Resonance

Many of X, Y, Z are scattering states.



How to solve scattering problem? (review)



$$\varphi^{(+)}(r,\theta) = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)c_{\ell}\frac{\chi_{\ell}(r)}{kr}P_{\ell}(\cos\theta)$$

asymptotic state (phase shift): $\chi_{\ell}(r) \sim \sin\left(kr - \frac{\pi\ell}{2} + \delta_{\ell}\right) \quad r \to \infty$

① Partial wave decomposition

$$\varphi^{(+)}(r,\theta) = e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\frac{e^{2i\delta_{\ell}}-1}{2ik}P_{\ell}(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty}(2\ell+1)f_{\ell}(k)P_{\ell}(\cos\theta)$$

Partial wave amplitude $f_{\ell}(k) = \dfrac{1}{k(\cot\delta_{\ell} - i)}$

② Resonance (definition by phase shift)

$$\cot\delta_{\ell}(k_r) = 0 \Leftrightarrow \delta_{\ell}(k_r) = \frac{\pi}{2}$$

$$\cot\delta_{\ell} = \cot\frac{\pi}{2} + \frac{\mathrm{d}}{\mathrm{d}E}\cot\delta_{\ell}\Big|_{\delta_{\ell}=\pi/2}(E - E_r) + \ldots$$

decay width: $\dfrac{\mathrm{d}}{\mathrm{d}E}\cot\delta_{\ell}\Big|_{\delta_{\ell}=\pi/2} \equiv -\dfrac{2}{\Gamma} \qquad E_r = \dfrac{k_r^2}{2m}$

Pole as complex energy $f_{\ell}(k) \sim -\dfrac{1}{k}\dfrac{\frac{\Gamma}{2}}{E - E_r + i\frac{\Gamma}{2}}$

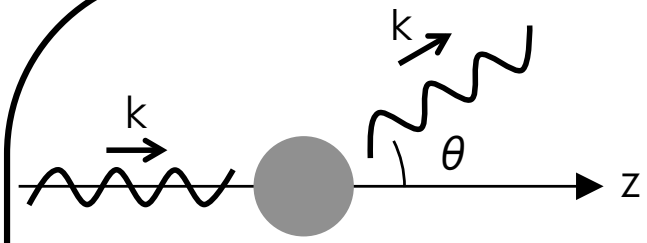
☑ Matching (direct) method: finding $\delta_l(k)$ → Exercise (toy model)

☑ Complex scaling method: finding $E_r + i\,\Gamma/2$ S. Aoyama T. Myom K. Kato, K. Ikeda, Prog. Theor. Phys. 116, 1 (2006)

Resonance

Many of X, Y, Z are scattering states.

How to solve scattering problem? (review)

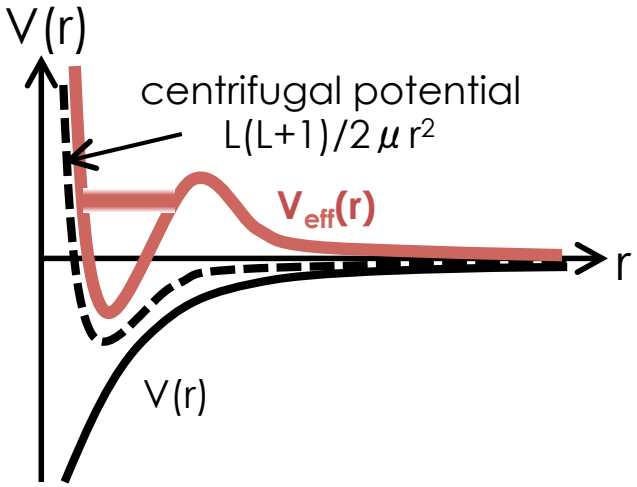


$$\varphi^{(+)}(r,\theta) = \sum_{\ell=0}^{\infty} i^{\ell}(2\ell+1)c_{\ell}\frac{\chi_{\ell}(r)}{kr}P_{\ell}(\cos\theta)$$

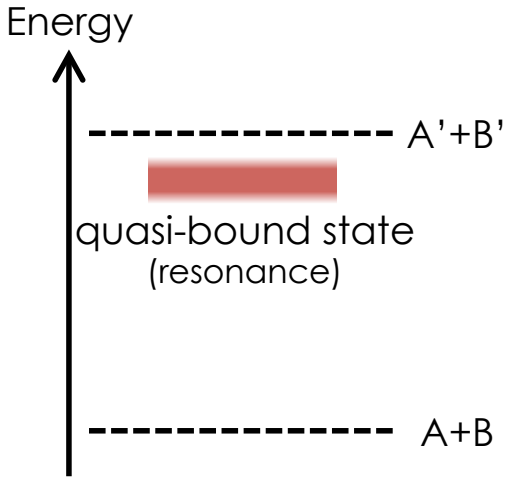
asymptotic state: $\chi_{\ell}(r) \sim \sin\left(kr - \frac{\pi\ell}{2} + \delta_{\ell}\right) \quad r \to \infty$
(phase shift)

Typical mechanism of resonances

① Centrifugal potential



② Feshbach resonance

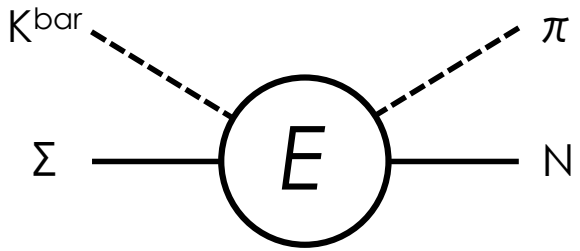


A+B → A'+B' → A+B

→ **Exercise**

③ *E*-dependent potential

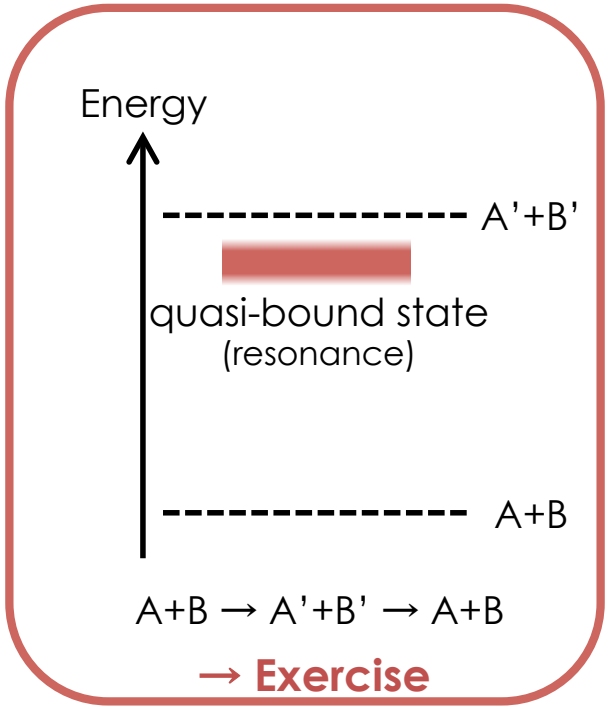
Λ(1405) baryon resonance



Weinberg-Tomozawa interaction
(Chiral symmerty for NG boson)

Resonance

Many of X, Y, Z are scattering states.



Resonance

S. Aoyama, T. Myo, K. Kato, K. Ikeda,
Prog. Theor. Phys. 116, 1 (2006)

$$\left[\frac{1}{2m}\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{l(l+1)}{r^2}\right)+V(r)\right]\psi_{kl}(r)=E\psi_{kl}(r)$$

$$\psi_l(k,r)\xrightarrow[r\to\infty]{}\frac{1}{2}\frac{i^{l+1}}{k^{l+1}}\left[\underbrace{f_l^+(k)e^{-ikr}}_{\text{incoming}}-(-)^l\underbrace{f_l^-(k)e^{ikr}}_{\text{outgoing}}\right]$$

Resonance

S. Aoyama, T. Myo, K. Kato, K. Ikeda,
Prog. Theor. Phys. 116, 1 (2006)

$$\psi_l(k,r) \xrightarrow[r\to\infty]{} \frac{1}{2}\frac{i^{l+1}}{k^{l+1}}\left[f_l^+(k)e^{-ikr} - (-)^l f_l^-(k)e^{ikr}\right]$$

incoming outgoing

S-matrix $S_l(k) = \dfrac{f_l^-(k)}{f_l^+(k)}$

Pole of S-matrix
$f_l^+(k)=0$

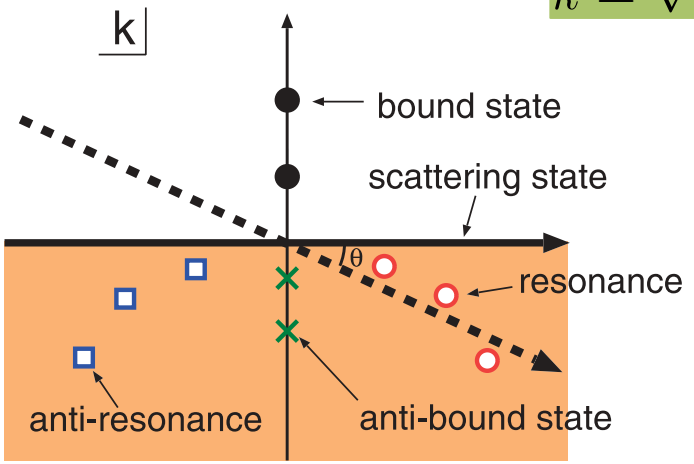
bound states: $k_B = i\gamma_b,$
anti-bound states: $k_{AB} = -i\gamma_{ab},$
(virtual states)
resonances: $k_R = \kappa_r - i\gamma_r,$
anti-resonances: $k_{AR} = -\kappa_r - i\gamma_r.$

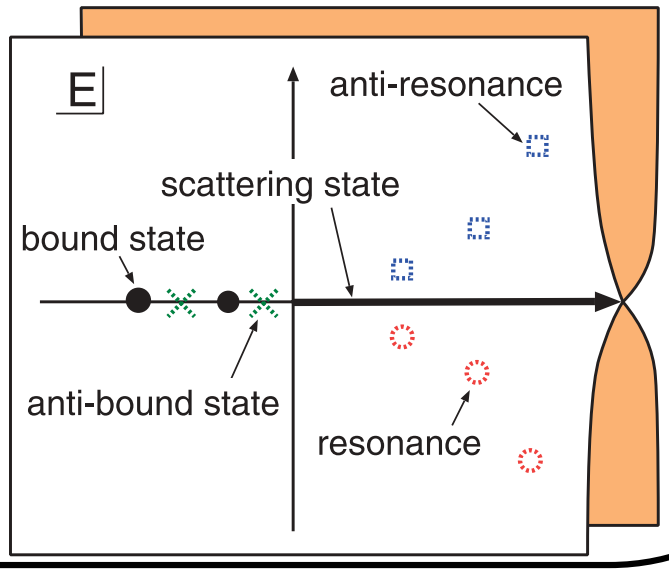
(The quantities κ_r and $\gamma_{r,\ b,\ ab}$ are all positive.)

$e^{ik_p r}$

(a) (b)

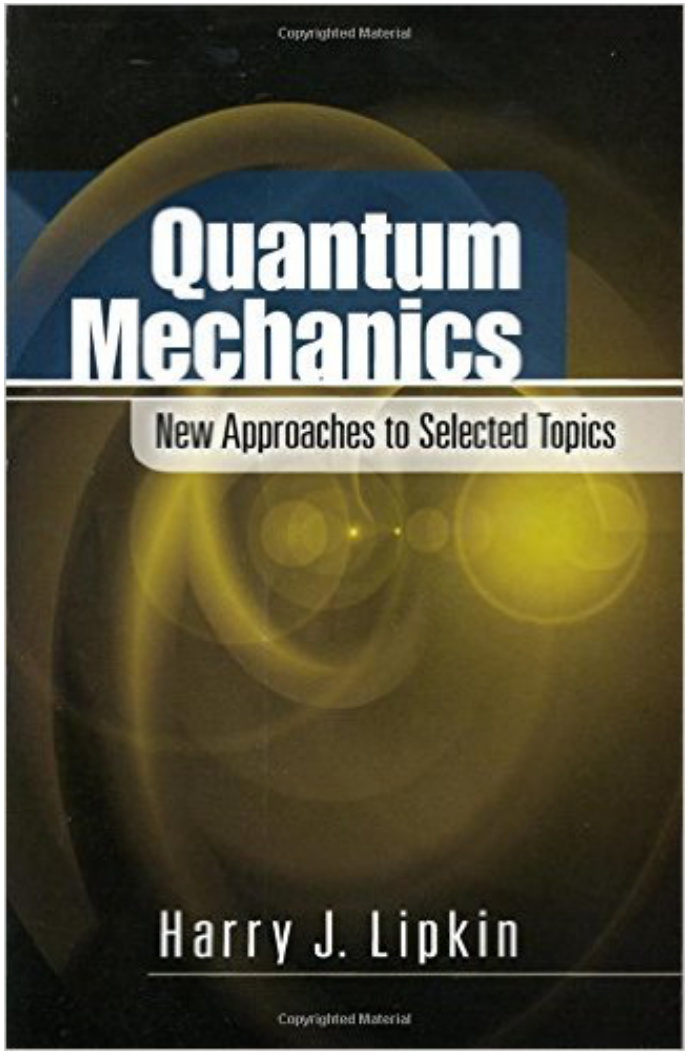
$k = \sqrt{2mE}$





Resonance

Scattering problem in one-dimension



Quantum Mechanics
New Approaches to Selected Topics

Harry J. Lipkin

Elsevier Science Publishing (1973)

Available in Dover Books

Resonance

Scattering problem in one-dimension

estimated time

⓪ Preparation — within *10 min.* → Answer

↓

① δ-function potential — within *15 min.* → Answer

↓

Skip Problem 2 (sorry....) ~~② Two δ-function potentials~~

↓

③ Feshbach resonance — within *20 min.* → Answer

Erratum
p. 5

Incorrect

3.1 Show the relation

remove

$$\left.\frac{\mathrm{d}\psi_g(x)}{\mathrm{d}x}\right|_{x=+\varepsilon} - \left.\frac{\mathrm{d}\psi_g(x)}{\mathrm{d}x}\right|_{x=-\varepsilon} + U_0\psi_g(0) + U_1\psi_e(0) = 0, \tag{31}$$

$$\left.\frac{\mathrm{d}\psi_e(x)}{\mathrm{d}x}\right|_{x=+\varepsilon} - \left.\frac{\mathrm{d}\psi_e(x)}{\mathrm{d}x}\right|_{x=-\varepsilon} + U_0\psi_e(0) = 0, \tag{32}$$

with a small positive number $\epsilon > 0$

Correct

3.1 Show the relation

$$\left.\frac{\mathrm{d}\psi_g(x)}{\mathrm{d}x}\right|_{x=+\varepsilon} - \left.\frac{\mathrm{d}\psi_g(x)}{\mathrm{d}x}\right|_{x=-\varepsilon} + U_1\psi_e(0) = 0, \tag{31}$$

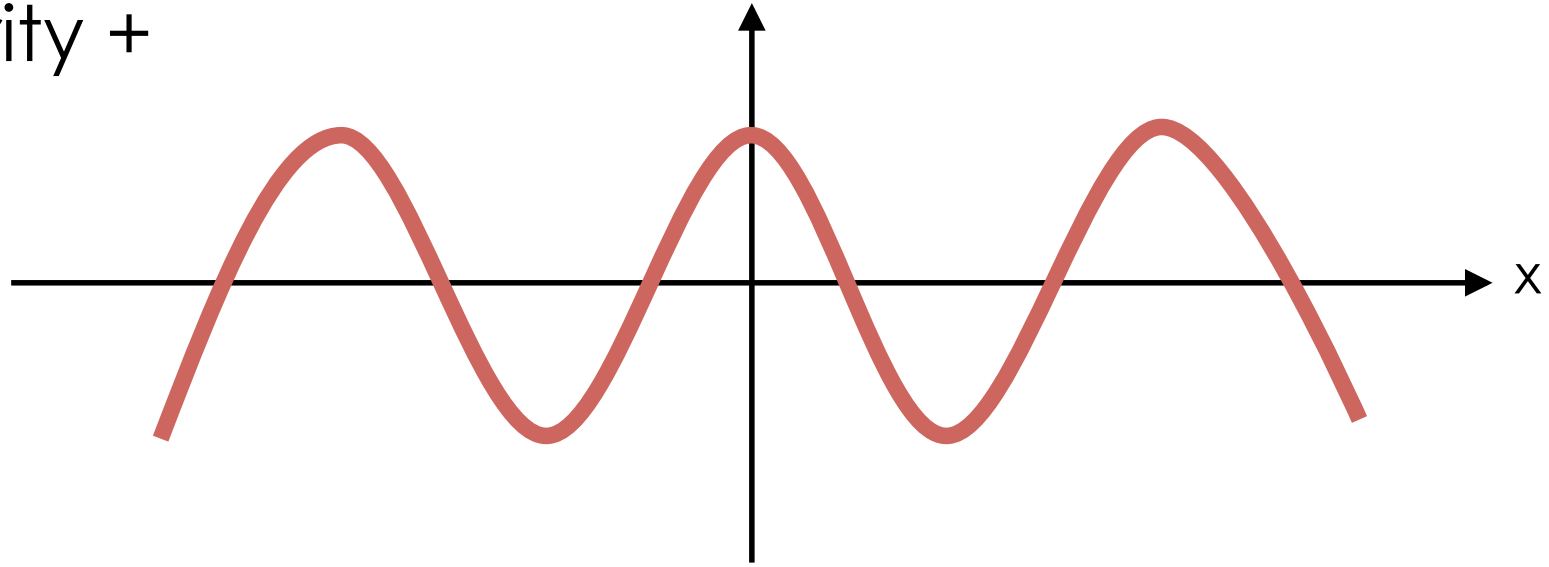
add

$$\left.\frac{\mathrm{d}\psi_e(x)}{\mathrm{d}x}\right|_{x=+\varepsilon} - \left.\frac{\mathrm{d}\psi_e(x)}{\mathrm{d}x}\right|_{x=-\varepsilon} + U_0\psi_e(0) + U_1\psi_g(0) = 0, \tag{32}$$

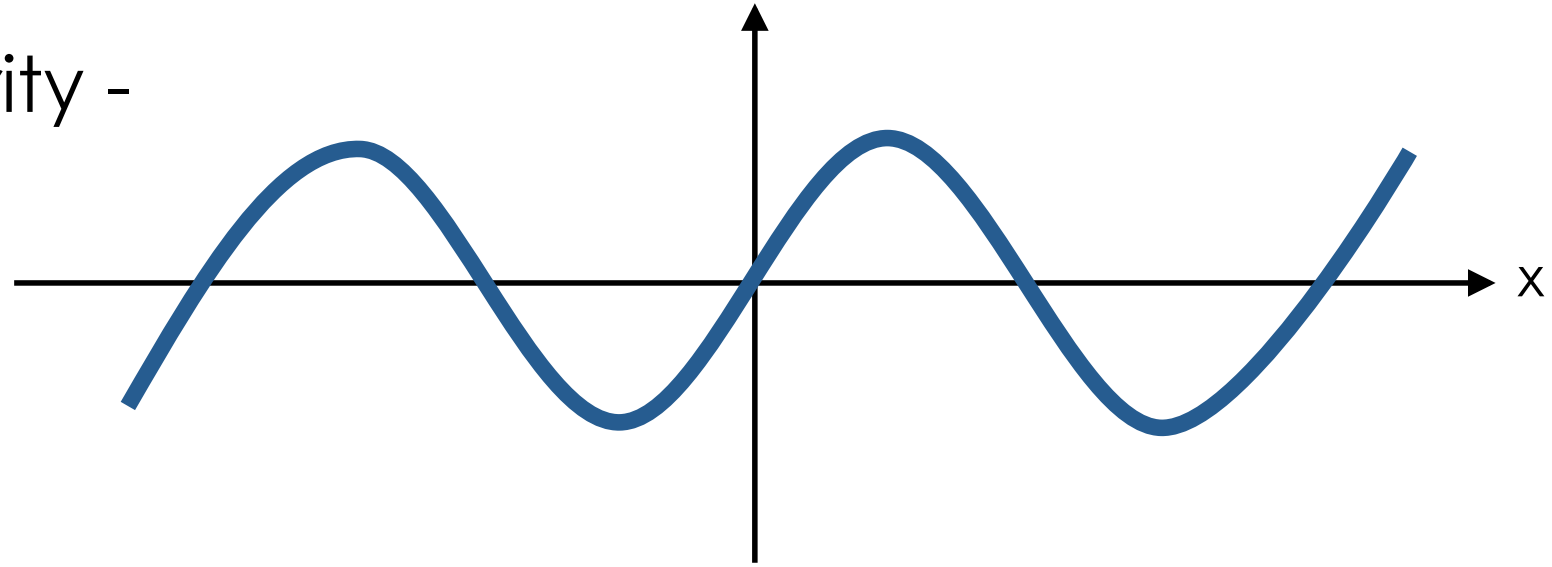
with a small positive number $\epsilon > 0$.

⓪ Preparation

parity +

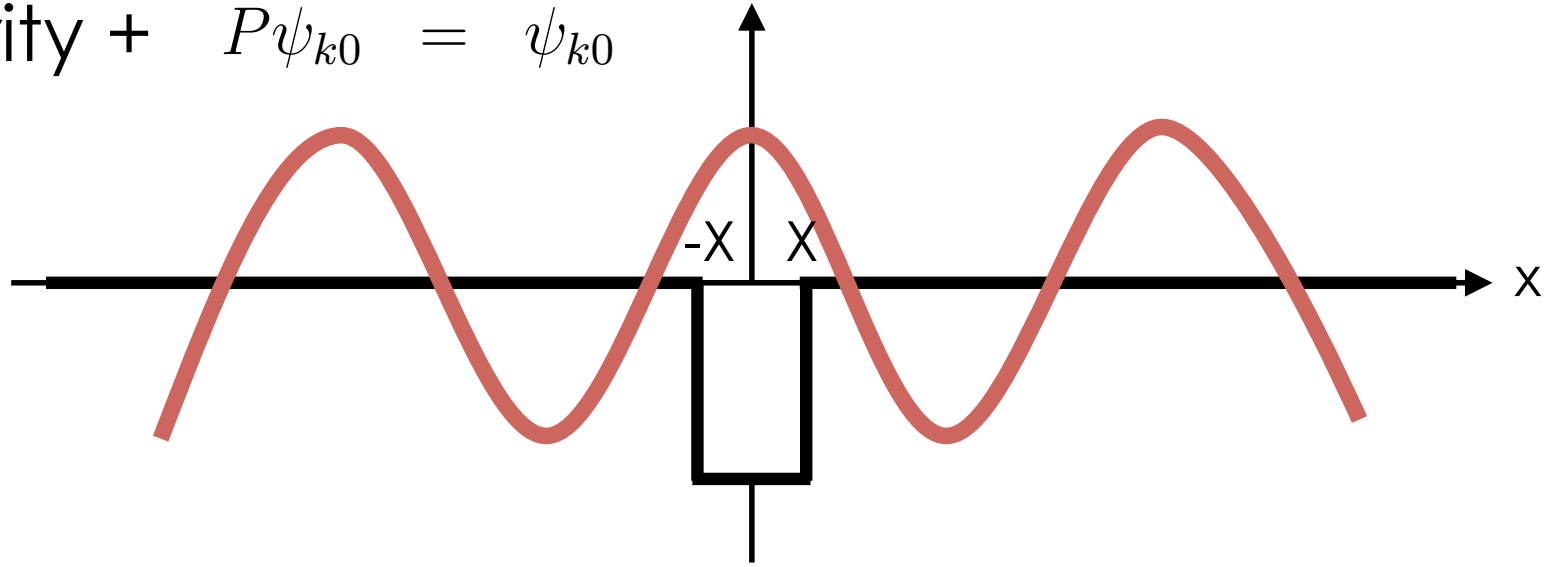


parity -



⓪ Preparation

parity + $P\psi_{k0} = \psi_{k0}$



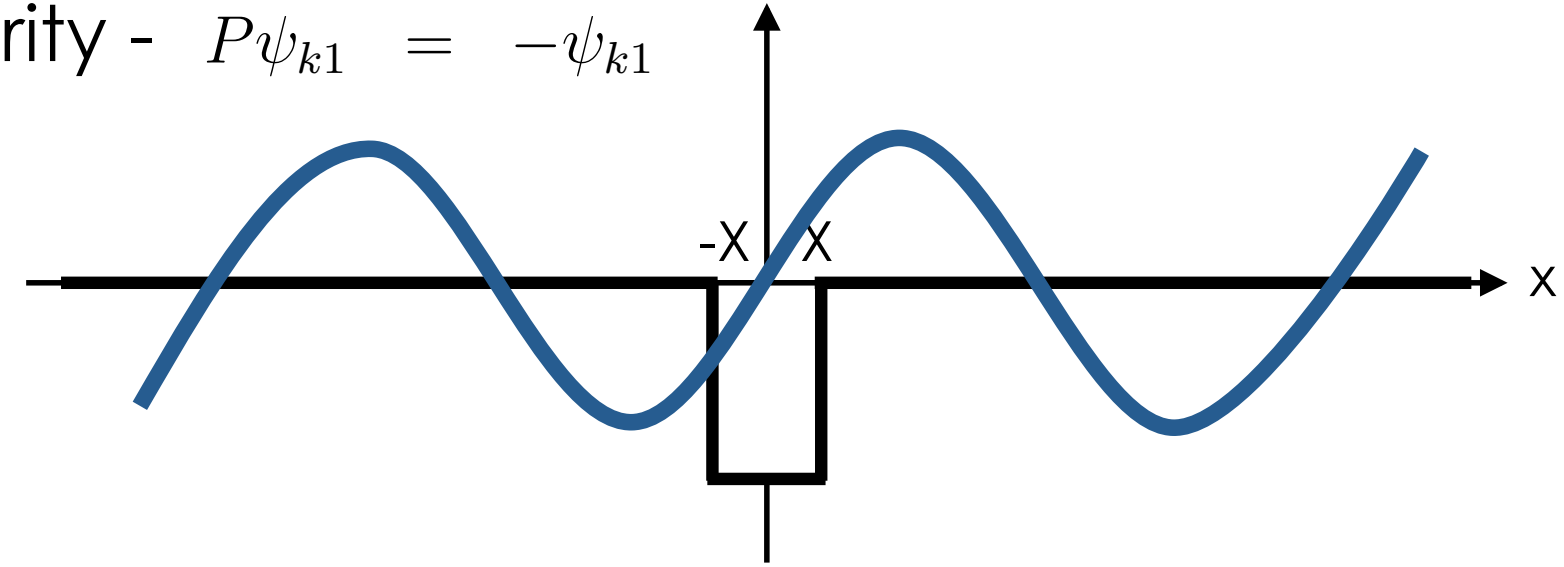
$$\psi_{k0}(x < -X) = \cos(kx - \delta_0) \qquad \psi_{k0}(x > X) = \cos(kx + \delta_0)$$

potential = 0 for x> | X |

Check! $\psi_{k0}(-x) = \psi_{k0}(x)$

⓪ Preparation

parity - $P\psi_{k1} = -\psi_{k1}$

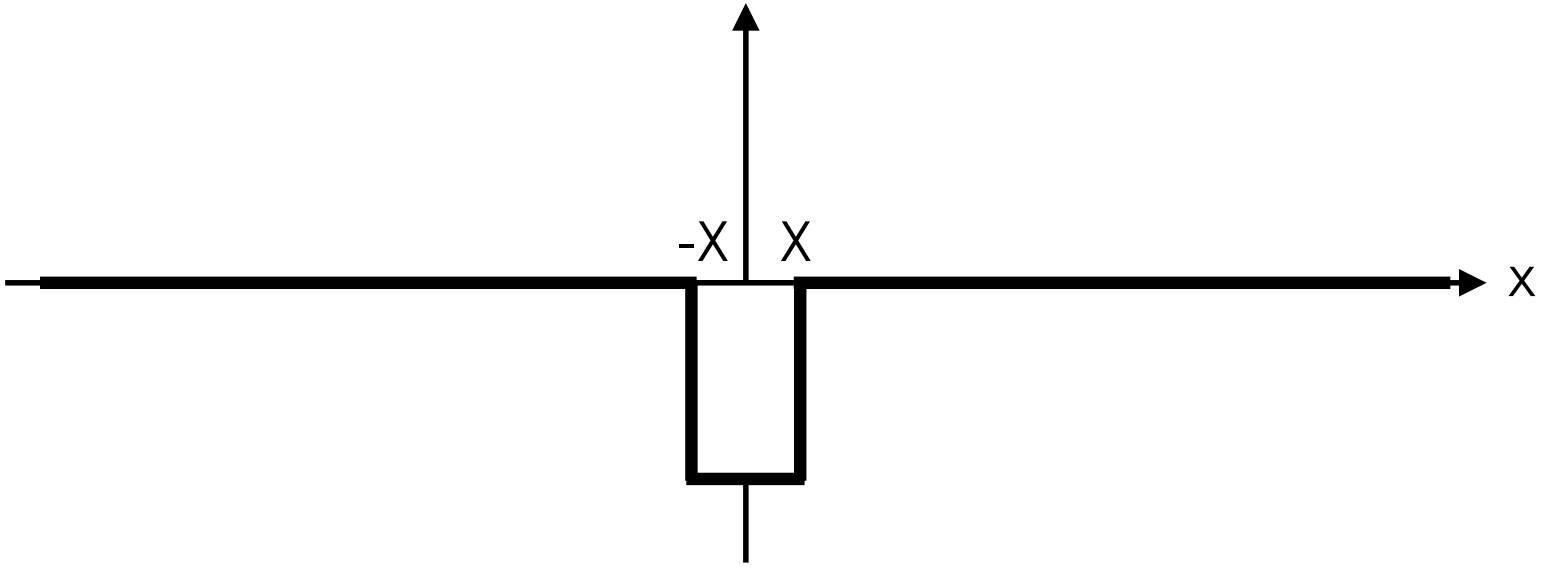


$$\psi_{k1}(x < -X) = \sin(kx - \delta_1) \qquad \psi_{k1}(x > X) = \sin(kx + \delta_1)$$

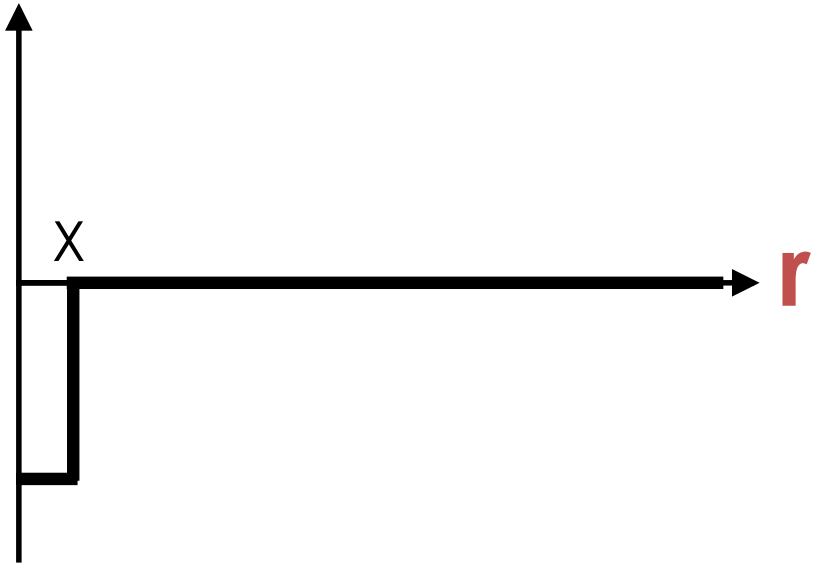
potential = 0 for x> | X |

Check! $\psi_{k1}(-x) = -\psi_{k1}(x)$

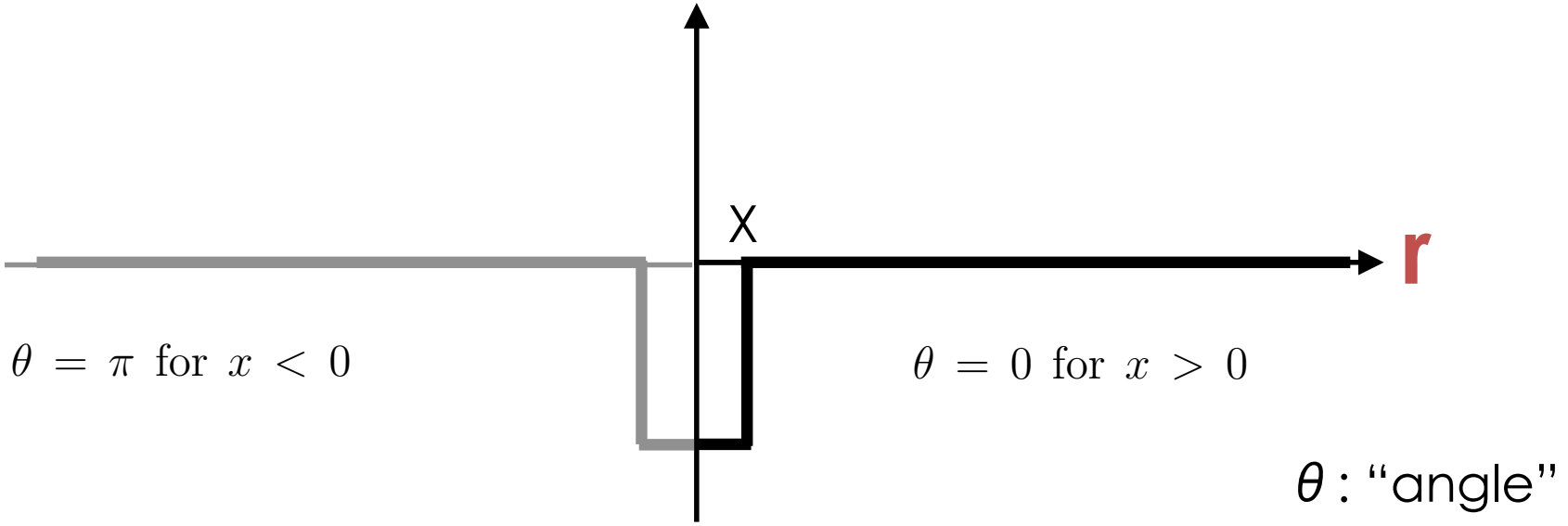
⓪ Preparation



⓪ Preparation



⓪ Preparation



0.1 Show that $\psi_{k0}(x)$ and $\psi_{k1}(x)$ are expressed by

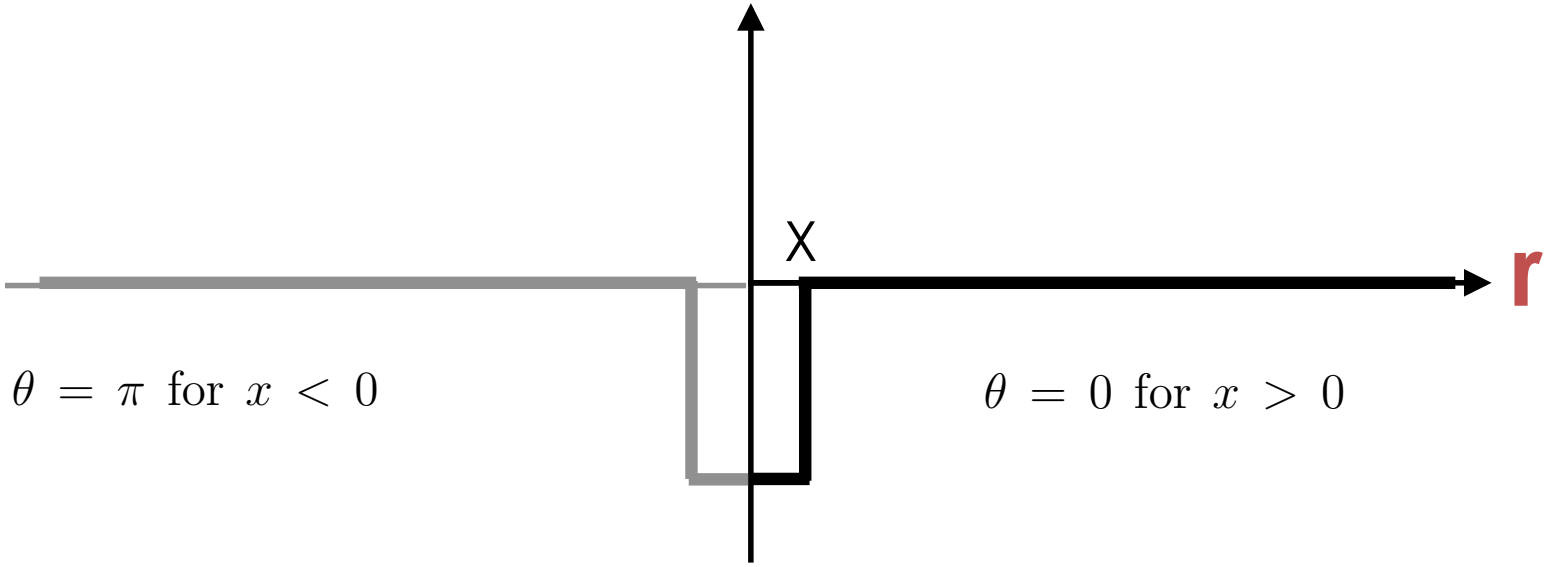
$$\psi_{k0}(r) = \cos(kr + \delta_0) = \frac{1}{2} e^{-i\delta_0} \left(e^{-ikr} + e^{2i\delta_0} e^{ikr} \right), \tag{9}$$

and

$$\psi_{k1}(r) = e^{i\theta} \sin(kr + \delta_1) = \frac{i}{2} e^{-i\delta_1} e^{i\theta} \left(e^{-ikr} - e^{2i\delta_1} e^{ikr} \right), \tag{10}$$

with $r = |x|$. We define $\theta = 0$ for $x > 0$ and $\theta = \pi$ for $x < 0$ corresponding to the odd parity. [We use the same symbols ψ_{k0} and ψ_{k1} for r in correspondence to $\psi_{k0}(x)$ and $\psi_{k1}(x)$, respectively.]

⓪ Preparation



0.2 Show that the above two equations are expressed by

$$\psi_{k\ell}(r) = e^{i\ell\theta}\left(e^{-ikr} + (-1)^{\ell}e^{2i\delta_{\ell}}e^{ikr}\right), \tag{11}$$

with $\ell = 0, 1$. [Notice that $e^{i\ell\theta}$ is the "angular part" in one-dimensional system, i.e. $\ell = 0$ (1) correspond to even (odd) angular momentum in three-dimensional system.]

⓪ Preparation

0.3 For the phase shift $\delta_\ell(k)$ as a function of momentum k, we define the S-matrix by

$$S_\ell(k) = e^{2i\delta_\ell(k)}, \tag{12}$$

for $\ell = 0, 1$. The partial wave amplitude $f_\ell(k)$ is defined by

$$f_\ell(k) = \frac{S_\ell(k) - 1}{2ik} = \frac{1}{k} e^{i\delta_\ell} \sin\delta_\ell. \tag{13}$$

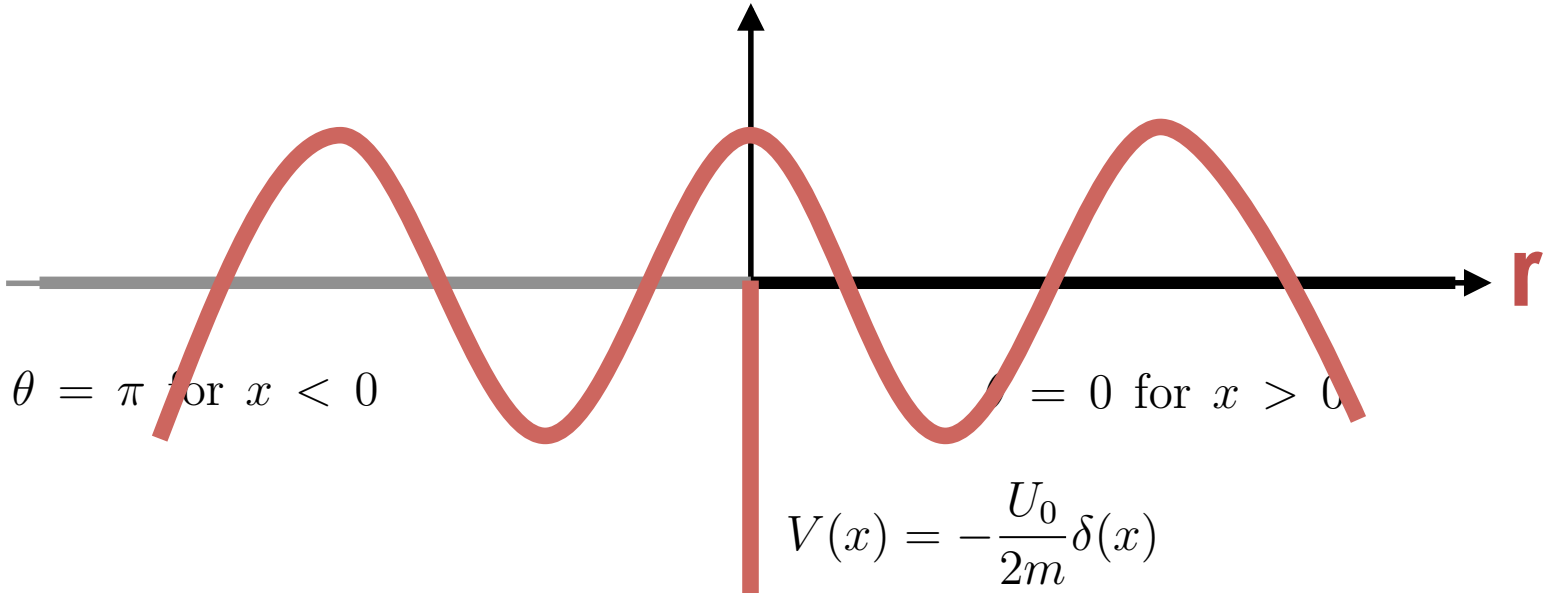
Notice that the second term -1 in $S_\ell(k) - 1$ is the subtraction of the free part, and that $1/2ik$ was multiplied to mimic the three-dimensional case. The cross section which is proportional to $|f_\ell(k)|^2$ has the large value for k_r such that $\sin\delta_\ell(k_r) \simeq 1$ (or $\cot\delta_\ell(k_r) \simeq 0$). Show that $f_\ell(k)$ is expanded in terms of $E = k^2/2m$ around $E_r = k_r^2/2m$ (resonance energys)

$$f_\ell(k) \simeq -\frac{1}{k} \frac{\Gamma/2}{E - E_r + i\Gamma/2}, \tag{14}$$

with the decay width Γ defined by

$$-\frac{2}{\Gamma} = \left.\frac{\mathrm{d}}{\mathrm{d}E} \cot\delta_\ell \right|_{E=E_r}. \tag{15}$$

① δ-function potential

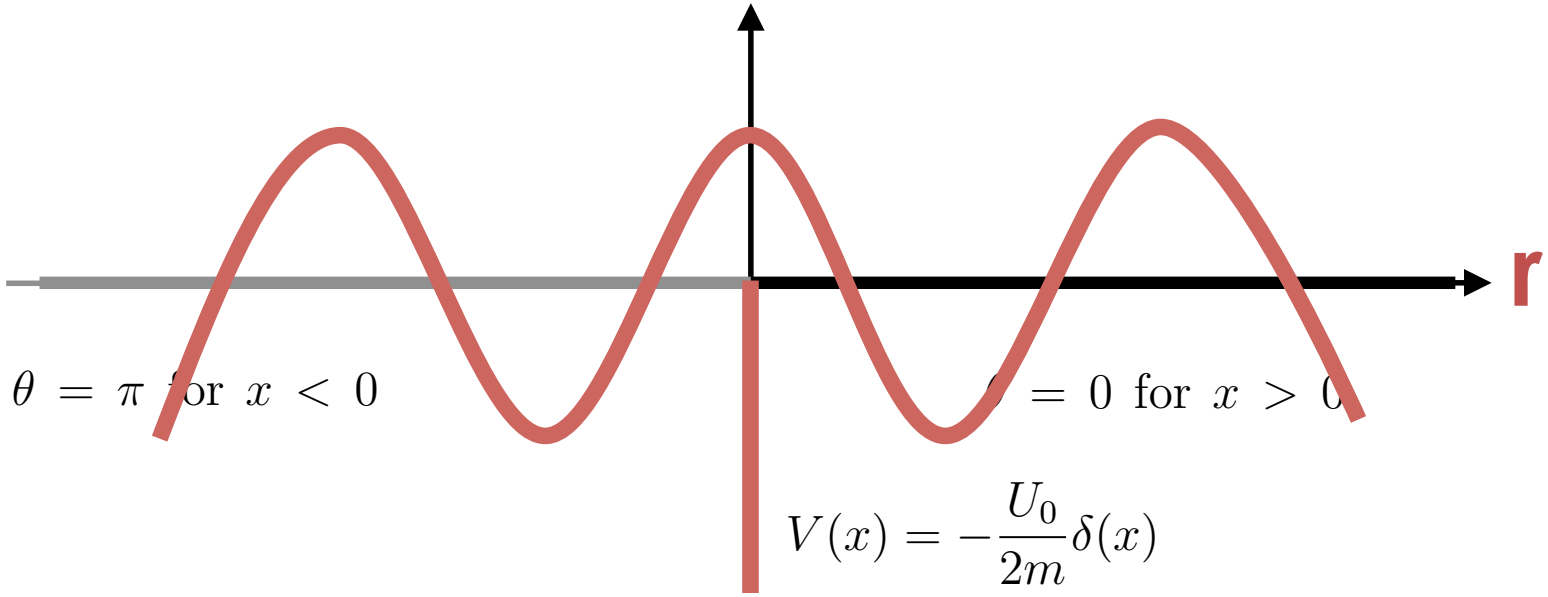


1.1 Show the relation

$$\left.\frac{\mathrm{d}\psi_{k0}(x)}{\mathrm{d}x}\right|_{x=+\varepsilon} - \left.\frac{\mathrm{d}\psi_{k0}(x)}{\mathrm{d}x}\right|_{x=-\varepsilon} + U_0\psi_{k0}(0) = 0, \tag{17}$$

with a small positive number $\epsilon > 0$ for the wave function $\psi_{k0}(x)$ with momentum k ($E = k^2/2m$), as an eigenstate of $H(x) = H_0(x) + V(x)$.

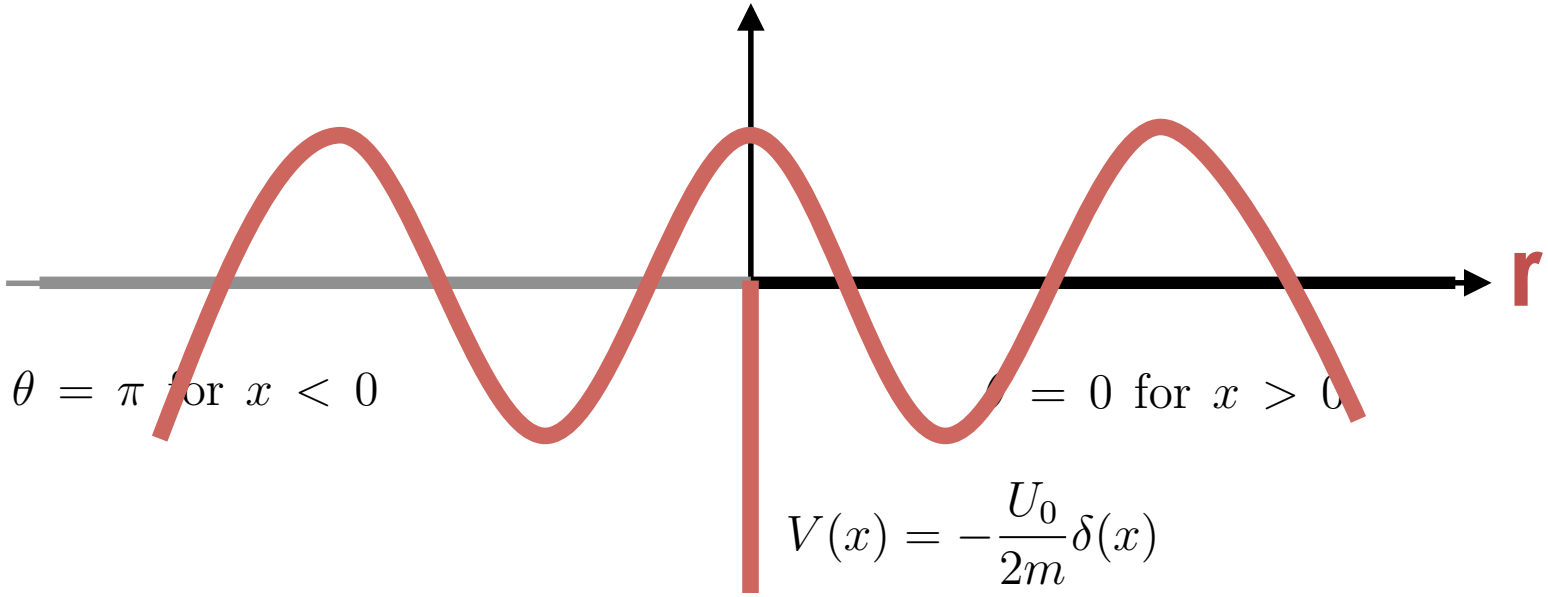
① δ-function potential



1.2 We consider the even parity solution with momentum k, $\psi_{k0}(x) = \cos(k|x| + \delta_0)$. Show that the phase shift $\delta_0(k)$ is given by

$$\tan \delta_0(k) = \frac{U_0}{2k}. \tag{18}$$

① δ-function potential

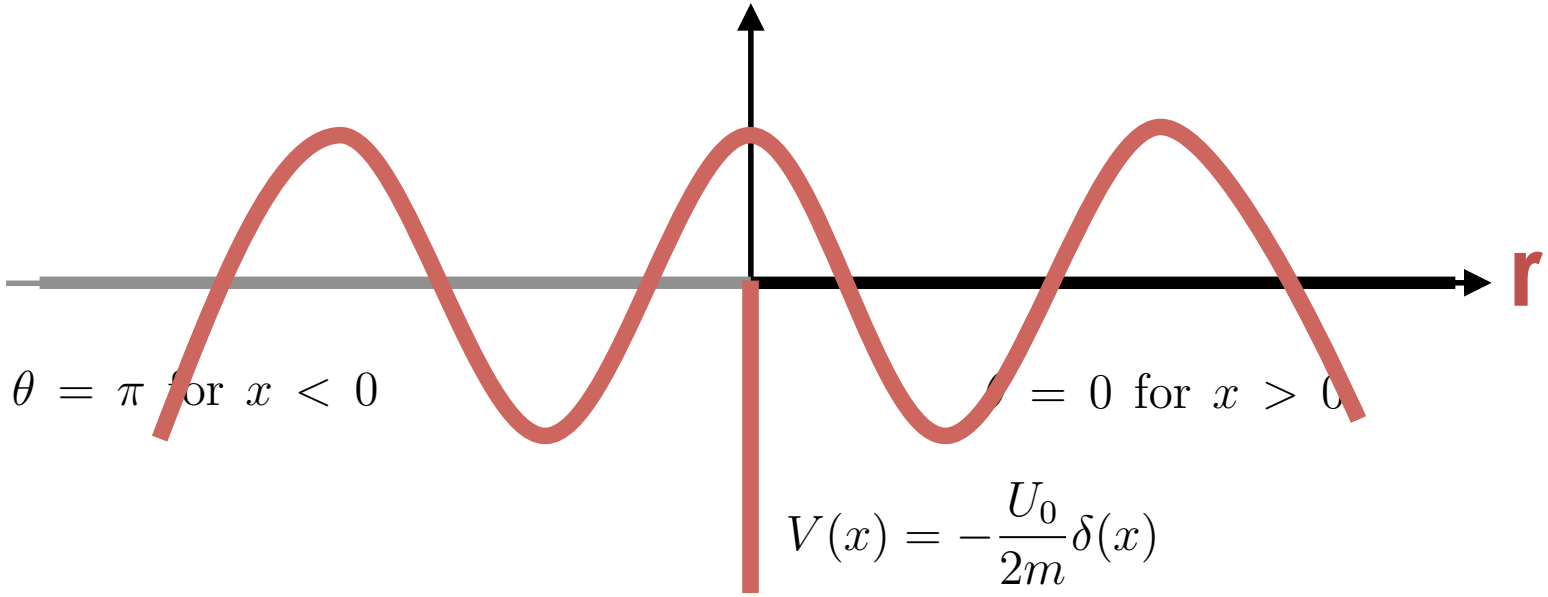


1.3 The S-matrix element is defined by

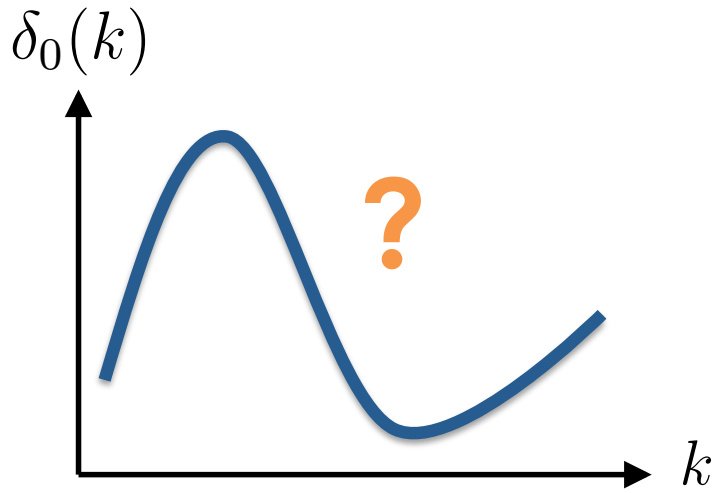
$$S_0(k) = e^{2i\delta_0(k)}. \tag{19}$$

Show that the $S_0(k)$ gives a pole for $k = iU_0/2$, and show also that there exists one bound state with binding energy $U_0^2/(8m)$.

① δ-function potential

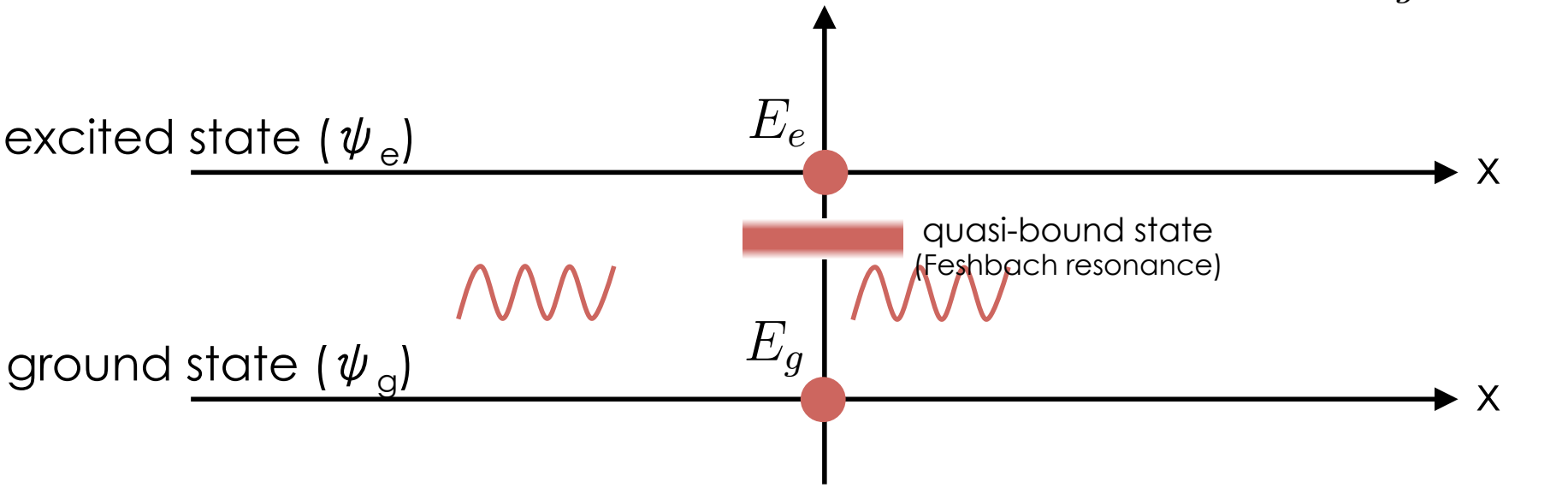


1.4 Draw the graph of $\delta_0(k)$ as a function of k.



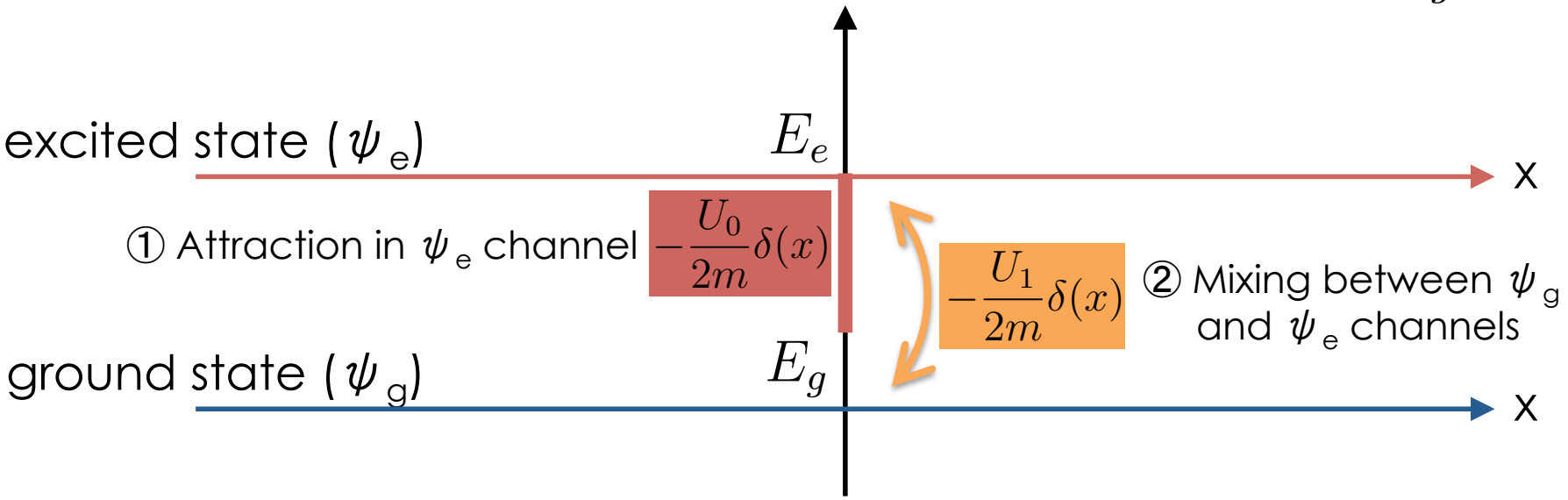
③ Feshbach resonance

$E_g < E_e$



③ Feshbach resonance

$E_g < E_e$

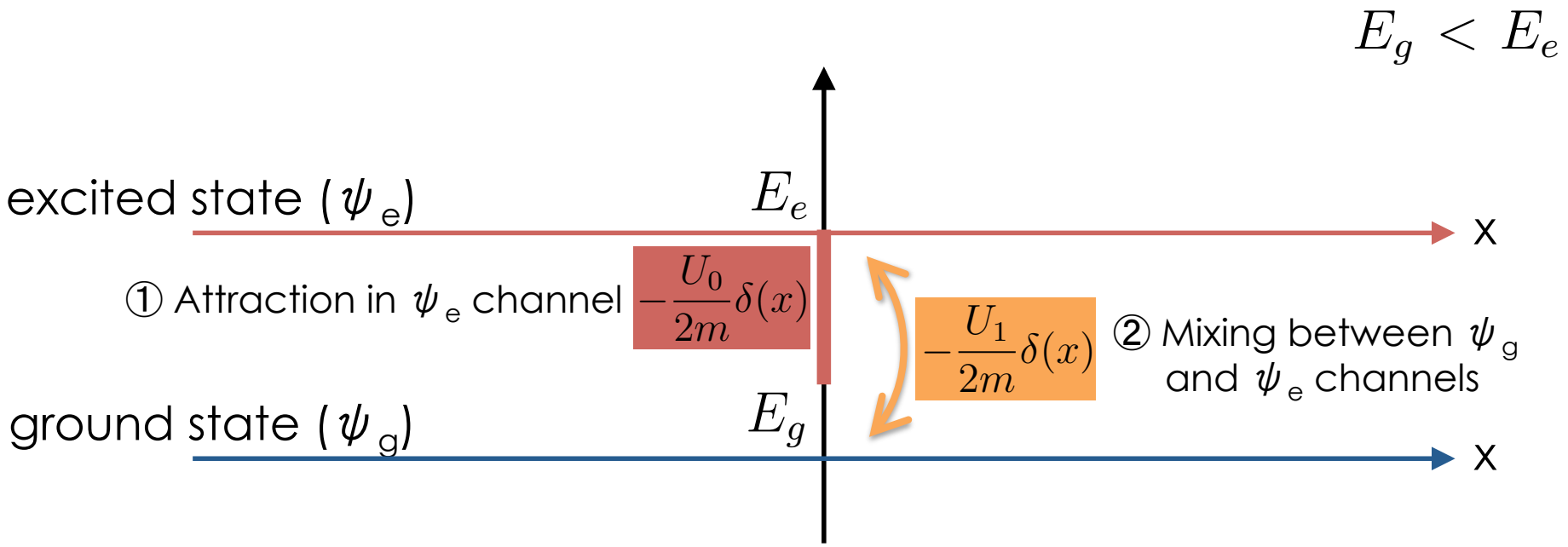


$$H(x) = H_0(x) + V(x)$$

$$H_0 = \begin{pmatrix} \hat{p}^2/2m + E_g & 0 \\ 0 & \hat{p}^2/2m + E_e \end{pmatrix}$$

$$V(x) = -\frac{1}{2m}\delta(x) \begin{pmatrix} 0 & U_1 \\ U_1 & U_0 \end{pmatrix}$$

③ Feshbach resonance



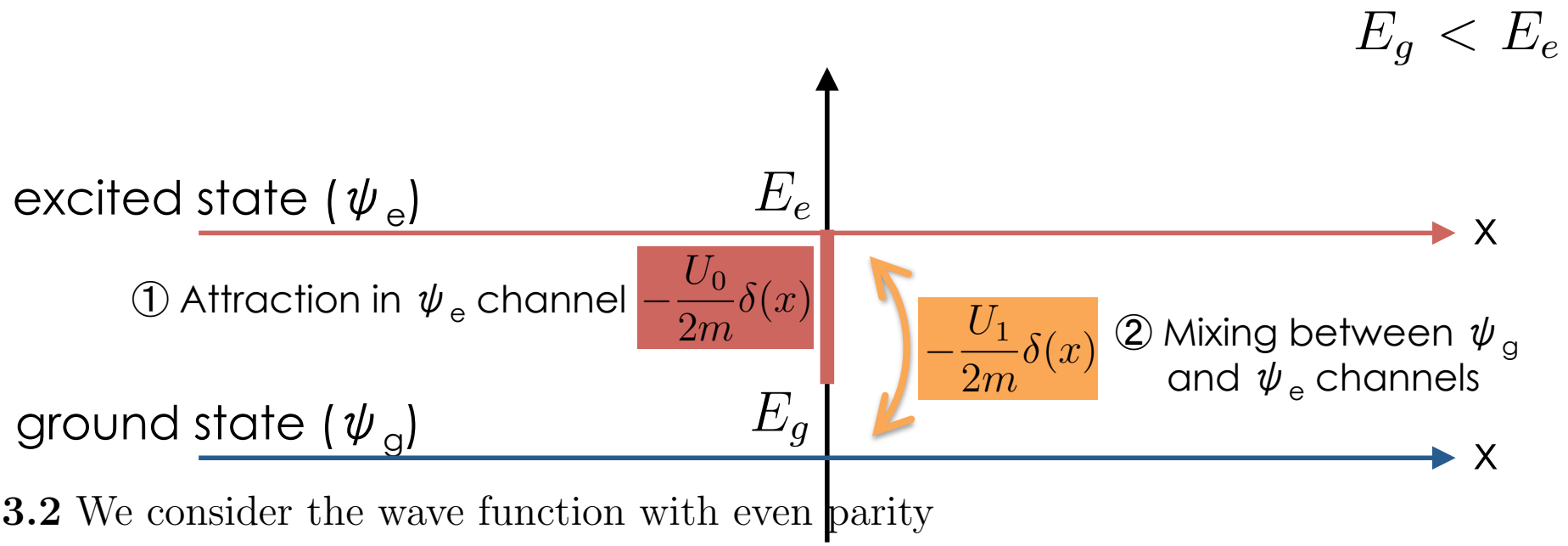
3.1 Show the relation

$$\left.\frac{\mathrm{d}\psi_g(x)}{\mathrm{d}x}\right|_{x=+\varepsilon} - \left.\frac{\mathrm{d}\psi_g(x)}{\mathrm{d}x}\right|_{x=-\varepsilon} + U_1\psi_e(0) = 0, \tag{31}$$

$$\left.\frac{\mathrm{d}\psi_e(x)}{\mathrm{d}x}\right|_{x=+\varepsilon} - \left.\frac{\mathrm{d}\psi_e(x)}{\mathrm{d}x}\right|_{x=-\varepsilon} + U_0\psi_e(0) + U_1\psi_g(0) = 0, \tag{32}$$

with a small positive number $\epsilon > 0$.

③ Feshbach resonance



3.2 We consider the wave function with even parity

$$\psi_g(x) = \alpha \cos(k|x| + \delta_0), \tag{33}$$
$$\psi_e(x) = \beta e^{ik_e|x|}, \tag{34}$$

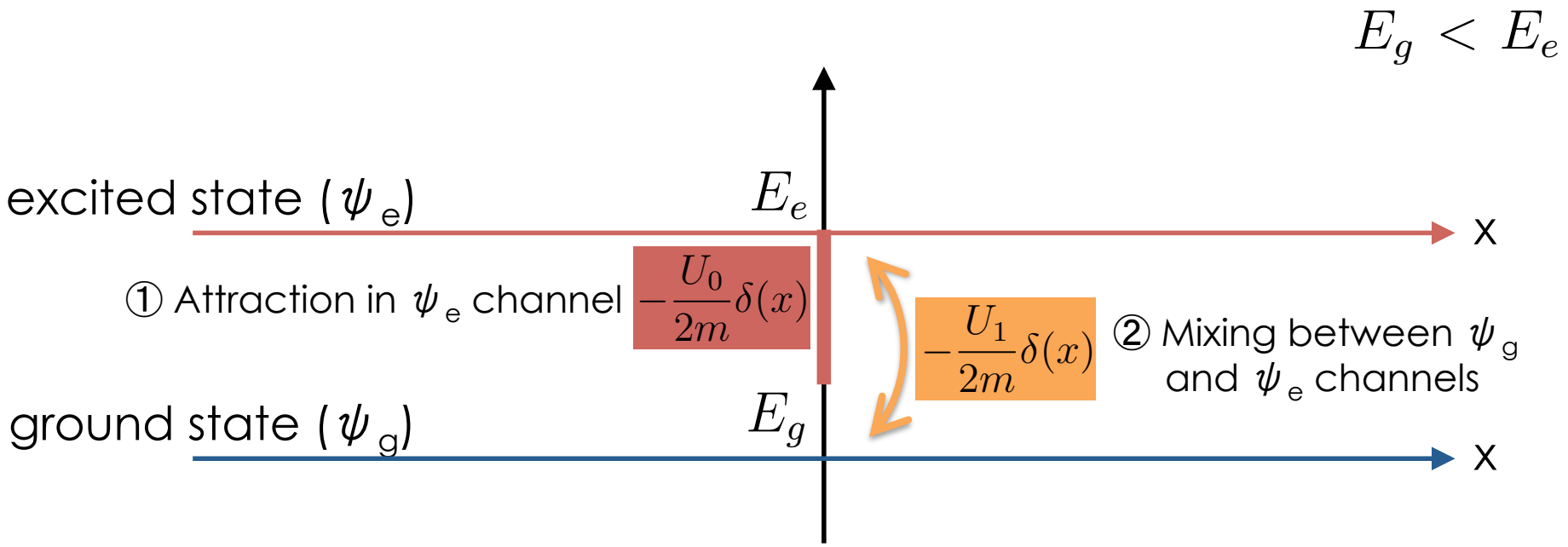
with coefficients α and β and momentum k and k_e, respectively, for ψ_g and ψ_e. The k and k_e are related to the energy E is by

$$k^2 = 2m(E - E_g), \tag{35}$$
$$k_e^2 = 2m(E - E_e), \tag{36}$$

respectively. In the followings, we consider "the bound state solution" for ψ_e with $E < E_e$ and set $k_e = i\kappa_e$ with a positive real number $\kappa_e > 0$. Show the relation

$$\tan \delta_0 = \frac{U_1^2}{2k(2\kappa_e - U_0)}. \tag{37}$$

③ Feshbach resonance



3.3 Show the resonance energy E_r and the decay width Γ are given by

$$E_r = E_e - \frac{U_0^2}{8m}, \tag{38}$$

$$\Gamma = \frac{U_0 U_1^2}{8\sqrt{2m^3}\sqrt{E_e - E_g - U_0^2/8m}}, \tag{39}$$

in the energy scale $E_g < E < E_e$. [Notice that the resonance energy is provided by U_0 and the decay width is proportional to U_1^2. What does this mean?]

おわり

Fin.

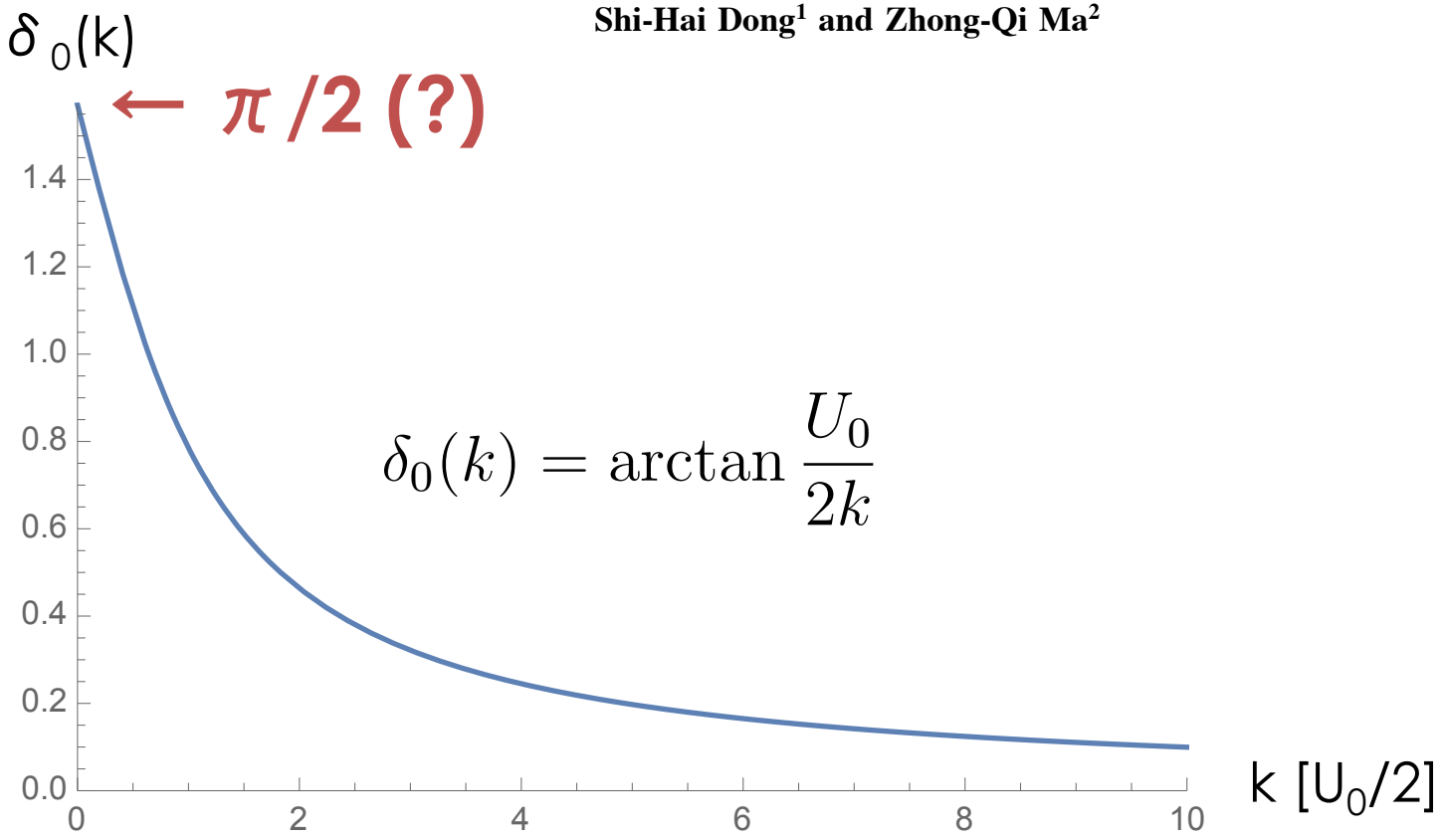
Part of answer

① δ-function potential

International Journal of Theoretical Physics, Vol. 39, No. 2, 2000

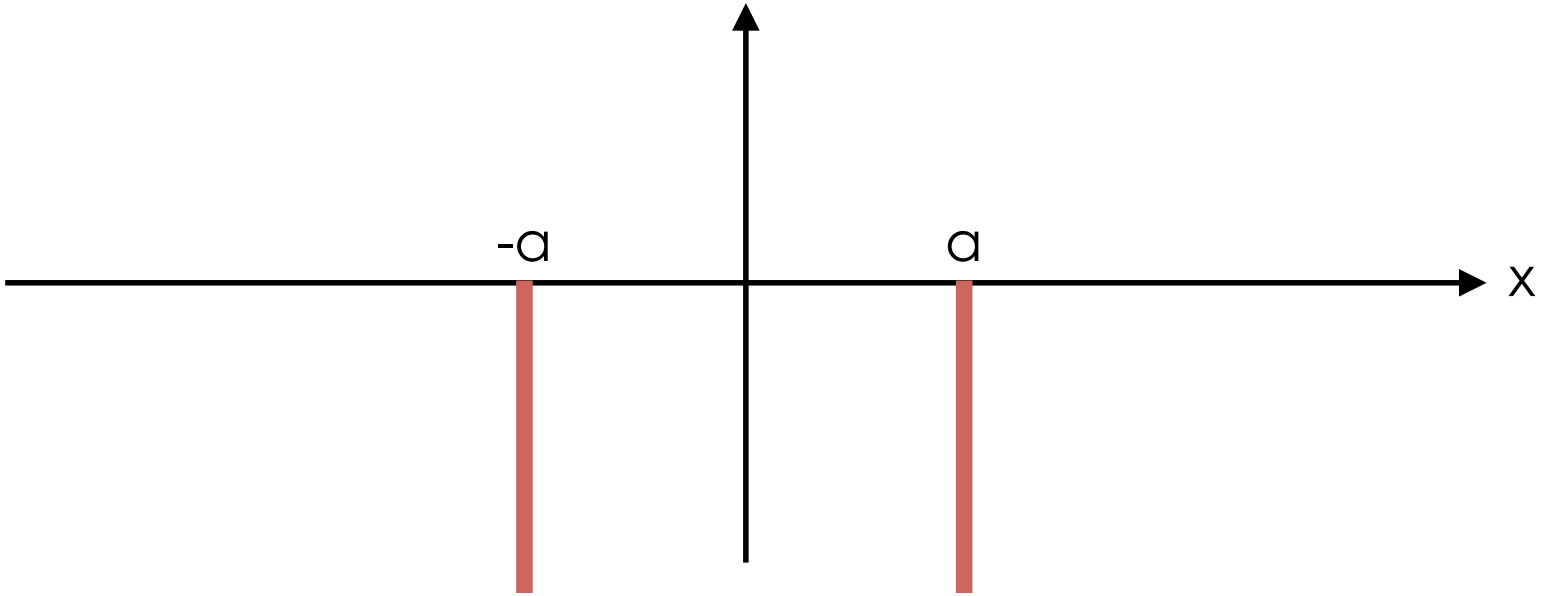
Levinson's Theorem for the Schrödinger Equation in One Dimension

Shi-Hai Dong[1] and Zhong-Qi Ma[2]



$$\delta_0(k) = \arctan \frac{U_0}{2k}$$

② Two δ-function potentials



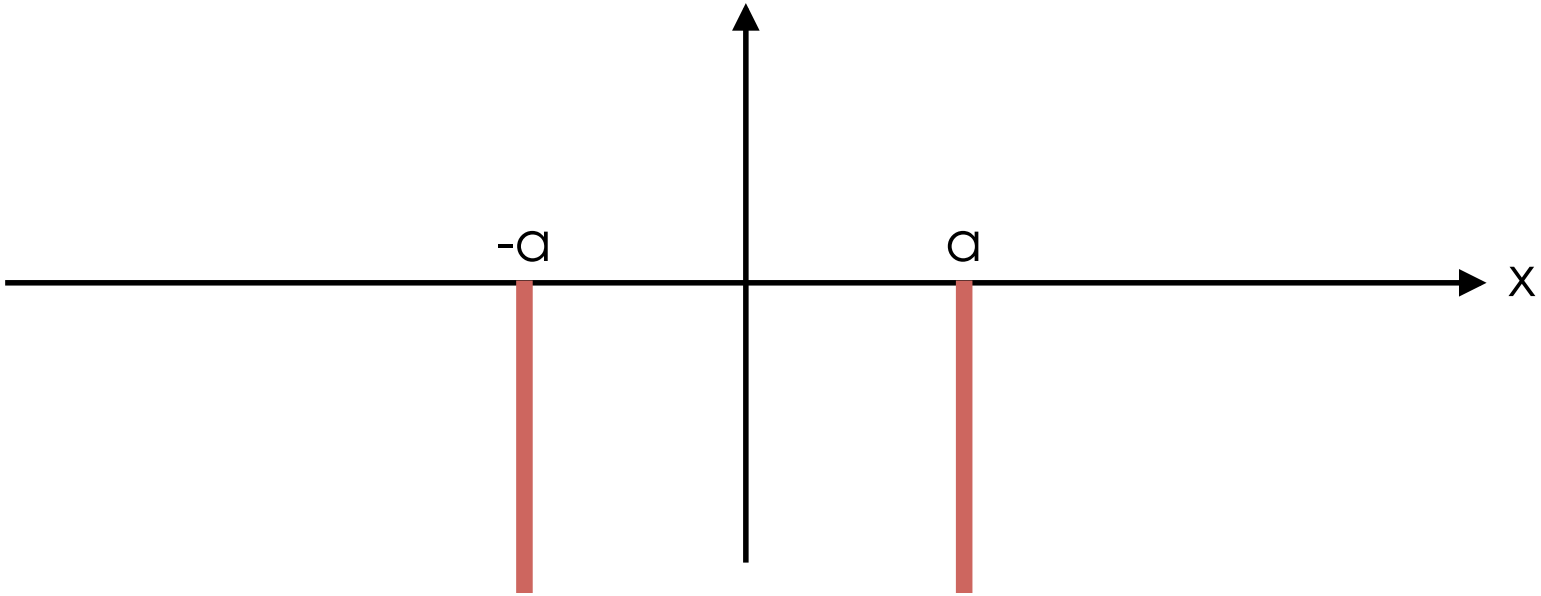
$$V(x) = -\frac{U_0}{4m}\left(\delta(x - a/2) + \delta(x + a/2)\right)$$

2.1 We consider the wave function ψ_{k0} with even parity and momentum k. Show the relation

$$\left.\frac{\mathrm{d}\psi_{k0}(x)}{\mathrm{d}x}\right|_{x=\pm a/2+\varepsilon} - \left.\frac{\mathrm{d}\psi_{k0}(x)}{\mathrm{d}x}\right|_{x=\pm a/2-\varepsilon} + U_0\psi_{k0}(\pm a/2) = 0, \tag{21}$$

with a small and positive number ε at $x = \pm a/2$, respectively.

② Two δ-function potentials



$$V(x) = -\frac{U_0}{4m}\left(\delta(x - a/2) + \delta(x + a/2)\right)$$

2.2 We consider the wave functions for $|x| \leq a/2$ and $|x| > a/2$,

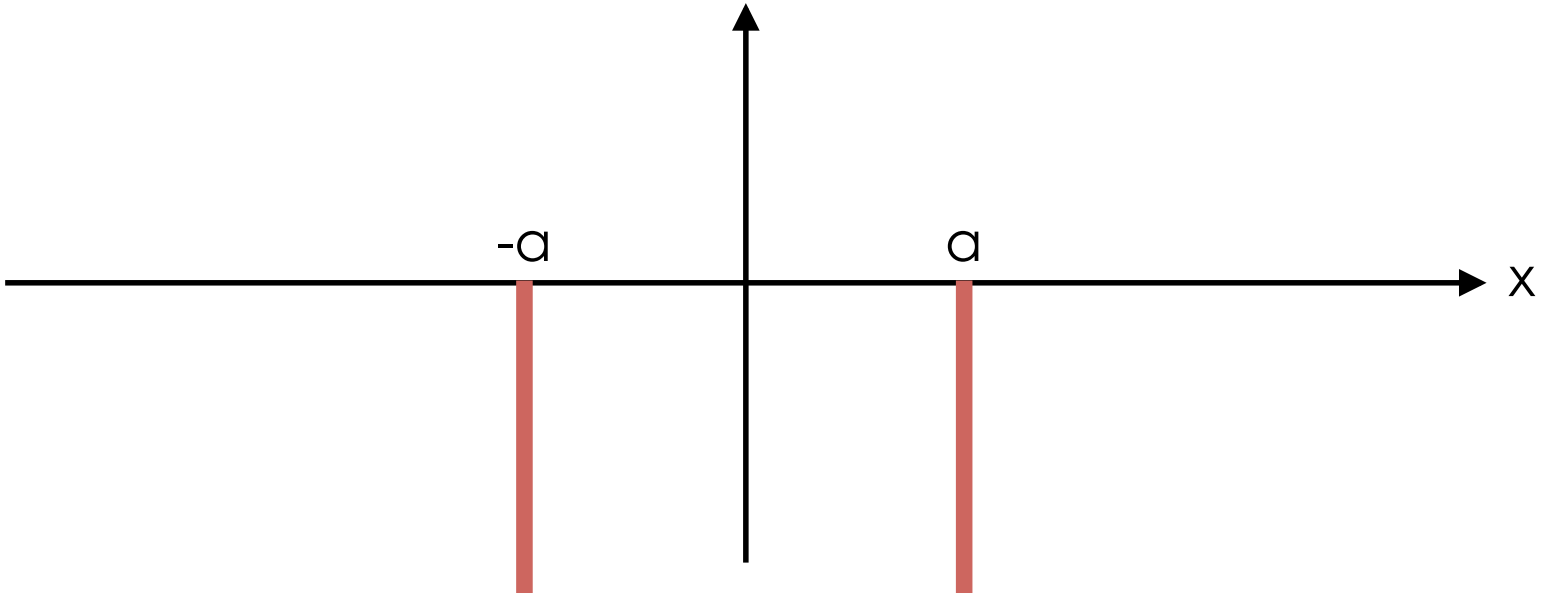
$$\psi_{k0}(x) = \cos kx \quad (|x| \leq a/2), \tag{22}$$

$$\psi_{k0}(x) = \cos(kx \pm \delta_0) \quad (|x| > a/2), \tag{23}$$

with the phase shift δ_0 for the latter. Show that δ_0 as a function of k is give by the relation

$$\tan\delta_0 = \frac{4k/U_0 + \sin ka}{1 + \cos ka}. \tag{24}$$

② Two δ-function potentials



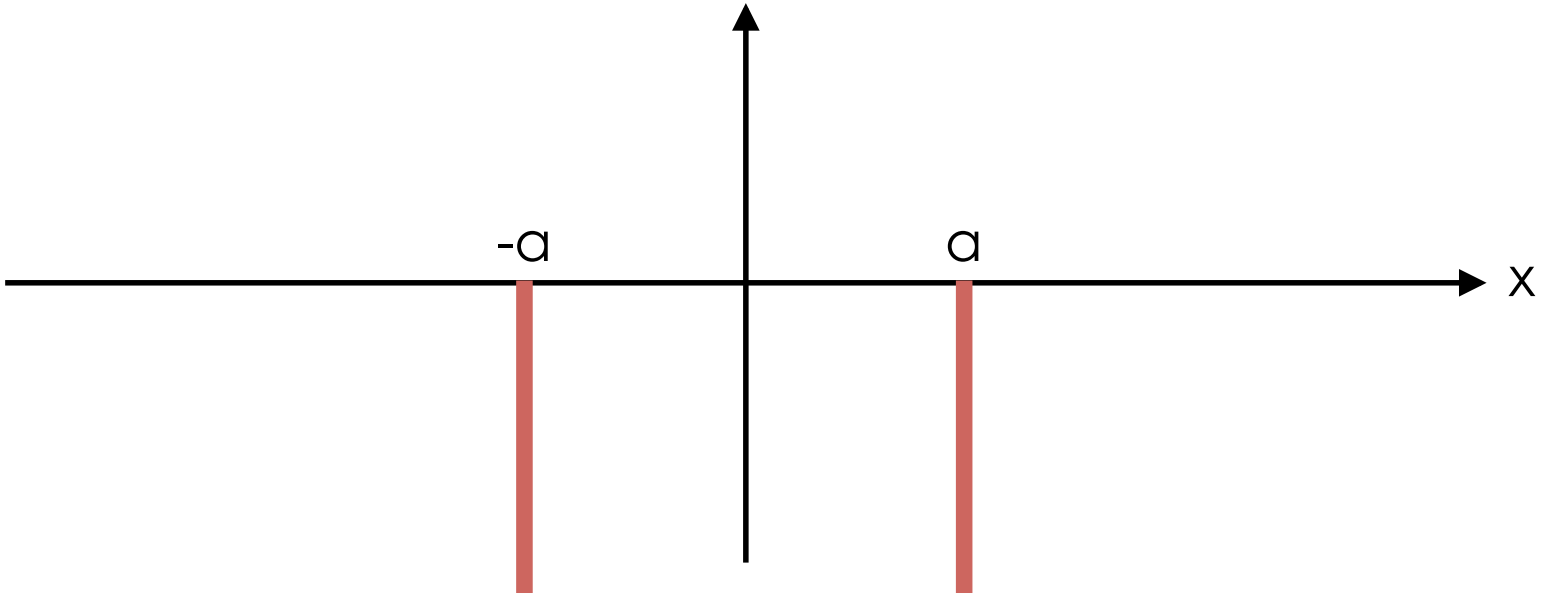
$$V(x) = -\frac{U_0}{4m}\left(\delta(x - a/2) + \delta(x + a/2)\right)$$

2.3 Show that δ_0 in limit $k \to 0, \infty$ behaves like

$$\tan\delta_0 \simeq \frac{U_0}{2k(1 + U_0 a/2)} \quad (k \to 0), \tag{25}$$

$$\tan\delta_0 \simeq \frac{U_0}{4k}(1 + \cos ka) \quad (k \to \infty). \tag{26}$$

② Two δ-function potentials



$$V(x) = -\frac{U_0}{4m}\left(\delta(x - a/2) + \delta(x + a/2)\right)$$

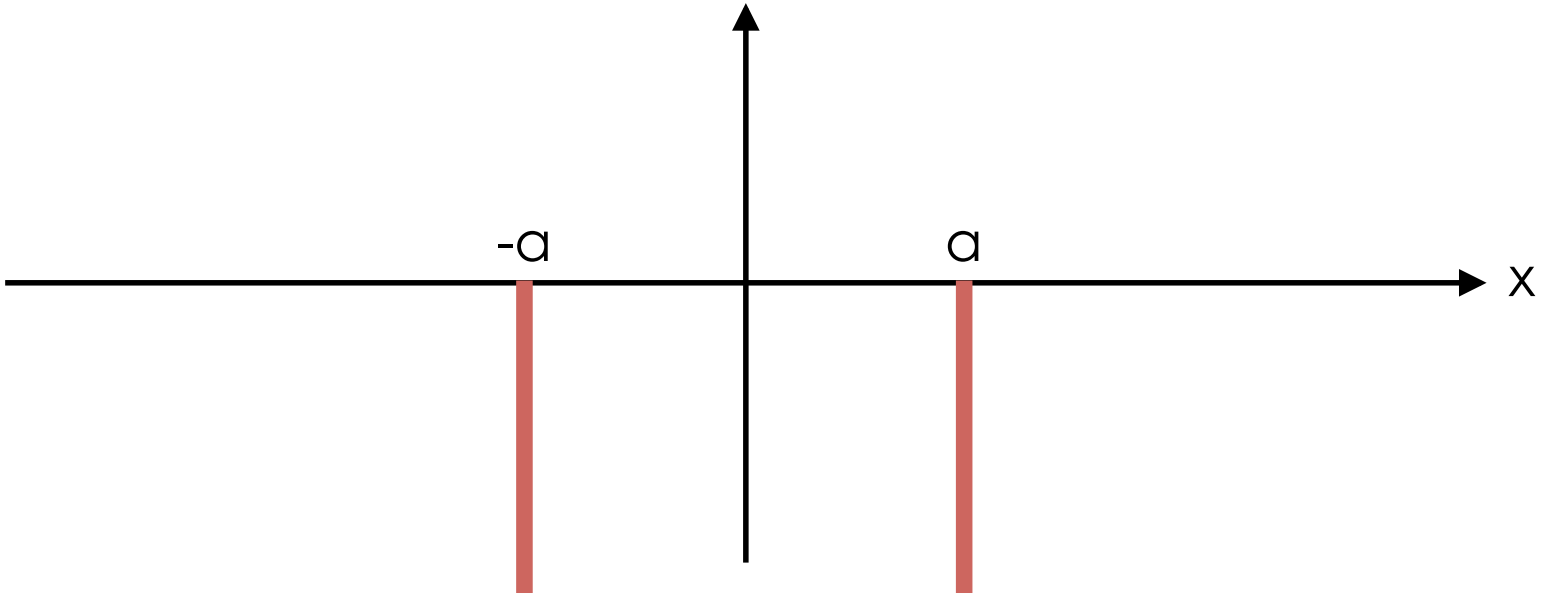
2.4 The scattering amplitude

$$f_0(k) = \frac{1}{k}\frac{1}{\cot\delta_0 - i}, \tag{27}$$

has a large value for momentum k where $\cot\delta_0 = 0$. Show that the k satisfying $\cot\delta_0 = 0$ is given by

$$\sin ka = -\frac{4k}{U_0}. \tag{28}$$

② Two δ-function potentials



$$V(x) = -\frac{U_0}{4m}\left(\delta(x - a/2) + \delta(x + a/2)\right)$$

2.5 We consider the resonance energy $E_r = k_r^2/2m$ for the solution k_r of $\sin k_r a = -4k_r/U_0$. Calculate the decay width Γ for E_r.