Positive representations of oscillatory integrals

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Motivation

- In path integral approach to quantum mechanics, amplitudes are calculated as sums over paths with complex weights e^{iS}.
- Computing path integral involves integration of highly oscillatory complex functions. Numerically this is very difficult.
- Monte Carlo methods are useful, but they are only applicable for averages with respect to (positive) probability distributions.
- This "sign problem" is one of the obstructions to increasing accuracy in lattice QFT calculations.

Idea

- Given a complex weight function $\rho(x)$ we want to find a (normalized and positive) probability distribution function of twice as many variables $P(x, y)$ such that the relation

$$\frac{\int \rho(x)\mathcal{O}(x)\mathrm{d}x}{\int \rho(x)\mathrm{d}x} = \int P(x, y)\mathcal{O}(x + iy)\mathrm{d}x\mathrm{d}y \tag{1}$$

holds at least for all polynomials $\mathcal{O}$.
- In complex Langevin [1] method P is determined by asymptotics of certain stochastic process. This is known to work in some cases and fail in other [2]. In our approach relation to any stochastic processes is not assumed.
- This idea is not new [3, 4]. The purpose of our study is to develop new techniques of finding positive representations.

Simple example

- For integrable complex gaussian weights ρ complex Langevin method succeeds [5] in providing a positive representation. We are not aware of a successful application of these methods to pure phase weights such as e^{ix^2}.
- One parameter manifold of positive representations in terms of real gaussians was obtained in [6] using complex analytic methods. It reduces to the complex Langevin representation only for specific choice of parameters.
- Numerical evaluation of integrals such as

$$\int e^{ix^2}\mathcal{O}(x)\mathrm{d}x \tag{2}$$

requires use of regulators (e.g. damping factor $\exp\left(-\epsilon x^2\right)$). When the regulating parameter ϵ is decreased one needs to rapidly increase numerical precision to get more accurate results. This difficulty is not present if a positive representation is used.
- Extension to many variables is straightforward. For simple quantum mechanical systems with quadratic actions positive representation of path integral without Wick's rotation was established.

Method of complex variables

- In the method of complex variables we analytically continue ρ to complex plane and assume it arises from integrating out another complex variable,

$$\rho(z) = \int_\Gamma P(z, w)\mathrm{d}w. \tag{3}$$

- $P(z, w)$ is assumed to be holomorphic on $\mathbb{C}^2$. When restricted to the complex plane $w = z^*$ it should be positive and normalized.
- In the next step $\rho(z)\mathcal{O}(z)$ is integrated on another contour in $\mathbb{C}^2$. Iterated contour integral of P is converted to surface integral on the subspace $w = z^*$ using Stokes theorem and the master formula (1) is established.
- Interesting examples can be generated by assuming some form of P and doing the integral (3). Finding the inverse of this operation is one of our goals.

Method of moments

- Systematic procedure for approximating unknown $P(x, y)$ was proposed by J. Wosiek. It is being developed by A. Wyrzykowski. In this approach we perform an expansion

$$P(x, y) = |\Psi(x, y)|^2, \tag{4a}$$

$$\Psi(x, y) = \sum_{n=0}^{n_{\text{cutoff}}} c_n\psi_n(x, y), \tag{4b}$$

where ψ_n is some orthonormal basis in L^2 space. Demanding that the master relation (1) holds for all polynomials up to order r_{cutoff} yields system of quadratic equations for coefficients c_n.
- Questions about convergence and efficiency of this approach are subjects of an ongoing study.

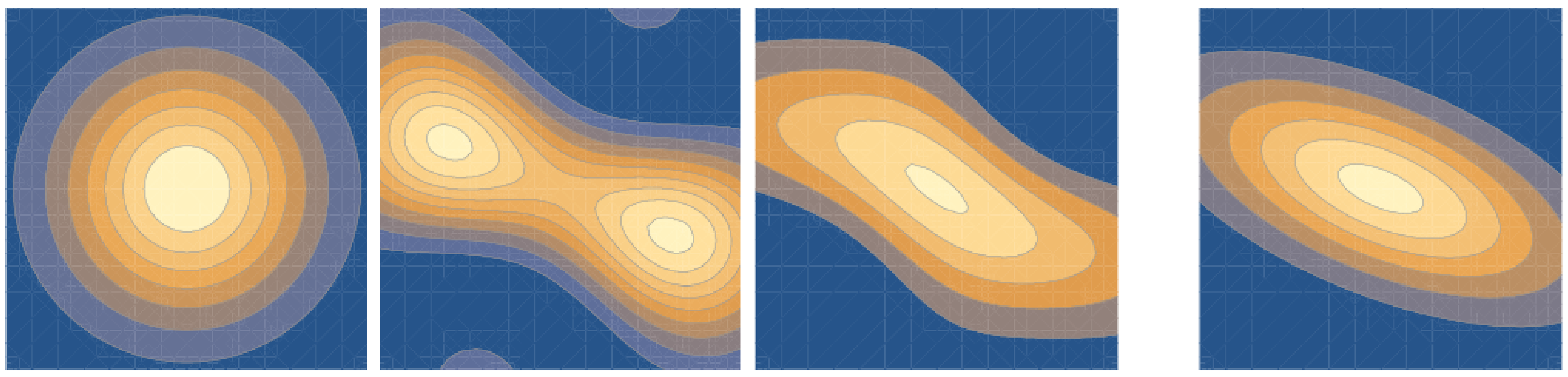

Fig. 1: Contour plots of probability distribution functions $P(x, y)$ obtained by method of moments with increasing (but still very small) truncation parameters $n_{\text{cutoff}}, r_{\text{cutoff}}$ (first three plots) and the exact result obtained using the complex Langevin method (the rightmost plot). The results are promising, but further study is required.

Summary

- New techniques for expressing oscillatory integrals as averages with respect to probability distributions are being developed.
- Our methods have been successfully applied in special cases.
- We are working on approximate solution for the general case. Results obtained thus far are promising.

References

[1] G. Parisi, Phys. Lett. B 131 (1983) 393,

[2] G. Aarts, E. Seiler, I.-O. Stamatescu, Phys. Rev. D 81, 054508,

[3] D. Weingarten, Phys. Rev. Lett. 89 (2002) 240201-1,

[4] L. L. Salcedo, J. Math. Phys. (N.Y.) 38 (1997) 1710,

[5] J. Ambjørn, S.K. Yang, Phys. Lett. B 165 (1985) 140

[6] J. Wosiek, J. High Energy Phys. (2016) 146