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Deformation quantization of principal bundles

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Construct noncommutative principal bundles deforming commutative principal bundles with a Drinfeld twist.

If the twist is related to the structure group then we have a deformation of the fiber, that becomes noncommutative.

If the twist is related to the automorphism group of the principal bundle, then in general we have noncommutative deformations of the base space as well.

The twist deformation of the fiber is compatible with the twist deformation of the base space so that we have noncommutative principal bundles with both noncommutative fiber and base space.

New Hopf-Galois extensions from twisting of Hopf-Galois extensions.

Motivations

Noncommutative Principal Bundles are not understood as well as noncommutative Vector Bundles are. There are however relevant examples of NC principal bundles (as Hopf-Galois extensions), e.g. the $SU(2)$ fibration on Connes-Landi noncommutative 4-sphere S^4_θ (instanton bundle).

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In the context of Drinfeld twist we provide a general theory, construct new examples and recover in particular the instanton bundle on S^4_θ.

The notion of **gauge group in NC geometry** can be considered from many different viewpoints:

- In NC vector bundles, gauge transformations are elements of the automorphism group of the vector bundle. Tipically unitary operators (if we have hermitian NC vector bundles).

- In gauge theories on NC spaces gauge groups are again mainly $U(N)$ or $GL(N)$ groups.

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- A way to consider NC gauge transformations based on more general groups (e.g. $SU(N)$ or $SO(N)$) is via the *Seiberg-Witten map* between commutative and noncommutative gauge transformations.

In geometry the gauge group is the group of authomorphisms of a Principal bundle (that are the identity on the base space). Then it is interesting to study NC gauge groups as authomorphisms of NC principal bundles.

Twist techniques in deformation quanization allow to construct a quite wide class of noncommutative algebras, and to consider differential calculi on these algebras. With these techniques it is possible to define **NC connections on NC vector bundles** [P.A, Schenkel 2012] (in the special case of NC equivariant connections we recover the bimodule connections of Dubois-Violette)
I am interested in the study of NC connections on NC principal bundles.

Given a Hopf algebra $\mathcal{U}$ and a twist $\mathcal{F}$, there is Twist quantization functor from commutative $\mathcal{U}$-module algebras $A = C^\infty(M)$ to noncommutative $H^\mathcal{F}$-module algebras $A_\star$.

It extends to quantization of commutative vector bundles to noncommutative vector bundles, and to their tensor products (functor of monoidal categories of $\mathcal{U}$-modules and A-bimodules and $\mathcal{U}^\mathcal{F}$-modules and $A_\star$-bimodules).

Vector bundle maps are also canonically quantized.

Right connections on commutative vector bundles are similarly quantized in NC right connections on NC vector bundles.

These quantized connections turn out to be also twisted Left connections

[Wess Group 2006, Aschieri Castellani 2010, Aschieri Schenkel 2012]

Theorem Let $(H, \mathcal{R})$ be a quasi-triangular Hopf algebra and $\big(\Omega^\bullet, \wedge, \mathrm{d}\big)$ be graded quasi-commutative. A right connection ∇ on a quasi-commutative A-bimodule $W \in {}^H_A\mathscr{M}_A$ is also a twisted left connection:

$$\nabla(a \cdot w) = (\bar{R}^\alpha \triangleright a) \cdot (\bar{R}_\alpha \blacktriangleright \nabla)(w) + (R_\alpha \triangleright w) \otimes_A (R^\alpha \triangleright \mathrm{d}a) \,. \qquad (1)$$

Rmk. If ∇ is H-equivariant we recover the notion of A-bimodule connection:

$$\nabla(a \cdot w) = a \cdot \nabla(w) + (R_\alpha \triangleright w) \otimes_A (R^\alpha \triangleright \mathrm{d}a) \,.$$

[Mourad], [Dubois-Violette Masson], [Bresser, Mueller-Hoisssen, Dimakis, Sitarz], [Madore]

Connections on tensor product modules

Let $\nabla_V : V \to V \otimes \Omega$ and $\nabla_W : W \to W \otimes \Omega$

$$\nabla_V \oplus_{\mathcal{R}} \nabla_W : V \otimes_A W \to V \otimes_A W \otimes_A \Omega \,,$$

defined by: $\nabla_V \oplus_{\mathcal{R}} \nabla_W := \nabla_V \otimes id + id \otimes_{\mathcal{R}} \nabla_W$.

Associativity: $\big(\nabla_V \oplus_{\mathcal{R}} \nabla_W\big) \oplus_{\mathcal{R}} \nabla_Z = \nabla_V \oplus_{\mathcal{R}} \big(\nabla_W \oplus_{\mathcal{R}} \nabla_Z\big)$.

H-action compatibility: $\xi \blacktriangleright (\nabla_V \oplus_{\mathcal{R}} \nabla_W) = (\xi \blacktriangleright \nabla_V) \oplus_{\mathcal{R}} (\xi \blacktriangleright \nabla_W)$.

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Applications:
-new topological invariants? or combination of know ones (e.g. from Poisson and de Rham cohomology).

-study of vierbein Gravity on NC spacetimes. (The principal bundle being the $SO(3,1)$-bundle of orthonormal frames).

For NC vierbein gravity using Seiberg-Witten map see [P.A., Castellani, Dimitrijević, 2012-14]

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Remark: Noncommutativity from Twist deformation is not the most general noncommutatity one can consider. It is however nontrivial (already semiclassically, at the Poisson level). We restrict the range of noncommutative algebras we consider, as a bonus we are able to deform also the differential geometry.

Princ. G-Bundle

If the bundle $P \longrightarrow M$ is a principal G-bundle:
The G-action on P, $P \times G \to P$ is fiber preserving
The G-action is free on P and
The G-action is transitive on the fibers
$M \simeq P/G$

i.e., the map

$$\begin{aligned} P \times G &\longrightarrow P \times_M P \\ (p, g) &\longmapsto (p, pg) \end{aligned} \quad \text{is injective and surjective}$$

Description in terms of algebras

$A \sim C^\infty(P) \qquad A \otimes A \ \sim \ C^\infty(P \times P)$ (Completion $\hat{\otimes}$ is understood or consider A the coordinate ring of an affine variety).

$H \sim C^\infty(G) \qquad A \otimes H \ \sim \ C^\infty(P \times G)$

$P \times G \to P$ dualizes to $A \longrightarrow A \otimes H$

$(p, g) \mapsto pg \qquad a \mapsto \delta^A(a) = a_0 \otimes a_1 \qquad (a_0 \otimes a_1)(p, g) = a(pg)$

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$B \sim C^\infty(M) \simeq C^\infty(P/G)$ i.e. B is the subalgebra of $A \sim C^\infty(P)$ of functions constant along the fibers

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Then $P \times G \longrightarrow P \times_M P$ is bijective iff

$$\chi : A \otimes_B A \longrightarrow A \otimes H$$

$$a \otimes_B a' \longmapsto a a'_0 \otimes a'_1 \text{ is bijective}$$

A is an H-comodule algebra because of the compatibility: $\delta^A(a\tilde{a}) = \delta^A(a)\delta^A(\tilde{a})$.

Def. of Hopf-Galois extension

Let H be a Hopf algebra and A be an H-comodule algebra,

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Equivariance property of χ

If H and A are commutative alg. then χ is an algebra map, this is no more true in the NC case

We show that χ is compatible with the H-coaction (the G-action).

A is an H-comodule, we write $A \in \mathcal{M}^H$
H is also an H-comodule with the Ad-action of H on H

$$\begin{aligned} Ad : H &\to H \otimes H \\ h &\mapsto h_2 \otimes S(h_1) h_3 \end{aligned} \qquad \begin{aligned} G \times G &\to G \\ (g, g') &\mapsto {g'}^{-1} g\, g' \end{aligned}$$

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$$h \mapsto h_2 \otimes S(h_1)h_3 \qquad\qquad (g, g') \mapsto g'^{-1} g\, g'$$

Since $A, H \in \mathcal{M}^H$ then also $A \otimes H \in \mathcal{M}^H$, $A \otimes A \in \mathcal{M}^H$, $A \otimes_B A \in \mathcal{M}^H$.

Hence χ is an H-comodule map (it is equivariant).

Moreover χ is compatible with multiplication of $A \otimes H$ and of $A \otimes_B A$ from the left with elements of A, i.e. it is a left A-module map

Twist and 2-cocycles

Consider a Hopf algebra $\mathcal{U}$, a twist $\mathcal{F} \in \mathcal{U} \otimes \mathcal{U}$ deforms $\mathcal{U}$ in $\mathcal{U}^{\mathcal{F}}$, i.e.

$$(\mathcal{U}, \cdot, \Delta, \varepsilon, S) \quad \text{is twisted to} \quad (\mathcal{U}, \cdot, \Delta^F, \varepsilon, S^F)$$

If H is paired to $\mathcal{U}$ the twist $\mathcal{F}$ defines a 2-cocycle (co-twist)

$$\gamma : H \otimes H \to \mathbb{C}[[\hbar]]$$

$$h \otimes h' \mapsto \gamma(h \otimes h') = \langle \mathcal{F}, h \otimes h' \rangle$$

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The notion of 2-cocycle γ of a Hopf alg. H doesn't require H to be paired to $\mathcal{U}$.

$$(H, \cdot, \Delta, \varepsilon, S) \quad \text{is twisted to} \quad (H, \cdot_\gamma, \Delta, \varepsilon, S_\gamma)$$

Let A be an H-comodule algebra, a 2-cocycle γ deforms A in A_γ, i.e.,

$$(A, \cdot, \delta^A) \quad \text{is twisted to} \quad (A, \cdot_\gamma, \delta^A)$$

Deformation of the structure group H

We can therefore consider in $(\mathcal{M}^{\mathcal{H}_\gamma}, \otimes^\gamma)$

$$\chi_\gamma : A_\gamma \otimes_B^\gamma A_\gamma \longrightarrow A_\gamma \otimes^\gamma H_\gamma$$

$$a \otimes_B^\gamma a' \longmapsto a \cdot_\gamma a'_0 \otimes^\gamma a'_1$$

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Theorem

$\chi_\gamma : A_\gamma \otimes_B^\gamma A_\gamma \longrightarrow A_\gamma \otimes^\gamma H_\gamma$ is bijective iff $\chi : A \otimes_B A \longrightarrow A \otimes H$ is bijective.

Proof

Follows from equivalence of monoidal categories $\{\mathcal{M}^{\mathcal{H}}, \otimes\}$ and $\{\mathcal{M}^{\mathcal{H}_\gamma}, \otimes^\gamma\}$ and from establishing a canonical isomorphism of the comodule coalgebras $(\underline{H}_\gamma, \Delta_\gamma, Ad)$ and $(\underline{H_\gamma}, \Delta_\gamma, Ad_\gamma)$.

For a different proof, without use of canonical maps, see [Montgomery Schneider]

Deformation of the base manifold M

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Let $Aut(P \to M)$ be the group of Principal bundle automorphism (ϕ, f):

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{f} & M \end{array}$$

$$\phi(pg) = \phi(p)g$$

$Aut(P \to M)$ and G actions commute

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In the dual picture we then consider a Hopf algebra K.
A is a left K-module algebra $\rho^A : A \to A \otimes K$
H is trivially a left K-module ($h \to 1_K \otimes h$)
K is trivially a right H-module ($k \to k \otimes 1_H$)

Since the coactions

$$\rho_A : A \to K \otimes A \quad \text{and} \quad \delta^A : A \to A \otimes H$$

are compatible (commute) we have $A \in {}^K\mathcal{M}^H$.
The A algebra structure is compatible with the K and H module structures.

We now consider a 2-cocycle

$$\sigma : K \otimes K \to \mathbb{C}[[\hbar]] \ ,$$

the corresponding deformations ${}_\sigma A, {}_\sigma B$, and

$${}_\sigma\chi : {}_\sigma A \ \otimes_{{}_\sigma B} \ {}_\sigma A \longrightarrow {}_\sigma A \otimes H$$

Theorem
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Example: $SU(2)$ instanton on Connes-Landi S^4_θ is the principal fibration

$$S^7_\theta \to S^4_\theta$$

Has been proven to be Hopf-Galois in [Landi Suijlekom], [Brain Landi].

Both Deformations: Base manifold and structure group

Finally we can deform the ${}_\sigma A$ and H with the 2-cocycle γ and obtain

$${}_\sigma\chi_\gamma : {}_\sigma A_\gamma \otimes_{{}_\sigma B} {}_\sigma A_\gamma \longrightarrow {}_\sigma A_\gamma \otimes H_\gamma$$

Theorem

${}_\sigma\chi_\gamma$ is bijective iff χ is bijective.