The Verlinde formula and higher-dimensional black hole entropy

Norbert Bodendorfer

Institute of Theoretical Physics
University of Warsaw

largely based on works in collaboration with
Yasha Neiman, Thomas Thiemann, Andreas Thurn

XXXIII Max Born Symposium, Wroclaw, Poland

08.07.2014

Unterstützt von / Supported by

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Stiftung / Foundation

Plan of the talk

Outline

Loop quantum gravity (LQG) in a nutshell

What is LQG

Non-perturbative, background-independent **quantisation of** general relativity

- Canonical (Dirac) quantisation $\leftarrow$ this talk (Dirac)
- Path integral (spinfoams)
- Group field theory

More generally: Techniques for background-independent & non-perturabative QFT

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Where are these techniques applicable?

- d-dimensional general relativity + standard matter fields, $d \geq 3$
- Supergravity (e.g. $d = 4$, 10, 11)

Focus of work: pure general relativity in $d = 3, 4$

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Example applications

- Quantum cosmology (singularity resolution, CMB corrections, ...)
- Black hole entropy (microscopic state counting) $\leftarrow$ this talk

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Choice of variables: Connection dynamics

Starting point: ADM formulation of $D+1$-dim. GR [Arnowitt, Deser, Misner '62]

spatial metric q_{ab}, momentum P^{ab} $\rightarrow \{q_{ab}, P^{cd}\} = \delta_a^{(c}\delta_b^{d)}$ $a = 1, \ldots, D$ spatial tensor indices

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Choice of variables $3+1$: [Ashtekar '86; Barbero '94] , $D+1$: [NB, Thiemann, Thurn '11]

SO$(D+1)$ connection A_{aIJ}, "hybrid vielbein" π^{aIJ}, $\rightarrow \{A_{aIJ}, \pi^{bKL}\} = \delta_a^b \delta_{IJ}^{KL}$

spatial metric q_{ab}, $\rightarrow qq^{ab} = \beta^2/2\ \pi^{aIJ}\pi^b{}_{IJ}$ $\beta \in \mathbb{R}\backslash\{0\}$ = free parameter

extrinsic curvature K_{ab} hidden in $A_{aIJ} = \Gamma_{aIJ}(\pi) + \beta K_{ab}\pi^b_{IJ}/\sqrt{q} + \text{gauge}$

$I, J = 1, \ldots D+1$ internal SO$(D+1)$ indices, $[IJ]$ so$(D+1)$ Lie algebra indices, Γ_{aIJ} "hybrid" spin connection

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Canonical phase space extension

$$\{q_{ab}[A,\pi], P^{cd}[A,\pi]\}_{\{A,\pi\}} \approx \delta_a^{(c}\delta_b^{d)}$$

Up to

- Gauß constraint $D_a\pi^{aIJ} = 0$ ($SO(D+1)$ gauge transformations)
- Simplicity constraint $\pi^{a[IJ}\pi^{b|KL]} = 0$ ($\pi^{aIJ} \propto n^{[I}\sqrt{q}e^{a|J]}$, Plebanski constr.)

Holonomy-flux algebra and quantisation

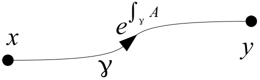
Canonical Dirac quantisation (rigorous QFT methods)

[Rovelli, Smolin, Ashtekar, Isham, Lewandowski, Marolf, Mourao, Thiemann, ...]

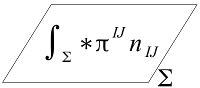
1. Holonomy-Flux algebra (point-splitting subalgebra of phase space functions)
2. Ashtekar-Lewandowski measure (defines state, i.e. positive linear functional on the above algebra)
3. GNS construction (state ⇔ representation)
4. Imposition / regularisation of constraints (anomaly free for matter coupled 3+1)

⇒ Kinematical Hilbertspace $L^2(\bar{\mathcal{A}}, d\mu_{\mathrm{AL}})$ of "generalised connections"

Holonomy along path γ



Flux through surface Σ, smearing function n_{IJ}



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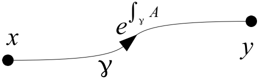

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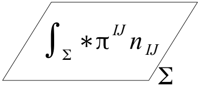

Simplicity constraints

Holonomies evaluated in a representation with heighest weight $\vec{\Lambda}$.

Simplicity constraint acting on holonomies $\Leftrightarrow \tau^{[IJ}\tau^{KL]} = 0$ equation on generators τ^{IJ}

$$\Leftrightarrow \vec{\Lambda} = (\lambda, 0, \ldots, 0), \quad \lambda \in \mathbb{N}_0.$$ [Freidel, Krasnov, Puzio, '99]

Spin Networks: A basis in $L^2(\bar{\mathcal{A}}, d\mu_{\mathrm{AL}})$

Colored graph embedded in D-dim. spatial slice σ:

- Edge: "Simple" representation λ of $SO(D+1)$
- Vertex: Invariant Map ι (gen. Clebsch-Gordan)

$$|\psi\rangle = \mathrm{Tr}\left[\left(\bigotimes_{m\,:\,\text{edges}} \pi_{\lambda_m}(h_{\gamma_m})\right)\left(\bigotimes_{n\,:\,\text{vertices}} \iota_n\right)\right]$$

h_{γ_m}: Parallel transporter along γ_m
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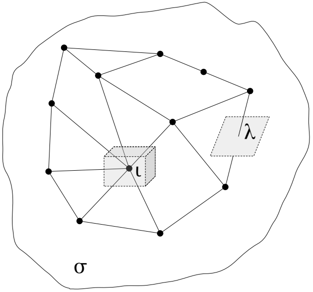

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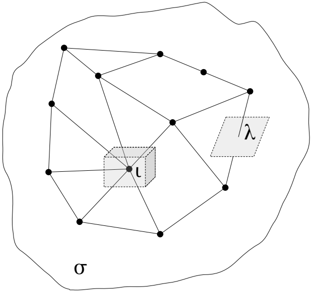

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Geometric operators:

- Area: $= \hat{A}(j)$ (spatial codimension 1)
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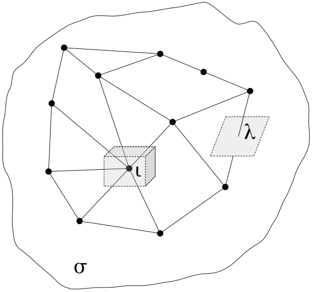

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- Hints at discrete spacetime interpretation of spin network basis states
- Spins, invariant maps, and graphs are dynamical

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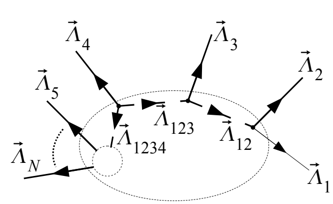

$\vec{\Lambda} = (\lambda, 0, ..., 0) \qquad \lambda = 0, 1, 2, ...$

Induces unitary, recoupling structure preserving map to SU(2)-intertwiners via

$$\mathcal{I}^{SO(D+1)}_{\text{simple}, N} \to \mathcal{I}^{SU(2)}_{N}$$

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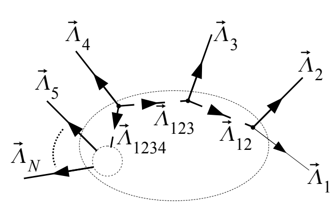

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- Solution depends on choice of recouping scheme (dynamical stability unclear)
- Dimension of intertwiner space independent of this choice (only this enters the black hole entropy computation)

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Black Hole Entropy: Boundary states

Recall (e.g. from condensed matter): Gauge fields + boundary $\Rightarrow$ **edge states**

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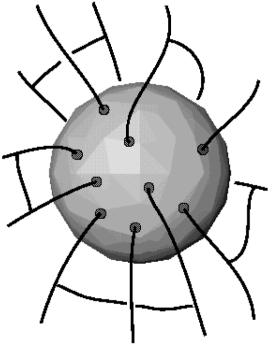
Sketch of derivation: [Ashtekar, Baez, Corichi, Krasnov '97-; ...; Engle, Noui, Perez, '09; NB, Thiemann, Thurn '13, NB '13]

1. **Model black hole as isolated horizon**
   → spacetime manifold with boundary
2. **Boundary term in canonical transformation**
   → boundary ⇒ boundary symplectic structure
3. **Poisson-algebra of boundary observables**
   → needed for quantisation
4. **Compute boundary condition**
   → relates bulk and boundary degrees of freedom
5. **Quantise** boundary observable algebra and boundary condition



picture taken from
[http://math.ucr.edu/home/baez/blackhole.html]

Black Hole Entropy

Idea: Black hole entropy $=$ Log$\big(\#$ boundary states with total area $A\big)$

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Counting results:

1. Quantise horizon theory [Ashtekar, Baez, Krasnov '00; Engle, Noui, Perez '09]
2. In $3+1$ dimensions: **SU(2) Chern-Simons theory** [Engle, Noui, Perez '09]
   Verlinde formula gives dimension of horizon Hilbert space for a spin network

$$N(j_1, \dots, j_P, k) = \frac{2}{k+2} \sum_{d=1}^{k+1} \sin^2\left(\frac{\pi d}{k+2}\right) \prod_{i=1}^{P} \frac{\sin\left(\frac{\pi d(2j_i+1)}{k+2}\right)}{\sin\left(\frac{\pi d}{k+2}\right)}$$

   j_i = representation labels, k = Chern-Simons level (cut-off for $\sum_i j_i$)

3. In higher dimensions: **Same result for dimension of Hilbert space**,
   with $\lambda = 2j$ labelling the simple representation puncturing the horizon [NB '13, NB '14]
   (Follows from implementing simplicity constraint on horizon states)

   Area of horizon: $8\pi G\beta \sum_{i=1}^{P} \sqrt{\lambda_i(\lambda_i + D - 1)}$

Black Hole Entropy

Selected results:

1. **Standard counting** (all states with given total area) $S \propto A$
[Smolin '95; Krasnov '96; Rovelli '96; Ashtekar, Baez, Corichi, Krasnov '97; ...]
2. **Thermodynamic** derivation: $S = A/4G$ [Ghosh, Perez '11]
3. Analytic continuation to imaginary free parameter, large quantum numbers:
$S = A/4G$ [Frodden, Geiller, Noui, Perez '12; Han '14]
Agreement with semiclassical **effective action** in same regime [NB, Neiman '13]
4. Calculations in **higher dimensions** reduce almost to $3+1$-dimensional case [NB '13]
(Only difference: precise form of the area spectrum)
5. For Lanczos-Lovelock gravity: **Wald entropy** [NB, Neiman '13]
(quantised area $\rightarrow$ quantised Wald entropy)
6. Computation works for **general boundaries** [NB '14]
$\rightarrow$ "weak" holography (information available at boundary)
7. Dual way of computing **entanglement entropy** of gravitational field
[Balachandran, Momen, Chandar '95; Husain '98; Donnelly '08; NB '14]

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Conclusion

- **Reformulation of general relativity** on a Yang-Mills phase space
  - Gauge group $SO(D+1)$
  - Constraints: ADM constraints + gauge invariance + simplicity
- **Dirac quantisation** by loop quantum gravity methods
  - $L^2(\bar{\mathcal{A}}, d\mu_{\text{AL}})$
  - Spin networks as basis
  - Intertwiner spaces map unitarily to those of SU(2)
- Boundaries lead to **boundary (edge) states**
  - Derive boundary symplectic structure
  - Quantise boundary observable Poisson algebra
- Entropy via **state counting**
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