Generally covariant Lagrangian formulation of Relative Locality

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Summary

- Test particle
  - in GR ,
  - in RL.
- Test particle in curved spacetime and momentum space:
  - Non-local variables,
  - Equations of motion,
  - Symmetries,
  - Interaction.
- Conclusions

Test particle in GR

The action for a free test-particle in GR can be written as

$$S = \int d\tau \, \left[\dot{x}^\mu(\tau)\, e^a_\mu(x(\tau))\, p_a(\tau) - N(\eta^{ab} p_a p_b - m^2)\right] \, . \tag{1}$$

$$\delta N \Rightarrow \eta^{ab} p_a p_b = m^2 \qquad\qquad \delta p_a \Rightarrow p_a \propto \dot{x}^\mu(\tau)\, e^a_\mu(x(\tau))$$

$$S \propto \int d\tau \, g_{\mu\nu}(x(\tau))\dot{x}^\mu(\tau)\, \dot{x}^\nu(\tau)\, , \tag{2}$$

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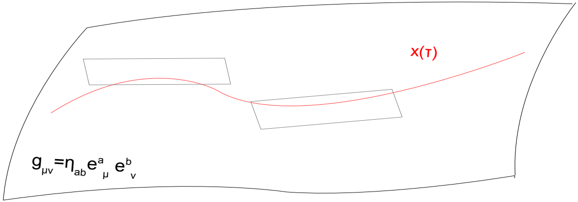
$\delta x^\mu(\tau) \Rightarrow \dot{x}^\nu \nabla_\nu x^\mu(\tau) = 0$ Geodesic motion



Test particle in RL

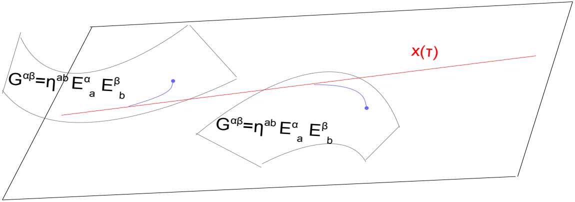
The action for a free test-particle in RL can be written as

$$S = \int d\tau \, \left[\dot{p}_\alpha(\tau)\, E^\alpha_a(p(\tau))\, x^a(\tau) + N(\mathcal{C}(p) - m^2)\right] . \qquad (3)$$

$\mathcal{C}(p) = \int_{\mathrm{geo},0}^{p(\tau)} G^{\alpha\beta} \frac{dp_\alpha}{d\lambda} \frac{dp_\beta}{d\lambda} d\lambda$ square of geodesic distance from O to $p(\tau)$

$\delta N \Rightarrow \mathcal{C}(p) = m^2$

$\delta p_\alpha, \delta x^a \Rightarrow \dot{p}_\alpha = 0, \quad \dot{x}^a(\tau) = 0$



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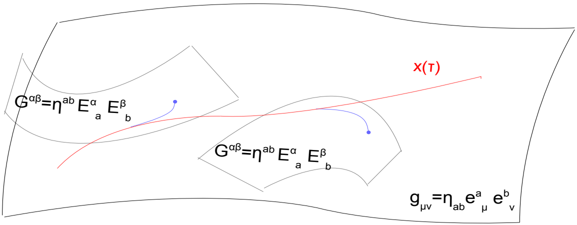

x(τ)
$G^{\alpha\beta}=\eta^{ab}E^{\alpha}_{a}E^{\beta}_{b}$
$G^{\alpha\beta}=\eta^{ab}E^{\alpha}_{a}E^{\beta}_{b}$
$g_{\mu\nu}=\eta_{ab}e^{a}_{\mu}e^{b}_{\nu}$

Non-local variables

Given a curve $\Gamma : x = x(\tau)$, one can always choose tetrads with vanishing spin connections on Γ

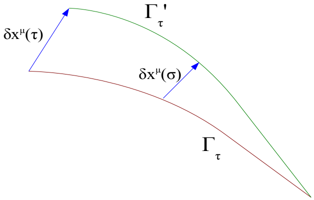
$$(\partial_\nu \bar{e}^a_\mu(x) - \partial_\mu \bar{e}^a_\nu(x))|_{x=x(\tau)} = 0, \tag{4}$$

Let us define the nonlocal variable X^a as

$$X^a(\Gamma; x(\tau)) = \int_{\Gamma_\tau} d\sigma\, \bar{e}^a_\mu(x(\sigma))\, \frac{dx^\mu}{d\sigma} = \int_0^\tau d\sigma\, \bar{e}^a_\mu(x(\sigma))\, \dot{x}^\mu\,, \tag{5}$$

and let us study its variation

$$\delta X^a = \int_{\Gamma'} \bar{e}^a_{\mu\,\Gamma'}(x+\delta x)(\dot{x}^\mu + \delta\dot{x}^\mu)\, d\sigma - \int_\Gamma \bar{e}^a_{\mu\,\Gamma}(x)\dot{x}^\mu\, d\sigma\,. \tag{6}$$



The variation of the tetrads $\bar{e}^a_\mu$ can be decomposed into two parts:

$$\delta\, \bar{e}^a_{\mu\,\Gamma}(x) = \delta_1\, \bar{e}^a_\mu(x) + \delta_2\, \bar{e}^a_\mu(x), \tag{7}$$

$$\delta_1\, \bar{e}^a_{\mu\,\Gamma}(x) = \bar{e}^a_{\mu\,\Gamma'}(x+\delta x) - \bar{e}^a_{\mu\,\Gamma}(x+\delta x), \quad \delta_2\, \bar{e}^a_{\mu\,\Gamma}(x) = \bar{e}^a_{\mu\,\Gamma}(x+\delta x) - \bar{e}^a_{\mu\,\Gamma}(x) = \delta x^\nu\, \bar{e}^a_{\mu,\nu\,\Gamma}.$$

There exist a local Lorentz transformation Λ such that

$$\bar{e}^a_{\mu\,\Gamma'}(x) = \Lambda^a_b(x)\, \bar{e}^b_{\mu\,\Gamma}(x), \tag{8}$$

For an infinitesimal local Lorentz gauge transformation $\Lambda(x) \simeq I + \lambda(x)$, keeping the leading order terms we get

$$\frac{d\lambda^{ab}}{d\sigma} = -\delta x^\nu\, \bar{\omega}^{ab}_{\mu,\nu\,\Gamma}\, \dot{x}^\mu = -\, R^{ab}_{\nu\mu}\, \delta x^\mu\, \dot{x}^\nu \;\rightarrow \lambda^{ab} = \int^\sigma R^{ab}_{\mu\nu}(\sigma')\, \delta x^\mu(\sigma')\, \dot{x}^\nu\, d\sigma', \tag{9}$$

Hence, the total variation reads

$$\begin{aligned}\delta X^a(\tau) &= \int_{\Gamma_\tau} d\sigma\, \lambda^a_{\ b}\, \bar{e}^b_\mu\, \dot{x}^\mu + \int_{\Gamma_\tau} d\sigma\, (\bar{e}_{\mu,\nu}\, \delta x^\nu + \bar{e}^a_\nu\, \delta x^\nu{}_{,\mu})\, \dot{x}^\mu = \\ &= \int_{\Gamma_\tau} d\sigma\, \lambda^a_{\ b}\, \bar{e}^b_\mu\, \dot{x}^\mu + \int_{\Gamma_\tau} d\sigma\, \frac{d}{d\sigma}(\bar{e}^a_\nu\, \delta x^\nu) = \\ &= \bar{e}^a_\nu(x(\tau))\, \delta x^\nu(x(\tau)) + \int_{\Gamma_\tau} d\sigma\, \lambda^a_{\ b}\, \bar{e}^b_\mu\, \dot{x}^\mu\,, \end{aligned} \tag{10}$$

where we used $\delta x^\mu(0) = 0$.
As a by-product for $\delta x^\mu(\tau) = x^\mu(\tau + d\tau) - x^\mu(\tau) = \dot{x}^\mu\, d\tau$, $\lambda = 0$ and one gets

$$\frac{dX^a}{d\tau} = \bar{e}^a_\mu(x(\tau))\, \dot{x}^\mu\,. \tag{11}$$

The total variation can be rewritten via an integration by part as

$$\begin{aligned}\delta X^a =& \bar{e}^a_\nu(x(\tau))\, \delta x^\nu(x(\tau)) + \int_{\Gamma_\tau} d\sigma\, \lambda^a_b(\sigma)\, \dot{X}^b(\sigma) = \\ =& \bar{e}^a_\nu(x(\tau))\, \delta x^\nu(x(\tau)) + \int_0^\tau d\sigma\, (X_b(\tau) - X_b(\sigma))\, R^{ab}_{\mu\nu}\, \delta x^\mu(\sigma)\, \dot{x}^\nu\,. \end{aligned} \tag{12}$$

This expression provides a linear map between δx^μ and δX^a.

$$\delta X^a = 0 \Leftrightarrow \delta x^\mu = 0$$

Proof $\delta X^a = 0 \Rightarrow \delta x^\mu = 0$

$x^\mu = x^\mu(\tau)$ is C^∞, thus $X^a(\tau)$ and $\delta X^a(\tau)$ are C^∞ too and one can shown

$$\left[\frac{d^n}{d\tau^n}\delta X^a(\tau)\right]\Bigg|_0 = 0 \Rightarrow \left[\frac{d^n}{d\tau^n}\delta x^\mu(\tau)\right]\Bigg|_0 = 0, \qquad \forall n \in \mathbb{N}\,. \tag{13}$$

Equations of motion

We propose the following action

$$S = \int_\Gamma d\tau \left\{ X^a[\Gamma, x(\tau)]\, E^\alpha_a\, \dot{p}_\alpha + N(\mathcal{C}(p) - m^2) \right\} , \qquad (14)$$

It can be shown that it has the proper limits (GR in flat momentum space and RL in flat spacetime).
The variation with respect to $x^\mu(\tau)$ gives

$$\delta_x S = \int d\tau\, E^\alpha_a\, \dot{p}_\alpha\, \delta X^a[x(\tau)] ,$$

thus the only solution of $\delta_x S = 0$ is

$$E^\alpha_a\, \dot{p}_\alpha = 0 \to \dot{p}_\alpha = 0 .$$

From the variations with respect to p_α and N, one gets

$$\dot{X}^a E^\alpha_a = N \frac{\partial \mathcal{C}}{\partial p_\alpha} , \qquad \mathcal{C}(p) - m^2 = 0 .$$

Hence, the following relation holds

$$\ddot{X}^a = \frac{d}{d\tau}(\dot{x}^\mu\,\bar{e}^a_\mu) = 0 \Leftrightarrow \ddot{x}^\mu + \bar{e}^\mu_a\,\partial_\rho\bar{e}^a_\nu\,\dot{x}^\nu\,\dot{x}^\rho = 0$$

and by noting that

$$\Gamma^\mu_{\nu\rho} = \bar{e}^\mu_a\,\partial_{(\rho}\bar{e}^a_{\nu)}\,,$$

it follows that the trajectory in spacetime is a GEODESIC one (as in GR!)

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\,\dot{x}^\nu\,\dot{x}^\rho = 0\,.$$

Symmetries

The action is manifestly invariant under general coordinate transformations, in both momentum space and spacetime.

It is also invariant under residual, global Lorentz transformations.

From the Relative Locality perspective we are especially interested in translational symmetries: in the case of the model of a particle moving in flat spacetime, the main features of relativity of spacetime locality are encoded in that the fact that the translations become momentum-dependent.

$$\delta x^a = E^a_\alpha(p)\, \xi^\alpha \,, \quad \dot{\xi}^\alpha = 0 \,.$$

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The analogous transformation in curved spacetime (leaving the action invariant up to a boundary term) is

$$\delta X^a = E^a_\alpha(p)\, \xi^\alpha \,, \quad \dot{\xi}^\alpha = 0 \,.$$

It maps the original geodesic, being the particle worldline, to another one (GEODESIC DEVIATION), with the magnitude of translation depending on the momentum carried by the particle.

Let us look for the infinitesimal shift of the particle trajectory δx^μ corresponding to the translation above

$$E^a_\alpha(p)\,\xi^\alpha \;=\delta X^a=\bar{e}^a_\nu(x(\tau))\,\delta x^\nu(\tau)+\int_0^\tau d\sigma\;(X_b(\tau)-X_b(\sigma))\;R^{ab}_{\mu\nu}\,\delta x^\mu(\sigma)\,\dot{x}^\nu\,. \quad (15)$$

This equation implies

$$\frac{D^2}{D\tau^2}\delta x^\mu-R^\mu{}_{\nu\rho\sigma}\,\dot{x}^\nu\,\dot{x}^\rho\,\delta x^\sigma=0\,, \quad (16)$$

where $D/D\tau\equiv\dot{x}^\mu\nabla_\mu$ is the covariant derivative projected along the worldline, subject to the initial conditions

$$\delta x^\mu(0)=\bar{e}^\mu_a(x(0))\,E^a_\alpha(p)\,\xi^\alpha\,,\qquad \left.\frac{D}{D\tau}\delta x^\mu\right|_0=0\,. \quad (17)$$

We can introduce the parameter ζ, for which $\delta x^\mu=\bar{e}^\mu_a(x)\,E^a_\alpha(p)\,\zeta^\alpha$, and get the following equation

$$\frac{D^2}{D\tau^2}\zeta^\alpha-\left(\bar{e}^a_\mu(x)\,E^\alpha_a(p)R^\mu{}_{\nu\rho\sigma}\,\bar{e}^\sigma_b(x)\,E^b_\beta(p)\right)\,\dot{x}^\nu\,\dot{x}^\rho\,\zeta^\beta=0\,, \quad (18)$$

and it describes a congruence of particle worldlines in the spacetime whose curvature is momentum-dependent. It might serve as a starting point of more phenomenologically oriented investigations (gravitational lensing,..).

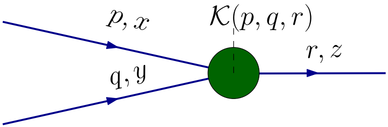
Interaction

Let us consider a process with two incoming particles and an outgoing one. The action for two particles this process is

$$\begin{aligned} S = & \int_{-\infty}^{t} d\tau \left[X^a E_a^\alpha \dot{p}_\alpha + N_p \left(\mathcal{C}(p) - m_p^2 \right) \right] \\ & + \int_{-\infty}^{t} d\tau \left[Y^a E_a^\alpha \dot{q}_\alpha + N_q \left(\mathcal{C}(q) - m_q^2 \right) \right] \\ & + \int_{t}^{\infty} d\tau \left[Z^a E_a^\alpha \dot{r}_\alpha + N_r \left(\mathcal{C}(r) - m_r^2 \right) \right] - k^\alpha \mathcal{K}_\alpha \left(p, q, r \right) \Big|_{\bar{\tau}} . \end{aligned} \tag{19}$$

k^α being a Lagrange multiplier enforcing on the worldlines endpoints the constraint (deformed law of energy-momentum conservation at the vertex)

$$\mathcal{K}_\alpha \left(p, q, r \right) = \left(p \oplus q \oplus (\ominus r) \right)_\alpha = 0.$$



The boundary term contributes with the constraints

$$\mathcal{K}_\alpha\left(p,q,r\right)\Big|_t = \left(p\oplus q\oplus(\ominus r)\right)_\alpha\Big|_t = 0, \tag{20}$$

$$X^a\left(t\right) = k^\beta E^a_\alpha(p)\frac{\partial\mathcal{K}_\beta}{\partial p_\alpha}\Big|_t, \tag{21}$$

$$Y^a\left(t\right) = k^\beta E^a_\alpha(q)\frac{\partial\mathcal{K}_\beta}{\partial q_\alpha}\Big|_t. \tag{22}$$

$$Z^a\left(t\right) = k^\beta E^a_\alpha(r)\frac{\partial\mathcal{K}_\beta}{\partial r_\alpha}\Big|_t. \tag{23}$$

When k^α changes, the X^a transforms as

$$\delta X^a\Big|_t = E^a_\alpha(p)\frac{\partial\left(p\oplus q\oplus(\ominus r)\right)_\beta}{\partial p_\alpha}\delta k^\beta\Big|_t, \tag{24}$$

with analogous relations holding for the other particles. Assuming that we take the initial condition for the geodesic deviation equation (17) at the interaction point, we find that

$$\delta x^\mu\Big|_t = \bar{e}^\mu_a\, E^a_\alpha(p)\frac{\partial\left(p\oplus q\oplus(\ominus r)\right)_\beta}{\partial p_\alpha}\delta k^\beta\Big|_t\,. \tag{25}$$

We see therefore that the structure of the interaction vertex in the case of curved spacetime is essentially the same as in the flat spacetime case of Relative Locality.

Conclusions

We gave a Lagrangian formulation for test particles in curved spacetime and momentum space in which

- the limits of flat momentum space and flat spacetime are described by GR and RL, respectively,
- the trajectory in spacetime is a geodesic one,
- the generalization of momentum-dependent translations maps spacetime geodesics among themeselves.

Perspective: phenomenology

- in astrophysics (gravitational lensing),
- in cosmology (CMB, neutrino cosmology).