On covariance of QFT on Moyal NC spaces under finite deformed Poincare' transformations, and "quantum reference" frames

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Introduction

Simplest noncommutative spacetime: **constant commutators (GMW):**

$$[\hat{x}^\mu, \hat{x}^\nu] = i\mathbf{1}\theta^{\mu\nu} \qquad \Leftrightarrow \qquad [x^\mu \stackrel{\star}{,} x^\nu] = i\mathbf{1}\theta^{\mu\nu} \tag{1}$$

$\theta^{\mu\nu} = -\theta^{\nu\mu}$. Privileged laboratory to investigate "noncommutative" QFT. Algebra $\widehat{\mathscr{X}}$ of functions on Moyal space: generated by $\mathbf{1}, \hat{x}^\mu$ fulfilling (1).

Various inequivalent approaches to QFT since 1996:

- Path-integral: Filk 96, Douglas,Schwarz,Nekrasov, Seiberg-Witten,..
- Operator approaches (canonical, algebraic, or á la Wightman? Standard or twisted Poincaré covariance? ...?) Chaichian *et al*,Balachandran *et al*, Lizzi-Vitale, Abe, Zahn, F.-Wess, F.; Aschieri-Lizzi-Vitale, Lukierski *et al*,.... Our framework

Chaichian *et al* 2004, Wess 2004, Koch *et al* 2004, Oeckl 2000:
(1) are not Poincaré -invariant; *but "twisted Poincaré"* invariant.

(Alternative: Doplicher-Fredenhagen-Roberts 94-95, *et* Bahns, Piacitelli,...: $\theta^{\mu\nu} \mapsto Q^{\mu\nu}$ central Lorentz-tensor. $\Rightarrow$ Poincaré-covariant.)

q1) **How to implement twisted Poincaré covariance in QFT? Namely, what is the deformed analog of Wightman axiom R?**

R. Relativistic transformations of $\mathscr{H}$ are represented by a strongly continuous unitary operator map $U:(A,y)\in \mathrm{P_s}\equiv SL(2,\mathbb{C})\ltimes\mathbb{R}^4 \mapsto U(I,y)\,U(A,0)$. $\exists!$ invariant state Ψ_0 (the vacuum). Each field operator φ^α belongs to an irred. finite-dim representation S of $SL(2,\mathbb{C})$ and transforms as follows:

$$U(A,y)\,\varphi^\alpha(x)\,U^{-1}(A,y) = S^\alpha_\beta(A^{-1})\,\varphi^\beta\left[\Lambda(A)x+y\right]. \qquad (2)$$

Energy-momentum and Lorentz operators: self-adjoint $\sigma(P_\mu),\sigma(M_{\mu\nu})$ s.t.

$$U(I,y) = e^{i\sigma(P)\cdot y}, \qquad U(A,0) = e^{i\sigma(M_{\mu\nu})\omega^{\mu\nu}}, \qquad (3)$$

$A=e^{s(M_{\mu\nu})\omega^{\mu\nu}}$, $s\equiv$fund. repr. of $sl(2,\mathbb{C})$: $s(M_{ij})=\frac{1}{2}\varepsilon^{ijk}\tau^k$, $s(M_{i0})=\frac{i}{2}\tau^i$.

$\Lambda^\mu_\nu(A)=\frac{1}{2}\mathrm{tr}(\sigma^\mu A\sigma^\nu A^\dagger)$ is the projection $SL(2,\mathbb{C})\mapsto SO^+(1,3)$.
So far incomplete answers to q1:

1. Based on the action of $H'\equiv U_\theta\mathscr{P}$ =deformed Poincaré UEA; but within H' $\nexists$ (full) analog of $\mathscr{P}$, nor of *"infinitesimal" transformations; so describing finite ones by* e^g, $g\in\mathscr{P}$, *is not justified!*
2. No clear distinction of active/passive transformations [lhs/rhs(2)].

Related questions:
q2) do coordinates $\hat{x}, \hat{y}$ of different spacetime points commute?
q3) deform the commutation relations of $a_p, a_p^\dagger$ for free fields?

[F.-Wess 07, F. 08]: Wightman axioms with twisted Poincaré covariance under the action of (the passive) $H' = U_\theta \mathscr{P}$.

Here: sketch how to complete that work by **providing the analog of (2)**, using the Hopf algebra H deformation of $Fun(\mathrm{P})$. Stick to scalar fields. Noncommutativity of H, i.e. of the variables parametrizing changes of reference frames, require the latter to be "quantum objects".

Plan

1. Introduction
2. The dual Hopf $*$-algebras H, H', and their (co)module $*$-algebras
3. (Free) field: passive/active transf's and formulation of (2).
4. Discussion

The Hopf ∗-algebra $H \equiv Fun_\theta(\mathrm{P})$, $H' \equiv U_\theta \mathscr{P}$

$$x^\mu \mapsto x'^\mu = \Lambda^\mu_\nu x^\nu + y^\mu \equiv x^\nu \otimes \Lambda^\mu_\nu + \mathbf{1} \otimes y^\mu. \qquad (4)$$

regard $\mathbf{1}, \Lambda^\mu_\nu, y^\mu$ as functions on the group P. They actually generate Hopf ∗-algebra *Fun* (P); the counit ε, coproduct Δ, antipode S resp. give the identical, (twice) repeated, inverse change of frame:

$$\varepsilon(\Lambda^\mu_\nu) = \delta^\mu_\nu, \quad \Delta(\Lambda^\mu_\nu) = \Lambda^\rho_\nu \otimes \Lambda^\mu_\rho, \quad S(\Lambda^\mu_\nu) = (\eta \Lambda^T \eta)^\mu_\nu \equiv \Lambda^{-1\mu}{}_\nu,$$
$$\varepsilon(y^\mu) = 0, \quad \Delta(y^\mu) = y^\nu \otimes \Lambda^\mu_\nu + \mathbf{1}_H \otimes y^\mu, \quad S(y^\mu) = -\Lambda^{-1\mu}{}_\nu y^\nu \qquad (5)$$

$$\text{Coaction:} \quad \Delta^r : \mathscr{X} \to \mathscr{X} \otimes Fun(\mathrm{P}), \quad f(x) \mapsto f(x'),$$
$$(\mathrm{id} \otimes \varepsilon) \circ \Delta^r = \mathrm{id}, \quad (\Delta \otimes \mathrm{id}) \circ \Delta^r = (\mathrm{id} \otimes \Delta^r) \circ \Delta^r. \qquad (6)$$

NC analog? "Quantize" [Drinfeld 83] *Fun*(P), i.e. make it noncommut. Hopf dual to $H' \equiv U_\theta \mathscr{P}$, so that (1) & (4-6) imply $[\hat{x}'^\mu, \hat{x}'^\nu] = i\mathbf{1}\theta^{\mu\nu}$:

The Hopf $*$-algebra $H \equiv Fun_\theta(\mathrm{P})$, $H' \equiv U_\theta \mathscr{P}$

$$\hat{x}^\mu \mapsto \hat{x}'^\mu = \Lambda^\mu_\nu \hat{x}^\nu + \hat{y}^\mu \equiv \hat{x}^\nu \otimes \Lambda^\mu_\nu + \mathbf{1} \otimes \hat{y}^\mu . \tag{4}$$

regard $\mathbf{1}, \Lambda^\mu_\nu, \hat{y}^\mu$ as functions on the group P. They actually generate Hopf $*$-algebra $H = Fun_\theta(\mathrm{P})$; the counit ε, coproduct Δ, antipode S resp. give the identical, (twice) repeated, inverse change of frame:

$$\begin{aligned} &\varepsilon(\Lambda^\mu_\nu) = \delta^\mu_\nu, \quad &&\Delta(\Lambda^\mu_\nu) = \Lambda^\rho_\nu \otimes \Lambda^\mu_\rho, \quad &&S(\Lambda^\mu_\nu) = (\eta\Lambda^T\eta)^\mu_\nu \equiv \Lambda^{-1}{}^\mu_\nu, \\ &\varepsilon(\hat{y}^\mu) = 0, \quad &&\Delta(\hat{y}^\mu) = \hat{y}^\nu \otimes \Lambda^\mu_\nu + \mathbf{1}_H \otimes \hat{y}^\mu, \quad &&S(\hat{y}^\mu) = -\Lambda^{-1}{}^\mu_\nu \hat{y}^\nu \end{aligned} \tag{5}$$

$$\begin{aligned} &\text{Coaction:} \qquad \Delta^r : \hat{\mathscr{X}} \to \hat{\mathscr{X}} \otimes H, \qquad f(\hat{x}) \mapsto f(\hat{x}'), \\ &(\mathrm{id} \otimes \varepsilon) \circ \Delta^r = \mathrm{id}, \quad (\Delta \otimes \mathrm{id}) \circ \Delta^r = (\mathrm{id} \otimes \Delta^r) \circ \Delta^r . \end{aligned} \tag{6}$$

NC analog? "Quantize" [Drinfeld 83] $Fun(\mathrm{P})$, i.e. make it noncommut. Hopf dual to $H' \equiv U_\theta \mathscr{P}$, so that (1) & (4-6) imply $[\hat{x}'^\mu, \hat{x}'^\nu] = i\mathbf{1}\theta^{\mu\nu}$:

$$[\Lambda^\mu_\rho, \cdot] = 0, \quad \Lambda^\mu_\rho \Lambda^\nu_\sigma \eta^{\rho\sigma} = \eta^{\mu\nu}\mathbf{1}_H, \quad [\hat{y}^\mu, \hat{y}^\nu] = i(\theta^{\mu\nu}\mathbf{1}_H - \theta^{\rho\sigma}\Lambda^\mu_\rho \Lambda^\nu_\sigma); \tag{7}$$

[Oeckl 00]. **Restricted** Lorentz: add $\det\Lambda = \mathbf{1}$, $\Lambda^0_0 > 0$. $SL(2,\mathbb{C})$: [Podleś-Woronowicz 96].

H is noncommutative $\Rightarrow$ **"quantum" change of reference frame!**;
Central Λ^{μ}_{ν}: relative orientation & velocity of two frames "not quantized".

$$[\hat{y}^{\mu}, \hat{y}^{\nu}] = i(\theta^{\mu\nu} - \theta^{\rho\sigma}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma})\mathbf{1} \equiv i\vartheta^{\mu\nu}(\Lambda)\mathbf{1} \neq 0$$

the relative position of their space-time origins is quantized.
Irreducible $*$-representations (irreps) of H: parametrized by eigenvalues $\lambda \in M_4(\mathbb{R})$ (with $\lambda\eta\lambda^T = \eta$) of Λ, and depend on $r := rank\,\vartheta(\lambda)$:

- If $r(\lambda) = 4$, then after a Darboux transf. $\hat{y}^{\mu}$ fulfill Heisenberg CR in 4D phase space, and their exponentials the Weyl CR: up to unitary transformations, $\exists!$ irrep $\sim$ Schrödinger irrep on $L^2(\mathbb{R}^2)$.

  Maximal relative localization of the origins on coherent states.
- If $r(\lambda) = 2$, after a Darboux transf irreps further parametrized by 2 real numbers y^{μ}; up to unitary transf.s, each is $\sim$ Schrödinger irrep on $L^2(\mathbb{R})$. Maximal relative localization of origin on coherent states.
- If $r(\lambda) = 0$, irreps further parametrized by 4 real numbers y^{μ}, and all their carrier spaces have dim=1: "classical changes of frame". (note: $\lambda = I \Rightarrow r = 0$).

Chaichian et al, Wess, Koch et al 04: (1) are covariant under action of $H' \equiv U_\theta \mathscr{P} =$"twisted" $U\mathscr{P}$; $U\mathscr{P}, H'$ have:

- same $*$-algebra over $\mathbb{C}[[\theta]]$: generated by real $M_{\mu\nu}, P_\mu, \mathbf{1}_{H'}$ fulfilling

$$
\begin{aligned}
&[M_\omega, M_{\omega'}] = 2i\,(\omega\eta\,\omega' - \omega'\eta\,\omega)^{\mu\nu}, \\
&[M_\omega, P_\mu] = 2iP_\nu(\omega\eta)^\nu_\mu, \qquad [P_\mu, P_\nu] = 0;
\end{aligned}
\tag{8}
$$

here: $M_\omega := \omega^{\mu\nu} M_{\mu\nu}$, $\eta \equiv$Minkowski metric; and same counit ε'.
$\Rightarrow$ Same irreps, hence same classification of elementary particles!

- coproducts Δ'_0, Δ' related by $\Delta'_0(g) \longrightarrow \Delta'(g) = \mathscr{F}\Delta'_0(g)\mathscr{F}^{-1}$, unitary *twist* [Drinfel'd 83] $\mathscr{F} \equiv \mathscr{F}^\alpha \otimes \mathscr{F}_\alpha := \exp\left(\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu\right)$:

$$
\begin{aligned}
&\Delta'(M_\omega) = M_\omega \otimes \mathbf{1}_{H'} + \mathbf{1}_{H'} \otimes M_\omega + P_\mu [\omega\eta, \theta\eta]^\mu_\nu \otimes P^\nu, \\
&\Delta'(P_\mu) = P_\mu \otimes \mathbf{1}_{H'} + \mathbf{1}_{H'} \otimes P_\mu,
\end{aligned}
\tag{9}
$$

- Same antipode S', if $\mathscr{F}$ is as above.

Triangular structure $\mathscr{R} = \mathscr{F}_{21}\mathscr{F}^{-1} = \exp\left(-i\theta^{\mu\nu} P_\mu \otimes P_\nu\right)$.

As expected, in H Δ, ε, S are undeformed, only m is deformed: while in H' m', ε', S' are undeformed, only Δ' is deformed. In addition:

- **Lorentz** Hopf subalgebra $H_L \subset H$ generated by $\mathbf{1}_H, \Lambda^\mu_\nu$ **is undeformed** $\Rightarrow$ $H_L \simeq$ algebra of functions on $SO^+(1,3)$: only the position of the origin of a frame w.r.t. another one is "quantum".
- Whereas in H' **translation** Hopf subalgebra $H'_T \subset H'$ **undeformed**.

Mild deformations!

Sweedler notation:

$$\Delta(a) = a_{(1)} \otimes a_{(2)} \equiv \sum_I a^I_{(1)} \otimes a^I_{(2)}$$

$$[\Delta \otimes \mathrm{id}\,) \circ \Delta](a) = a_{(1)} \otimes a_{(2)} \otimes a_{(3)} \equiv \sum_I a^I_{(1)} \otimes a^I_{(2)} \otimes a^I_{(3)}$$

$$\Delta'_0(g) = g_{(1)} \otimes g_{(2)} \equiv \sum_I g^I_{(1)} \otimes g^I_{(2)}$$
$$\Delta'(g) = g_{(\hat{1})} \otimes g_{(\hat{2})} \equiv \sum_I g^I_{(\hat{1})} \otimes g^I_{(\hat{2})}$$

...

Dual H,H': $\exists$ nondegenerate bilinear map $\langle\,,\rangle : H'\otimes H\to\mathbb{C}$ (pairing) s.t.

$$\begin{aligned}
&\langle g,\mathbf{1}_H\rangle=\varepsilon'(g), && \langle\mathbf{1}_{H'},b\rangle=\varepsilon(b),\\
&\langle g,bc\rangle=\langle g_{(\hat{1})},b\rangle\langle g_{(\hat{2})},c\rangle, && \langle gh,b\rangle=\langle g,b_{(1)}\rangle\langle h,b_{(2)}\rangle,\\
&\langle S'(g),b\rangle=\langle g,S(b)\rangle, &&\\
&\langle g^{*'},b\rangle=\overline{\langle g,S(b)^*\rangle}, && \langle g,b^*\rangle=\overline{\langle S'(g)^{*'},b\rangle}.
\end{aligned}\tag{10}$$

Here $\langle\,,\rangle : H'\otimes H\to\mathbb{C}$ recursively determined from the relations

$$\begin{aligned}
&\langle\mathbf{1}_{H'},\mathbf{1}_H\rangle=1, && \langle\mathbf{1}_{H'},\Lambda^\mu_\nu\rangle=\delta^\mu_\nu, && \langle\mathbf{1}_{H'},\hat{y}_\mu\rangle=0,\\
&\langle M_\omega,\mathbf{1}_H\rangle=0, && \langle M_\omega,\Lambda^\mu_\nu\rangle=2i(\omega\eta)^\mu_\nu, && \langle M_\omega,\hat{y}^\nu\rangle=0,\\
&\langle P_\mu,\mathbf{1}_H\rangle=0, && \langle P_\mu,\Lambda^\mu_\nu\rangle=0, && \langle P_\mu,\hat{y}^\nu\rangle=i\delta^\mu_\nu,
\end{aligned}\tag{11}$$

as in the undeformed case. $H\simeq\star$-deformation [Drinfel'd] of $Fun(\mathrm{P})$.

H has coquasitriangular structure [Majid] $\langle\mathscr{R},\cdot\otimes\cdot\rangle$, and (7) amount to

$$b\,c=\langle\mathscr{R},c_{(1)}\otimes b_{(1)}\rangle\;c_{(2)}b_{(2)}\;\langle\mathscr{R}_{21},c_{(3)}\otimes b_{(3)}\rangle \tag{12}$$

Twisting $U\mathscr{P}$-module & $Fun(P)$-comodule $*$-algebras

Let $\mathscr{A}$ be a $U\mathscr{P}$-module $*$-algebra, $V(\mathscr{A})$ the underlying vector space. $V(\mathscr{A})[[\theta]]$ gets a H-module $*$-algebra $\mathscr{A}_\star$ when endowed with the product

$$a\star a' := \left(\overline{\mathscr{F}}^{\alpha}\triangleright_c a\right)\left(\overline{\mathscr{F}}_{\alpha}\triangleright_c a'\right). \qquad \overline{\mathscr{F}}^{\alpha}\otimes\overline{\mathscr{F}}_{\alpha}=\mathscr{F}^{-1} \tag{13}$$

I.e. $\star$ is associative by the cocycle condition for $\mathscr{F}$, fulfills $(a\star a')^*=a'^*\star a^*$ and

$$g\triangleright_c(a\star a')=\left[g_{(\hat{1})}\triangleright_c a\right]\star\left[g_{(\hat{2})}\triangleright_c a'\right]. \tag{14}$$

(deformed Leibniz rule). From $\mathscr{A}\otimes\mathscr{B}$ the H'-module $*$-algebra $(\mathscr{A}\otimes\mathscr{B})_\star$: setting $a\otimes_\star b:=(a\otimes\mathbf{1}_{\mathscr{B}})\star(\mathbf{1}_{\mathscr{A}}\otimes b)$ one finds

$$(a\otimes_\star b)\star(a'\otimes_\star b') = a\star(\mathscr{R}^{(2)}\triangleright_c a')\otimes_\star(\mathscr{R}^{(1)}\triangleright_c b)\star b', \tag{15}$$

so $\otimes_\star$ is the *braided tensor product* associated to $\mathscr{R}$; *involutive*, as $\mathscr{R}\mathscr{R}_{21}=\mathbf{1}\otimes\mathbf{1}$. **If $\mathscr{A}$ defined by generators a_i and relations, then also $\mathscr{A}_\star$ is**, with same PBW series. One can define a *linear map* $\wedge: f\in\mathscr{A}\to\hat{f}\in\mathscr{A}_\star$ (Weyl map) by the eq.

$$f(a_1,a_2,...)\star=\hat{f}(a_1\star,a_2\star,...) \qquad \text{in } V(\mathscr{A})=V(\mathscr{A}_\star) \tag{16}$$

Change notation: $a_i\star a_j \rightsquigarrow \hat{a}_i\hat{a}_j$, $\hat{f}(a_i\star)\rightsquigarrow\hat{f}(\hat{a}_i)$, $\mathscr{A}_\star\rightsquigarrow\widehat{\mathscr{A}}$.

Several spacetime variables; derivatives; integrals

$U\mathscr{P}$-module algebras $\mathscr{X}$ of polynomials in x, $\mathscr{X}^n := \mathscr{X}^{\otimes n}$, $P_\mu = i\sum_{i=1}^{n} \frac{\partial}{\partial x_i^\mu}$.
Let $x_1^\mu \equiv x^\mu \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1}$, $\quad x_2^\mu \equiv \mathbf{1} \otimes x^\mu \otimes \ldots \otimes \mathbf{1}, \ldots$ Choose $\mathscr{A} = \mathscr{X}^n$:

$$a(x_i) \star b(x_j) = m\left[\exp\left[-\tfrac{i}{2} P_\mu \theta^{\mu\nu} \otimes P_\nu\right] (\triangleright_c \otimes \triangleright_c)(a \otimes b)\right], \tag{17}$$

Applying it to power series expansions,

$$e^{ip\cdot x_i} \star e^{iq\cdot x_j} = e^{iq\cdot x_j} \star e^{ip\cdot x_i} e^{-ip^t\theta q} \quad \Leftrightarrow \quad e^{ip\cdot \hat{x}_i} e^{iq\cdot \hat{x}_j} = e^{iq\cdot \hat{x}_j} e^{ip\cdot \hat{x}_i} e^{-ip^t\theta q} \tag{18}$$

$\star$ defined on larger function spaces using Fourier transforms $\tilde{a}(p), \tilde{b}(p)$:

$$a(x_i) \star b(x_j) = \int d^4p \int d^4q \; e^{i\left(p\cdot x_i + q\cdot x_j - p^t\theta q/2\right)} \tilde{a}(p) \tilde{b}(q). \tag{19}$$

(19) makes sense for $a \in \mathscr{S} \equiv \mathscr{S}(\mathbb{R}^4)$, $b \in \mathscr{S}'$, if $i = j$; $a, b \in \mathscr{S}'$, if $i \neq j$.
(Note: (17) implies $[x_i^\mu \overset{\star}{,} x_j^\nu] = \mathbf{1} i\theta^{\mu\nu}$, not $[x_i^\mu \overset{\star}{,} x_j^\nu] = \mathbf{1} i\theta^{\mu\nu}\delta_{ij}$!)

Neither the differential nor the integral calculus are deformed.

A right H-comodule $*$-algebra is also a left H'-module $*$-algebra $\widehat{\mathscr{A}}$, and conversely. The action $\triangleright : H'\otimes\widehat{\mathscr{A}} \to \widehat{\mathscr{A}}$ and the coaction $\Delta^r : \widehat{\mathscr{A}} \to \widehat{\mathscr{A}}\otimes H$ are determined by each other through the formulae

$$\Delta^r(a) = a_{(\bar{1})}\otimes a_{(2)} \qquad \Rightarrow \qquad g\triangleright a = a_{(\bar{1})}\langle g, a_{(2)}\rangle \tag{20}$$

$$\Delta^r(a) = \mathscr{T}(\triangleright\otimes\cdot)(a\otimes\mathbf{1}) = \sum_{J\in\mathscr{J}} (e^J\triangleright a)\otimes e_J, \tag{21}$$

where $\{e_J\}_{J\in\mathscr{J}}$ is any basis of $V(H)$, $\{e^J\}_{J\in\mathscr{J}} \subset H'$ the dual basis and

$$\mathscr{T} = \sum_{J\in\mathscr{J}} e^J\otimes e_J \in H'\otimes H \qquad \text{(canonical element)}. \tag{22}$$

$$\mathscr{T}^{-1} = \sum_{J\in\mathscr{J}} e^J\otimes S(e_J) = \sum_{J\in\mathscr{J}} S'(e^J)\otimes e_J = \sum_{I,J\in\mathscr{J}} s_I^J e^I\otimes e_J \tag{23}$$

$$= \mathscr{T}^{*'\otimes *} \quad \text{if } H, H' \equiv *\text{-alg.}$$

$\widehat{\mathscr{X}}$ is a left H'-module and a right H-comodule $*$-algebra resp. under

$$P_\mu \triangleright \hat{x}^\rho = i\delta^\rho_\mu, \qquad M_{\mu\nu} \triangleright \hat{x}^\rho = i(\delta^\rho_\nu \hat{x}_\mu - \delta^\rho_\mu \hat{x}_\nu), \qquad \Delta^r(\hat{x}^\mu) = \Lambda^\mu_\nu \hat{x}^\nu + \hat{y}^\mu.$$

Extend Δ^r to $\widehat{\mathscr{S}}, \widehat{\mathscr{S}}'$ by the generalized basis $\{e_p\}_{p\in\mathbb{R}^4}$, $e_p(\hat{x}) = e^{-ip\hat{x}}$

$$e^{ip\cdot(\Lambda\hat{x}+\hat{y})} = e^{ip\cdot\hat{y}} e^{i\Lambda^{-1}p\cdot\hat{x}} = e^{ip\cdot\hat{y}} \int d^4q\, \delta(q-\Lambda^{-1}p) e^{iq\cdot\hat{x}} = e^{ip\cdot\hat{y}} \int d^4q \int \frac{d^4z}{(2\pi)^4} e^{i(\Lambda q-p)\cdot z} e^{iq\cdot\hat{x}};$$

this leads us to define $\Delta^r(e_p)$ as the following combination of e_q:

$$\Delta^r(e_p) = \text{``} e^{-ip\cdot\hat{y}} e_{\Lambda^{-1}p} \text{''} = e^{-ip\cdot\hat{y}} \int d^4q \int \frac{d^4z}{(2\pi)^4} e^{i(\Lambda q-p)\cdot z} e_q, \qquad (24)$$

provided we enlarge H so as to contain 'functions' of $\hat{y}, \Lambda$ like $e^{ip\cdot\hat{y}}$, $e^{i\Lambda q\cdot z}$.

$$\Delta\left(e^{ip\cdot\hat{y}}\right) \text{``} = e^{ip_\mu(y^\nu\otimes\Lambda^\mu_\nu + \mathbf{1}_H\otimes y^\mu)} \text{''} = e^{-ip\cdot\hat{y}} \int d^4q \int \frac{d^4z}{(2\pi)^4} e^{iq\cdot(\hat{y}-z)} \otimes e^{ip\cdot(\Lambda z+\hat{y})},$$

$$\Delta\left(e^{ip\cdot\Lambda a}\right) \text{``} = e^{ip_\mu(\Lambda^\rho_\nu\otimes\Lambda^\mu_\rho)a^\nu} \text{''} = \int d^4q \int \frac{d^4z}{(2\pi)^4} e^{ip\cdot(\Lambda a-z)} \otimes e^{iq\cdot\Lambda z}$$

$$(25)$$

Scalar field of mass m

First a scalar field. Abbreviate $L=(y,\Lambda)$, $Lx=\Lambda x+y$. (2) becomes

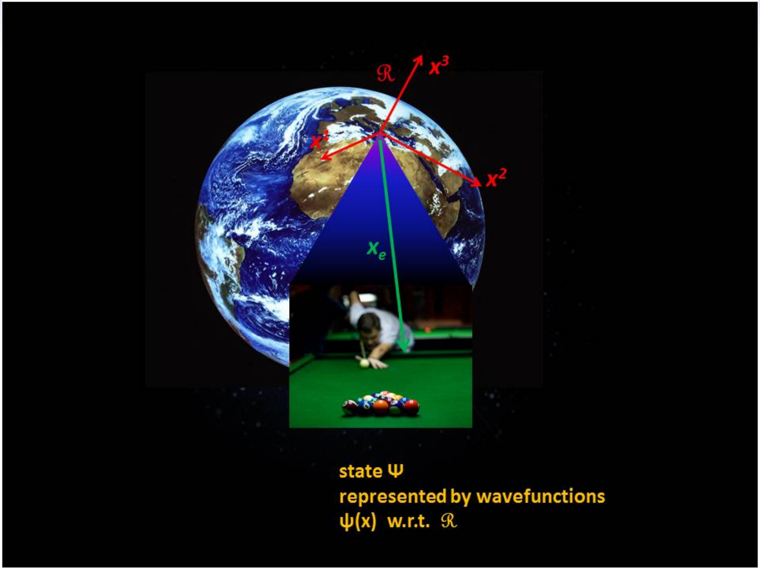
$$\varphi(Lx)=U(L)\,\varphi(x)\,U^{-1}(L), \qquad \Leftrightarrow \qquad \varphi\big(f^L\big)=U(L)\,\varphi(f)\,U^{-1}(L) \quad (26)$$

Lhs: **passive Poincaré transformation** due to reference frame change. It transforms the $\mathscr{S}'$ part of the field, or equivalently test functions $f\in\mathscr{S}\mapsto f^L\in\mathscr{S}$, $f^L(x)=f\big(L^{-1}x\big)$; not the state $\Psi\in\mathscr{H}$ of the physical system nor $\alpha\in\mathscr{A}\equiv$ algebra of operators on $\mathscr{H}$.

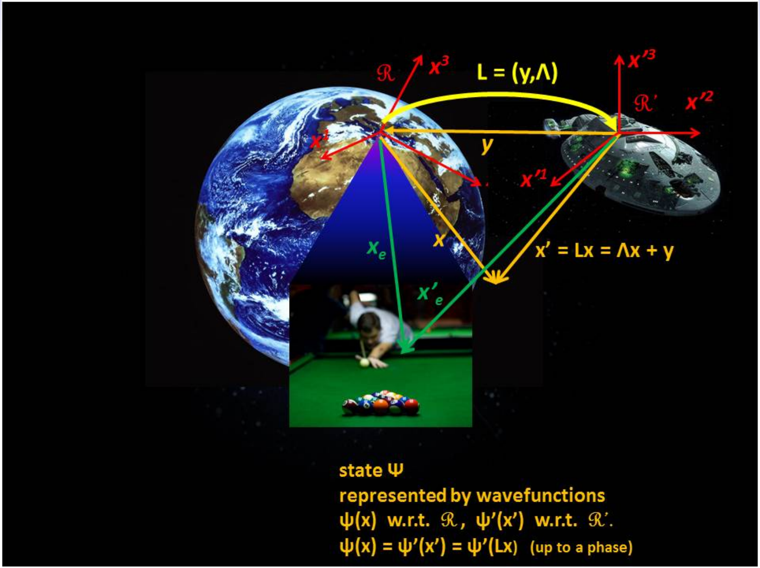
Rhs: **active Poincaré transformation**. It transforms $\Psi\mapsto U(L)\Psi$, $\alpha\mapsto U(L)\,\alpha\,U^{-1}(L)$, not $f\in\mathscr{S},\mathscr{S}'$, not the $\mathscr{S}'$ part of the field.

To deform (2) we should separaty deform passive, active transf., then impose equalities like (26) or (2) when active and passive transformations are parametrized by the *same L*.

First a small detour: we recall the physical difference between passive, active transf. thinking of an "experiment of elementary particle physics"...



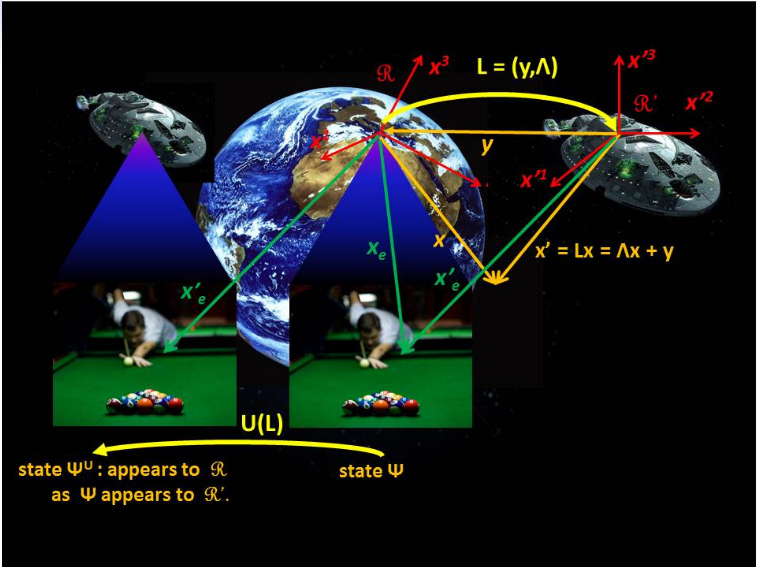
Why not? Particle physics has begun playing billiard....



$\psi \mapsto \psi' \equiv \psi^L, \;\; \psi^L(x) = \psi\big(L^{-1}x\big),$ is the passive transf. parametrized by L

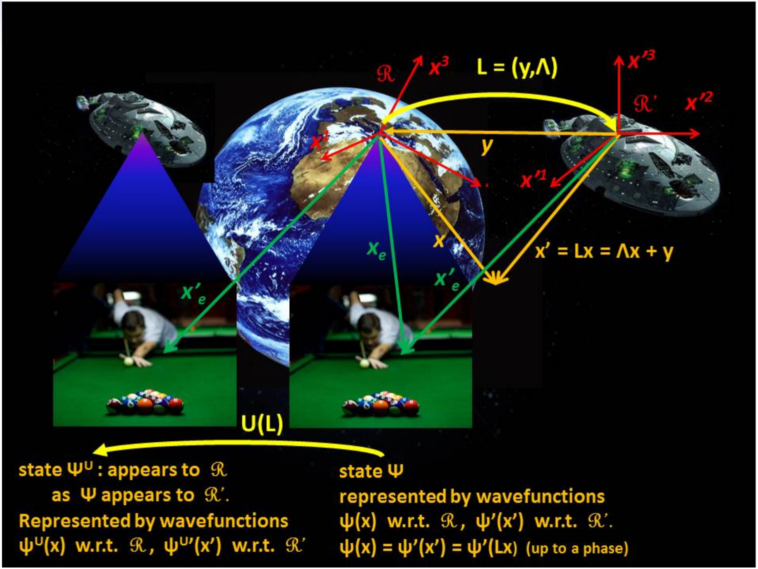
Its existence is independent of relativistic invariance of particle physics.

(It is defined in the same way for classical and quantum theories.)

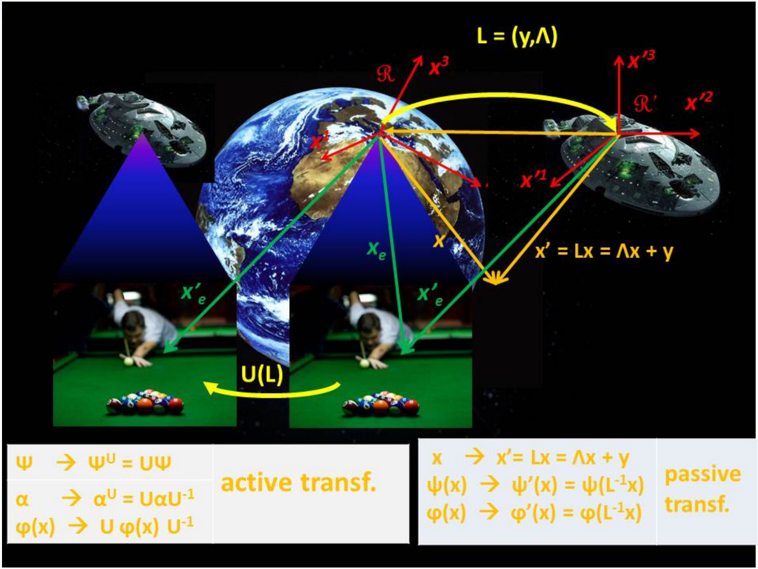


$U(L) : \Psi \mapsto \Psi^U$ is the unitary active transformation parametrized by L.
It exists because of the relativistic invariance of particle physics.

(in classical theories defined similarly.)



Consequently $\quad \psi^U(x') = \psi'(x')$


L = (y,Λ)
ℛ
x3
x'3
ℛ'
x'2
x1
x'1
xe
x
x' = Lx = Λx + y
x'e
x'e
U(L)
ψ → ψU = Uψ
α → αU = UαU-1
φ(x) → U φ(x) U-1
active transf.
x → x'= Lx = Λx + y
ψ(x) → ψ'(x) = ψ(L-1x)
φ(x) → φ'(x) = φ(L-1x)
passive transf.

By Stone thm $\exists$ self-adjoint operators $\sigma(P_\mu), \sigma(M_{\mu\nu})$ such that

$$U(I,y) = e^{i\sigma(P)\cdot y}, \qquad U(\Lambda,0) = e^{i\sigma(M_{\mu\nu})\omega^{\mu\nu}}. \tag{27}$$

Free hermitean scalar field: algebra $\mathscr{A}$ generated by a_p^+, a_p,

$$\varphi(x) = \int \frac{d^3p}{2p_0}\,[e^{-ip\cdot x}a^p + a_p^\dagger e^{ip\cdot x}] \tag{28}$$

(with $p^0 = \sqrt{\mathbf{p}^2+m^2}$), $\sigma(\mathscr{P}) \equiv$"Jordan-Schwinger realization" of $\mathscr{P}$ as operators on Fock space:

$$\sigma(P_\mu) = \int \frac{d^3p}{2p_0}\, p_\mu a_p^+ a^p \tag{29}$$

$$\sigma(M_\omega) = -i\int \frac{d^3p}{2p_0}\, a_p^+\, p_\mu \omega^{\mu j}\partial_{p^j} a^p = i\int \frac{d^3p}{2p_0}\left[p_\mu \omega^{\mu j}\partial_{p^j} a_p^+\right] a^p. \tag{30}$$

Basic properties:

$$[\sigma(P_\mu), a^p] = -p_\mu\, a^p, \qquad [\sigma(M_\omega), a^p] = ip_\mu \omega^{\mu j}\partial_{p^j} a^p, \tag{31}$$

and their hermitean conjugates.

Deformed: In [F.-Wess07] we found that the Ansatz

$$\hat{\varphi}(\hat{x}) = \int \frac{d^3p}{2p_0} [e^{-ip\cdot\hat{x}} \hat{a}^p + \hat{a}_p^\dagger e^{ip\cdot\hat{x}}], \tag{32}$$

may define a free scalar field compatible with Wightman axioms and H'-covariance *in two different ways*. In either case, *undeformed*

$$(\Box + m^2)\hat{\varphi} = 0 \tag{33}$$

$$[\hat{\varphi}(\hat{x}), \hat{\varphi}(\hat{x}')] = iF(\hat{x}-\hat{x}') = \int \frac{d^3p}{2p_0(2\pi)^3} [e^{-ip\cdot(\hat{x}-\hat{x}')} - e^{ip\cdot(\hat{x}-\hat{x}')}] \tag{34}$$

The **first way** is by plugging $\hat{a}^p, \hat{a}_p^\dagger$ satisfying the commutation relations

$$\begin{aligned} \hat{a}_p^\dagger \hat{a}_q^\dagger &= e^{ip^t\theta q} \hat{a}_q^\dagger \hat{a}_p^\dagger, \qquad \hat{a}^p \hat{a}^q = e^{ip^t\theta q} \hat{a}^q \hat{a}^p, \\ \hat{a}^p \hat{a}_q^\dagger &= e^{-ip^t\theta q} \hat{a}_q^\dagger \hat{a}^p + 2\omega_{\mathbf{p}} \delta^3(\mathbf{p}-\mathbf{q}), \\ [\hat{a}^p, f(\hat{x})] &= [\hat{a}_p^\dagger, f(\hat{x})] = 0, \end{aligned} \tag{35}$$

NB: (35) like (18) after $\theta \mapsto -\theta$!

The "functions" of a^p, a^+_p

$$\check{a}^p := a^p e^{-\frac{i}{2} p^t \theta \sigma(P)}, \qquad \check{a}^+_p := e^{\frac{i}{2} p^t \theta \sigma(P)} a^+_p, \tag{36}$$

fulfill (35) and $*$ structure $\dagger$. Together with $\mathbf{1}$, they provide a *realization* of generators $\hat{a}^p, \hat{a}^+_p, \mathbf{1}$ of $\widehat{\mathscr{A}}$ within $\mathscr{A}[[\theta]]$: $\mathscr{A}, \widehat{\mathscr{A}}$ isomorphic $*$-algebras. Moreover, they have the same vacuum and Fock space representation. Conversely,

$$\hat{\sigma}(P_\mu) := \int \frac{d^3p}{2p_0} p_\mu \, \hat{a}^+_p \, \hat{a}^p \in \widehat{\mathscr{A}} \tag{37}$$

fulfill $[\hat{\sigma}(P_\mu), \hat{a}^p] = -p_\mu \, \hat{a}^p$ and their hermitean conjugate; hence

$$\tilde{a}^p := \hat{a}^p \, e^{\frac{i}{2} p^t \theta \hat{\sigma}(P)}, \qquad \tilde{a}^+_p := e^{-\frac{i}{2} p^t \theta \hat{\sigma}(P)} \, \hat{a}^+_p, \tag{38}$$

together with $\mathbf{1}$, provide a *realization* of $a^p, a^+_p, \mathbf{1}$ of $\mathscr{A}$ within $\widehat{\mathscr{A}}[[\theta]]$.

Corollary: replacing a^p, a^+_p by $\tilde{a}^p, \tilde{a}^+_p$ in $\sigma(g)$ gives Jordan-Schwinger realization $\hat{\sigma}$ of $U\mathbf{g}$ within $\widehat{\mathscr{A}}[[\theta]]$. If $g = P_\mu$ one finds again (37).

Is $\widehat{\mathscr{A}}$ covariant? Under which Hopf algebra?

NB: (35) like (18) after $\theta \mapsto -\theta$! Accountable assuming $\widehat{\mathscr{A}}$ to be a right H^o-comodule $(*)$-algebra, hence also a left H'^o-module $(*)$-algebra:

$H^o = \left(V(H), m^o, *, \Delta, \varepsilon, S^{-1}\right) \equiv$ **opposite** Hopf $*$-algebra H, $m^o(a \otimes b) = ba$.

$H'^o = \left[V(H'), m', *', \Delta'^o, \varepsilon', S'^{-1}\right] \equiv$ co-opposite Hopf $*$-algebra, $\Delta'^o = \tau \circ \Delta'$.

H^o, H'^o are dual Hopf algebra w.r.t. the same paring $\langle\, ,\, \rangle$ as above.
As $e_p \sim \hat{a}^+_p|0\rangle$, then $\hat{a}^+_p, e_p$ transform in the same way, and so do $\hat{a}_p, e^*_p$:

$$\Delta^o_r(\hat{a}^+_p) = \text{“}e^{-ip\cdot\hat{y}}\, \hat{a}^+_{\Lambda^{-1}p}\text{''} = e^{-ip\cdot\hat{y}} \int d^4q \int \frac{d^4z}{(2\pi)^4}\, e^{i(\Lambda q - p)\cdot z}\, \hat{a}^+_q \tag{39}$$

$$\Delta^o_r(\hat{a}_p) = \text{“}e^{ip\cdot\hat{y}}\, \hat{a}^{\Lambda^{-1}p}\text{''} = e^{ip\cdot\hat{y}} \int d^4q \int \frac{d^4z}{(2\pi)^4}\, e^{-i(\Lambda q - p)\cdot z}\, \hat{a}_q \tag{40}$$

The action $\triangleright^o$ and coaction $\Delta^o_r(a) = a_{(\bar{1})} \otimes a_{(2)}$ are related by

$$g \triangleright^o a = a_{(\bar{1})}\langle g, a_{(2)}\rangle, \qquad \Delta^o_r(a) = \mathring{\mathscr{T}}(\triangleright^o \otimes \cdot)(a \otimes \mathbf{1}) = (\mathring{e}^I \triangleright^o a) \otimes \mathring{e}_I,$$

where the canonical element $\mathring{\mathscr{T}} = \mathring{e}^I \otimes \mathring{e}_I$ in $H'^o \otimes H^o$ and its inverse $\mathring{\mathscr{T}}^{-1} = \mathring{\mathscr{T}}^{*' \otimes *}$ are like before. We define

$$\mathscr{V} := (\hat{\sigma} \otimes \mathrm{id})(\mathring{\mathscr{T}}^{-1}) = s^H_K \hat{\sigma}\left(\mathring{e}^K\right) \otimes \mathring{e}_H \in \widehat{\mathscr{A}} \otimes H^o. \tag{41}$$

It is unitary: $\mathscr{V}^{* \otimes *} = \mathscr{V}^{-1} = (\hat{\sigma} \otimes \mathrm{id})(\mathring{\mathscr{T}}) = \hat{\sigma}\left(e^I\right) \otimes \mathring{e}_I$.

Lemma: $\mathscr{V}$ is unitary, and for all $\alpha \in \widehat{\mathscr{A}}$ $\quad \mathscr{V}^{-1}(\alpha \otimes \mathbf{1})\mathscr{V} = \Delta^o_r(\alpha)$.

Main result: field covariance under finite Poincaré Transformation

$$\Delta^r[\hat{\varphi}(\hat{x})] \equiv \hat{\varphi}\big(\Lambda\hat{x}+\hat{y}\big) = \mathscr{U}(\Lambda,\hat{y})\,\hat{\varphi}(\hat{x})\,\mathscr{U}^{-1}(\Lambda,\hat{y}), \tag{42}$$

where

$$\begin{aligned} \mathscr{U} &= \hat{\sigma}\left(e^{l}\right)\otimes S(e_l) \in \widehat{\mathscr{A}\otimes} H \\ \mathscr{U}^{*\otimes *} &= \mathscr{U}^{-1} = \hat{\sigma}\left(e^{l}\right)\otimes e_l; \end{aligned} \tag{43}$$

More explicitly, as the Lorentz Hopf subalgebra is undeformed [cf. (29)],

$$\begin{aligned} &\mathscr{U}(\Lambda,\hat{y}) = \mathscr{U}(\hat{y})\,\mathscr{U}(\Lambda), \\ &\mathscr{U}(\hat{y}) = e^{i\hat{\sigma}(P)\cdot\hat{y}} := \int d^4q \int \frac{d^4z}{(2\pi)^4}\, e^{i\hat{\sigma}(P)\cdot z + iq\cdot(z-\hat{y})}, \qquad (44) \\ &\mathscr{U}(\Lambda) = e^{i\hat{\sigma}(M_{\mu\nu})\omega^{\mu\nu}}, \quad \omega^{\mu\nu}(\Lambda) \text{ undeformed} \end{aligned}$$

$$\hat{\sigma}(P_\mu) = \int \frac{d^3p}{2p_0}\, p_\mu\, \hat{a}^+_p\, \hat{a}^p, \quad \text{etc.,} \quad \text{for free fields.} \tag{45}$$

(42) & (44) is reasonable also for interacting fields, using the correct realization $\hat{\sigma}$ of $\mathscr{P}$ on $\mathscr{H}$.

To describe a combination of *unrelated* passive, active transformation we have to introduce the Hopf $*$-algebras $\mathbb{H} := H \otimes H^o$, $\mathbb{H}' := H' \otimes H'^o$.
We abbreviate $a \otimes \mathbf{1} \equiv a$, $\mathbf{1} \otimes a \equiv \mathring{a}$, $g \otimes \mathbf{1} \equiv g$, $\mathbf{1} \otimes g \equiv \mathring{g}$, $\mathbb{H} \equiv HH^o$, $\mathbb{H}' \equiv H'H'^o$, etc. The product M of $\mathbb{H}$ and coproduct Δ of $\mathbb{H}'$ fulfill

$$M(a\mathring{b} \otimes c\mathring{d}) = ac\mathring{d}\mathring{b}, \qquad \Delta(g\mathring{h}) = g_{(1)}\mathring{h}_{(2')} \otimes g_{(2)}\mathring{h}_{(1')}.$$

$\mathbb{H}, \mathbb{H}'$ are dual Hopf $*$-algebras w.r.t. the pairing $\langle\langle u \otimes v, a \otimes b \rangle\rangle = \langle u, a \rangle \langle v, b \rangle$

Introduce the free field tensor algebra $\hat{\Phi}^e := \widehat{\mathscr{A}} \otimes \bigotimes_{i=1}^{\infty} \widehat{\mathscr{S}}^{+}$;
$fa = af$ for all $f \in \bigotimes_{i=1}^{\infty} \widehat{\mathscr{S}}^{+}$, $a \in \widehat{\mathscr{A}}$, by construction.
$\hat{\Phi}^e$ is a right $\mathbb{H}$-comodule algebra and a left $\mathbb{H}'$-module algebra.

We recover the Hopf algebra of finite *passive* spacetime transformations by projecting the coaction on the Hopf $*$-subalgebra $H \subset \mathbb{H}$:
$\Delta_p^r : \hat{\Phi}^e \mapsto \hat{\Phi}^e \otimes H$, $\Delta_p^r(a) := a \otimes \mathbf{1}$, $\Delta_p^r(f) := \Delta^r(f)$.

We recover the Hopf algebra of finite *active* spacetime transformations by projecting the coaction on the Hopf $*$-subalgebra $\mathring{H} \subset \mathbb{H}$:
$\Delta_a^r : \hat{\Phi}^e \mapsto \hat{\Phi}^e \otimes H^o$, $\Delta_a^r(f) = f \otimes \mathbf{1}$, $\Delta_a^r(a) = (\mathring{e}^l \triangleright^o a) \otimes \mathring{e}_l$.

Passive and active transformations commute with each other.

Discussion

Free field operators with arbitrary spin $\hat{\varphi}^{\alpha}$ transform according to

$$\hat{U}(\hat{L})\,\hat{\varphi}^{\alpha}(\hat{x})\,\hat{U}^{-1}(\hat{L}) = S^{\alpha}_{\beta}(A^{-1})\,\hat{\varphi}^{\beta}\left[\Lambda(A)\hat{x}+\hat{y}\right]. \tag{46}$$

Denoting $\omega(A)$ a matrix such that $A = e^{is(M_{\mu\nu})\omega^{\mu\nu}}$, completely analogous construction of self-adjoint operators $\hat{\sigma}(P_{\mu}), \hat{\sigma}(M_{\mu\nu})$ such that

$$\hat{U}(A,\hat{y}) = \hat{U}(\hat{y})\hat{U}(A), \qquad \hat{U}(\hat{y}) = e^{i\hat{\sigma}(P)\cdot y}, \qquad \hat{U}(A) = e^{i\hat{\sigma}(M_{\mu\nu})\omega^{\mu\nu}}.$$

One can formulate deformed covariance (42) also using the dual Hopf algebras $H', H'^{o}, \mathbb{H}'$. H' acts by $\triangleright$, H'^{o} acts by $\triangleright^{o}$, etc.

Up to our knowledge, distinguishing two different actions $\triangleright, \blacktriangleright$ was considered only in [Lukierski *et al* 11-12] in a different but related model.

Moreover, some authors [Balachandran et al], [Piacitelli] propose a formulation of deformed covariance under finite transformations obatined by "exponentiation" of the action of infinitesimal elements $\varepsilon g \in \mathscr{P} \subset U\mathscr{P} \simeq H'$. This was in fact used by Piacitelli to argue that true relativistic invariance of physical laws under deformed Poincaré transformations is impossible, in a way or the other broken by the choice of the deformation parameters $\theta\mu\nu$. We claim this is wrong because $U\mathscr{P} \simeq H'$ only as algebras, not Hopf algebras.

Caveat

The action $H' \triangleright \widehat{\mathscr{X}}$ is the deformed analog of the action $U\mathscr{P} \triangleright_c \mathscr{X}$. Both are algebra maps, i.e. when acting on a product the result is the product of the results. Then the infinitesimal variations $\delta_{\varepsilon g} a$ associated to the infinitesimal change of frame parametrized by $\varepsilon g \in H'$ fulfill

$$ab \mapsto (a+\delta_{\varepsilon g} a)(b+\delta_{\varepsilon g} b) \simeq ab+(\delta_{\varepsilon g} a)b+a(\delta_{\varepsilon g} b), \tag{47}$$

i.e. must be derivations, or equivalently g must be primitive. $\mathscr{P}$ is the only subspace of $U\mathscr{P}$ spanned by primitive generators of the whole $U\mathscr{P}$

$$\Delta'(g) = \mathbf{1} \otimes g + g \otimes \mathbf{1}, \qquad g \in \mathscr{P}. \tag{48}$$

We can recover the transformation associated to the finite change of frame parametrized by $g \in \mathscr{P}$ iterating infinitesimal transformations

$$a \mapsto e^g \triangleright_c a = \lim_{n\to\infty} (\mathbf{1} + g/n)^n \triangleright_c a. \tag{49}$$

If we replace $\Delta', \triangleright_c \mapsto \hat{\Delta}', \triangleright$ then (47-48) fail; $\nexists$ (full) analog of $\mathscr{P}$ within H'. Consequently, *we cannot interpret $a \mapsto e^g \triangleright a$, i.e. (49), as the transformation associated to a deformed finite change of frame*, as done by Balachandran et al, Piacitelli.

Change of frames are to be described by the H-coaction, as said before.

The second, "exotic" way to realize the free com. rel. (19) is: Assume $P_\mu \triangleright_c \hat{a}^\dagger_p = p_\mu \hat{a}^\dagger_p$, $P_\mu \triangleright_c \hat{a}^p = -p_\mu \hat{a}^p$. It amounts to $\theta \mapsto -\theta$ and nontrivial commutation relations between the $\hat{a}^p, \hat{a}^\dagger_p$ and functions:

$$\begin{aligned}
\hat{a}^\dagger_p \hat{a}^\dagger_q &= R^{sr}_{pq}\, \hat{a}^\dagger_r \hat{a}^\dagger_s = e^{-ip^t\theta q}\, \hat{a}^\dagger_q \hat{a}^\dagger_p, \\
\hat{a}^p \hat{a}^q &= R^{pq}_{rs}\, \hat{a}^s \hat{a}^r = e^{-ip^t\theta q}\, \hat{a}^q \hat{a}^p, \\
\hat{a}^p \hat{a}^\dagger_q &= \delta^p_q + R^{rp}_{qs}\, \hat{a}^\dagger_r \hat{a}^s = e^{ip^t\theta q}\, \hat{a}^\dagger_q \hat{a}^p + 2\omega_{\mathbf{p}} \delta^3(\mathbf{p}-\mathbf{q}), \\
\hat{a}^p e^{iq\cdot x} &= e^{-ip^t\theta q}\, e^{iq\cdot x} \hat{a}^p, \qquad \hat{a}^\dagger_p e^{iq\cdot x} = e^{ip^t\theta q}\, e^{iq\cdot x} a^\dagger_p.
\end{aligned} \tag{50}$$

Hence $[\hat{\varphi}(\hat{x}), f(\hat{x}')] = 0$.

It is covariant under a braided tensor product of an active copy and a passive copy of H, invariant under the diagonal active-passive transformation. So far, we can define purely active/passive trasformation appplying suitable projections, but they do not form Hopf sub-algebras (only co-ideals). **Work in progress** for an analog of (26').

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