# Hydrogen atom spectrum in rotationally invariant noncommutative space

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# Introduction

In the canonical version of noncommutative space the coordinate operators satisfy the following commutation relations

$$[X_i, X_j] = i\hbar\theta_{ij}, \tag{1}$$

where $\theta_{ij}$ is a constant antisymmetric object.
In two-dimensional space the rotational symmetry survives even in the canonical version of noncommutativity

$$[X, Y] = i\hbar\theta, \tag{2}$$

where $\theta$ is a constant.

$$[X_i, X_j] = 2i\lambda\varepsilon_{ijk}X_k, \tag{3}$$

V. Galikova, P. Presnajder, J. Phys: Conf. Ser. 343 (2012) 012096.

$$[X^i, X^j] = i\theta\varepsilon_{ijk}X^k f(X_l X^l), \tag{4}$$

V. G. Kupriyanov, J. Phys. A: Math. Theor. 46 (2013) 245303.

# Preserving of the rotational symmetry in noncommutative space

The generalization of the parameter of noncommutativity which gives the possibility to preserve the rotational symmetry

$$\theta_{ij} = \frac{\alpha}{\hbar}(a_i b_j - a_j b_i), \tag{5}$$

where $\alpha$ is a dimensionless constant, and $a_i$, $b_i$ are governed by the harmonic oscillators

$$H_{osc} = \frac{(p^a)^2}{2m} + \frac{(p^b)^2}{2m} + \frac{m\omega^2 a^2}{2} + \frac{m\omega^2 b^2}{2}. \tag{6}$$

We put

$$\sqrt{\frac{\hbar}{m\omega}} = l_p, \tag{7}$$

where $l_p$ is the Plank length. Independently of (7) we also consider the limit

$$\omega \to \infty.$$

So, we propose to consider the following commutation relations

$$[X_i, X_j] = i\alpha(a_i b_j - a_j b_i), \tag{8}$$

$$[X_i, P_j] = i\hbar\delta_{i,j}, \tag{9}$$

$$[P_i, P_j] = 0. \tag{10}$$

It is convenient to introduce dimensionless coordinates $\tilde{a}_i = \frac{a_i}{l_p}$, $\tilde{b}_i = \frac{b_i}{l_p}$.

$$\theta_{ij} = \frac{\alpha}{\hbar} l_p^2 (\tilde{a}_i \tilde{b}_j - \tilde{a}_j \tilde{b}_i). \tag{11}$$

Another generalization of the parameter of noncommutativity which gives the possibility to preserve the rotational symmetry

$$\theta_{ij} = \frac{\alpha}{\hbar} l_p^2 \varepsilon_{ijk} \tilde{a}_k, \tag{12}$$

# Hamiltonian of the hydrogen atom

The Hamiltonian of the hydrogen atom reads

$$H_h = \frac{P^2}{2M} - \frac{e^2}{R}, \tag{13}$$

where $R = \sqrt{\sum_i X_i^2}$. We use the following representation

$$X_i = x_i + \frac{1}{2}[\boldsymbol{\theta} \times \mathbf{p}]_i, \tag{14}$$

where $\boldsymbol{\theta} = \frac{\alpha l_p^2}{\hbar}[\tilde{\mathbf{a}} \times \tilde{\mathbf{b}}]$. Therefore, the distance $R$ reads

$$R = \sqrt{\sum_i (x_i + \frac{1}{2}[\boldsymbol{\theta} \times \mathbf{p}]_i)^2} = \sqrt{r^2 - (\boldsymbol{\theta} \cdot \mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta} \cdot \mathbf{p}]^2}, \tag{15}$$

here $r = \sqrt{\sum_i x_i^2}$ and $\mathbf{L} = [\mathbf{r} \times \mathbf{p}]$.

Let us find the expansion for the distance $R$.

$$R = r - \frac{1}{2r}(\boldsymbol{\theta}\cdot\mathbf{L}) - \frac{1}{8r^3}(\boldsymbol{\theta}\cdot\mathbf{L})^2 + \frac{1}{16}\left(\frac{1}{r}[\boldsymbol{\theta}\cdot\mathbf{p}]^2 + [\boldsymbol{\theta}\cdot\mathbf{p}]^2\frac{1}{r} + f(\mathbf{r})\right). \quad (16)$$

Squaring left and right hand sides of equation (16), we find

$$f(\mathbf{r}) = \frac{\hbar^2}{r^5}[\boldsymbol{\theta}\cdot\mathbf{r}]^2, \quad (17)$$

$$\frac{1}{R} = \frac{1}{r} + \frac{1}{2r^3}(\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{3}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2 - $$
$$-\frac{1}{16}\left(\frac{1}{r^2}[\boldsymbol{\theta}\cdot\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta}\cdot\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\boldsymbol{\theta}\cdot\mathbf{r}]^2\right) \quad (18)$$

So, we can rewrite Hamiltonian in the following form

$$H = \frac{p^2}{2M} - \frac{e^2}{r} + H_{osc} + V, \quad (19)$$

where $V$ is the perturbation caused by the noncommutativity of coordinates

$$V = -\frac{e^2}{2r^3}(\boldsymbol{\theta}\cdot\mathbf{L}) - \frac{3e^2}{8r^5}(\boldsymbol{\theta}\cdot\mathbf{L})^2 + \frac{e^2}{16}\left(\frac{1}{r^2}[\boldsymbol{\theta}\cdot\mathbf{p}]^2\frac{1}{r} + \frac{1}{r}[\boldsymbol{\theta}\cdot\mathbf{p}]^2\frac{1}{r^2} + \frac{\hbar^2}{r^7}[\boldsymbol{\theta}\cdot\mathbf{r}]^2\right).$$

# Perturbation of the spectrum of hydrogen atom

Let us find the corrections to $n, l$ levels of the hydrogen atom in the case when the internal oscillators are in the ground state. According to the perturbation theory, in the first order in $V$ we have

$$\Delta E^{(1)}_{n,l} = \langle \psi^{(0)}_{n,l,m,\{0\},\{0\}} | V | \psi^{(0)}_{n,l,m,\{0\},\{0\}} \rangle \tag{21}$$

where

$$\psi^{(0)}_{n,l,m,\{n^a\},\{n^b\}} = \psi_{n,l,m} \psi^a_{n^a_1,n^a_2,n^a_3} \psi^b_{n^b_1,n^b_2,n^b_3}, \tag{22}$$

We obtain

$$\Delta E^{(1)}_{n,l} = -\frac{\hbar^2 e^2 \langle \theta^2 \rangle}{a_B^5 n^5} \left( \frac{1}{6l(l+1)(2l+1)} - \frac{6n^2 - 2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)} + \right.$$
$$+ \frac{5n^2 - 3l(l+1) + 1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)} -$$
$$\left. - \frac{5}{6} \frac{5n^2 - 3l(l+1) + 1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)} \right).$$

In the second order of the perturbation theory we have

$$\lim_{\omega\to\infty} \Delta E^{(2)}_{n,l,\{0\},\{0\}} =$$

$$= \lim_{\omega\to\infty} \sum_{n',l',m'\{p^a\},\{p^b\}} \frac{\left|\left\langle \psi^{(0)}_{n',l',m',\{p^a\},\{p^b\}} \left|V\right| \psi^{(0)}_{n,l,m,\{0\},\{0\}} \right\rangle\right|^2}{E^{(0)}_n - E^{(0)}_p - \hbar\omega(p^a_1 + p^a_2 + p^a_3 + p^b_1 + p^b_2 + p^b_3)} = 0,$$

where $n' \neq n$, $l' \neq l$, $m' \neq m$, $p^a_i \neq 0$, $p^b_i \neq 0$, $i = 1, 2, 3$, and $E^{(0)}_n = -\frac{e^2}{2a_B n^2}$ is the unperturbed energy of the hydrogen atom.

$$\sqrt{\frac{\hbar}{m\omega}} = l_p. \qquad (23)$$

# Effective Hamiltonian

We propose to construct the effective Hamiltonian in the following form

$$H^{eff} = \langle \psi^a_{0,0,0}\psi^b_{0,0,0}|H|\psi^a_{0,0,0}\psi^b_{0,0,0}\rangle =$$
$$= \frac{p^2}{2M} - \frac{e^2}{r} - \frac{3e^2L^2}{8r^5}\langle\theta^2\rangle + \frac{e^2}{24}\left(\frac{1}{r^2}p^2\frac{1}{r} + \frac{1}{r}p^2\frac{1}{r^2} + \frac{\hbar^2}{r^5}\right)\langle\theta^2\rangle. \qquad (24)$$

Using the first order in $\langle\theta^2\rangle$ perturbation theory, we find

$$E^{eff}_{n,l} = -\frac{e^2}{2a_Bn^2} - \frac{\hbar^2e^2\langle\theta^2\rangle}{a_B^5n^5}\left(\frac{1}{6l(l+1)(2l+1)} - \right.$$
$$-\frac{6n^2-2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)} + \frac{5n^2-3l(l+1)+1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)} -$$
$$\left. -\frac{5}{6}\frac{5n^2-3l(l+1)+1}{l(l+1)(l+2)(2l+1)(2l+3)(l-1)(2l-1)}\right),$$

# Corrections to the ns levels of hydrogen atom

The perturbation caused by the noncommutativity of coordinates reads

$$V = -\frac{e^2}{R} + \frac{e^2}{r} = -\frac{e^2}{\sqrt{r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\cdot\mathbf{p}]^2}} + \frac{e^2}{r}. \quad (25)$$

Therefore, in the first order of the perturbation theory the corrections to the ns levels read

$$\Delta E^{(1)}_{ns} = \langle\psi^{(0)}_{n,0,0,\{0\},\{0\}}|\frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 - (\boldsymbol{\theta}\cdot\mathbf{L}) + \frac{1}{4}[\boldsymbol{\theta}\cdot\mathbf{p}]^2}}|\psi^{(0)}_{n,0,0,\{0\},\{0\}}\rangle. \quad (26)$$

Up to the second order of parameter of noncommutativity we can write

$$\Delta E^{(1)}_{ns} = \langle\psi_{n,0,0}|\frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 + \frac{1}{6}\langle\theta^2\rangle p^2}}|\psi_{n,0,0}\rangle. \quad (27)$$

$$\Delta E^{(1)}_{1s} = \frac{4e^2\gamma^2}{a_B}\int_0^\infty d\rho\rho^2 e^{-\gamma\rho}\left(\frac{1}{\rho} - \frac{1}{\sqrt{\rho^2+\tilde{p}^2}}\right)e^{-\gamma\rho}, \tag{28}$$

here $\rho = r\left(\frac{6}{\hbar^2\langle\theta^2\rangle}\right)^{\frac{1}{4}}$, $\gamma = \left(\frac{\hbar^2\langle\theta^2\rangle}{6a_B^4}\right)^{\frac{1}{4}}$

$$e^{-\gamma\rho} = \sum_n C_n\phi_{n,0,0}, \tag{29}$$

where $\phi_{n,l,m}$ are the eigenfunctions of $\rho^2 + \tilde{p}^2$

$$C_n = (-1)^n\sqrt{\frac{2n!}{\Gamma(n+\frac{3}{2})}}\int_0^\infty d\rho\rho^2 e^{-\gamma\rho}e^{-\frac{\rho^2}{2}}L_n^{\frac{1}{2}}\left(\rho^2\right), \tag{30}$$

$$\Delta E^{(1)}_{1s} = \frac{4e^2\gamma^2}{a_B}\sum_{n=0}^{\infty}\left(\frac{C_n^2}{\sqrt{\lambda_{n,0}}} - C_nI_n\right), \tag{31}$$

where $I_n = (-1)^n\sqrt{\frac{2n!}{\Gamma(n+\frac{3}{2})}}\int_0^\infty d\rho\rho e^{-\gamma\rho}e^{-\frac{\rho^2}{2}}L_n^{\frac{1}{2}}\left(\rho^2\right)$,

$$\lambda_{n,0} = 2n + \frac{3}{2}.$$

$$S_{1s}(\gamma) = 4\int_0^\infty d\rho\rho^2 e^{-\gamma\rho}\left(\frac{1}{\rho} - \frac{1}{\sqrt{\rho^2+\tilde{p}^2}}\right)e^{-\gamma\rho} = 4\sum_{n=0}^{\infty}\left(C_n I_n - \frac{C_n^2}{\sqrt{\lambda_{n,0}}}\right),$$

$$S_{1s}(0) = 16\sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\frac{\Gamma(n+\frac{3}{2})}{n!}\left({}_1F_2\left(-n,\frac{1}{2},\frac{3}{2},2\right) - \sqrt{\frac{\pi}{8n+6}}\right) = 1,72.$$

$$E_{1s}^{(1)} = \frac{e^2\gamma^2}{a_B}S_{1s}(0) = \frac{e^2}{a_B^3}\sqrt{\frac{\hbar^2\langle\theta^2\rangle}{6}}S_{1s}(0). \qquad (32)$$

Likewise, we can find the corrections to the ns levels

$$\Delta E_{ns}^{(1)} = \frac{e^2\gamma^2}{a_B n^5}S_{ns}(\gamma),$$

$$S_{ns}(\gamma) = 4\int_0^\infty d\rho\rho^2 e^{-\frac{\gamma\rho}{n}}L_{n-1}^1\left(\frac{2\gamma\rho}{n}\right)\left(\frac{1}{\rho} - \frac{1}{\sqrt{\rho^2+\tilde{p}^2}}\right)e^{-\frac{\gamma\rho}{n}}L_{n-1}^1\left(\frac{2\gamma\rho}{n}\right),$$

$$S_{ns}(0) = S_{1s}(0)n^2 = 1,72n^2,$$

$$\Delta E_{ns}^{(1)} = \frac{e^2}{a_B^3 n^3}\sqrt{\frac{\hbar^2\langle\theta^2\rangle}{6}}S_{ns}(0).$$

# Estimation of the upper bound of the parameter of noncommutativity

The correction to the energy of the 1S-2S transition reads

$$\Delta_{1,2} = \Delta E_{2,0} - \Delta E_{1,0} = -\frac{7e^2 S_{1s}(0)\hbar}{8\sqrt{6}a_B^3}\sqrt{\langle\theta^2\rangle}, \quad (33)$$

$$\frac{\Delta_{1,2}}{E_2^{(0)} - E_1^{(0)}} = \frac{7}{3\sqrt{6}} S_{1s}(0)\hbar\sqrt{\langle\theta^2\rangle}. \quad (34)$$

Assuming that $\frac{\Delta_{1,2}}{E_2^{(0)} - E_1^{(0)}}$ does not exceed $4.2 \cdot 10^{-15}$ $(f_{1S-2S} = 2466061413187035(10)Hz)$

C. G. Parthey, A. Matveev, J. Alnis at all., Phys. Rev. Lett. 107, (2011), 203001

we find

$$\hbar\sqrt{\langle\theta^2\rangle} \leq 1.6 \cdot 10^{-36} m^2. \quad (35)$$

M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett. 86 (2001) 2716.

## Conclusions

- The ways to preserve the rotational symmetry in noncommutative space have been proposed

$$\theta_{ij} = \frac{\alpha}{\hbar} l_p^2 (\tilde{a}_i \tilde{b}_j - \tilde{a}_j \tilde{b}_i), \tag{36}$$

$$\theta_{ij} = \frac{\alpha}{\hbar} l_p^2 \varepsilon_{ijk} \tilde{a}_k. \tag{37}$$

- We have founded the corrections to the energy levels of hydrogen atom in rotationally invariant noncommutative space.
- The problem of divergence of the corrections to the ns levels has been considered
- The upper bound of the parameter noncommutativity has been estimated

$$\hbar \sqrt{\langle \theta^2 \rangle} \leq 1.6 \cdot 10^{-36} m^2. \tag{38}$$

Thank you for your attention!