Noncommutative geometry and manifolds with boundary - Fuzzy disc revisited

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PLAN

Motivations: Manifolds with boundary

- Quantum physics on Riemannian manifolds with boundary exhibit novel phenomena even in commutative geometric framework. There is a physical need to understand them, motivated by quantum blackholes and edge states in quantum hall effect, topological insulators, ...

Motivations: Manifolds with boundary

- Quantum physics on Riemannian manifolds with boundary exhibit novel phenomena even in commutative geometric framework. There is a physical need to understand them, motivated by quantum blackholes and edge states in quantum hall effect, topological insulators, ...

- One expects interesting interplay between noncommutativity and boundary effects. We will explore with a specific example..

Riemannian manifolds with boundary

- We are interested in the Laplace-Beltrami operator on these manifolds to obtain appropriate energy functionals.

- We have to make sure that Laplacian is self adjoint operator on suitable domain of $\mathcal{L}^2$ functions. That will be translated to appropriate boundary conditions.

- The operator is symmetric if $\int \phi^* \{-\Delta^2\}\chi = \int \{-\Delta^2\phi\}^* \chi$.

- This leads to the boundary term which should vanish with suitable boundary conditions to ensure domains of the operator and its hermitian conjugate are same: $\mathcal{D}_{\Delta^2} = \mathcal{D}_{\Delta^{2\dagger}}$.

Laplace-Beltrami

- It was suggested by Asorey, Ibort, Marmo to use the structure of the boundary condions:

$$\phi \ - \ \mathrm{i}\partial_n\phi \ = \ U \ (\phi \ + \ \mathrm{i}\partial_n\phi)$$

  where ∂_n stands for the normal derivative to the boundary.

- It was pointed out by them that allowed self-adjoint extensions can be identified with the unitary matrix U.

- $U \ = \ \pm 1$ correspond to Neumann and Dirichlet conditions.

- They are extreme points and generic boundary conditions are interesting and bring novel features. One of the interesting feature is the possible existence of edge states localised close to the boundary with negative eigenvalue.

Conditions for edge states

- To appreciate further one should divide the Riemannian manifold into a collar neighbourhood and outside called bulk.

- The contribution of the bulk is always positive, whereas that of the collar is semibounded below if a gap conditions is obeyed.

- Without getting into details we will establish these in a simple example of $R^2 - $ (Disc). We will use a simplified condition known as 'Robin' boundary condition.

- This condition is characterised by

$$\psi(r, \theta) + \kappa\, \partial_r \psi(r, \theta) = 0.$$

2D Laplacian on $R^2 - $ Disc

- We are interested in finding the bound state solutions of the time-independent Schroedinger equation

$$-\triangle \psi(r, \Phi) = \Lambda \psi(r, \Phi).$$

  The solution to the above equation is to be obtained subject to the Robin boundary condition

$$\kappa \psi + \partial_{\overrightarrow{n}} \psi \mid_{R_b} = 0,$$

- R_b is radius of the disc. We note that κ has dimensions of $(\text{length})^{-1}$ and is a constant for rotational invariance. This parameter would not be present if we use the usual Dirichlet or Neumann boundary conditions (which are the limits of Robin boundary condition corresponding, respectively, to κ^{-1} or $\kappa \to 0$)
- We will see the interplay of R and κ, decides the number of 'boundstates'.

2+1 dimensional spacetime

- In terms of polar coordinates, our eigenvalue equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = \lambda \psi\,,$$

  where we have replaced $\Lambda \to -\lambda$. For bound states $\lambda > 0$.
- We solve using the ansatz $\psi(r, \theta) = e^{in\theta} R(r)$ (with n an integer) resulting in the equation

$$\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2 R}{r^2} = \lambda R\,.$$

- We know that the Laplacian (and this can directly be checked), $\left(\hat{O} = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)$ is self-adjoint on the domain

  $D(\hat{O}) = \{\psi | \psi \text{ continuous and } \psi \in \mathcal{L}^2(r_b, \infty), \kappa\psi(r_b) - \psi'(r_b) = 0\}$

2D Laplacian on $R^2 - D$: Eigenvalues and solutions

- Explicit solutions are given by modified Bessel's function $I_n(\sqrt{\lambda}r)$ and $K_n(\sqrt{\lambda}r)$. Square-integrability on (R_b, ∞) with measure rdr (R_b being the boundary), rules out the exponentially growing $I_n(\sqrt{\lambda}r)$ solution:

$$R(r) = cK_n(\sqrt{\lambda}r) \, ,$$

  where c is a normalization constant.

- Impose the boundary condition at $r = R_b$:

$$\kappa = \frac{\sqrt{\lambda}K_n'(\sqrt{\lambda}R_b)}{K_n(\sqrt{\lambda}R_b)} \, .$$

- Since $K_n'(r) < 0$, we find that there will be bound solutions only for $\kappa < 0$. Graphically the above equation (for different values of n) as a function of $\sqrt{\lambda}$ shows only a finite number of bound solutions for a given value of κ and R_b, given by the intersection of the above mentioned curves with the horizontal line representing the constant value of κ,

2D Laplacian on $R^2 - D$: Eigenvalues and solutions
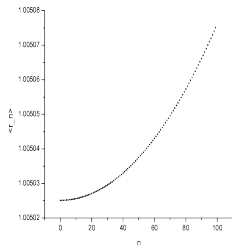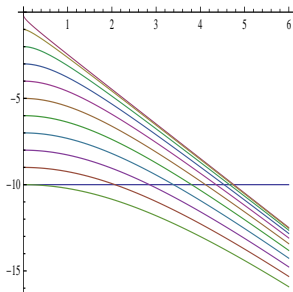
solutions and expectation values $< r_n >, \kappa = 100, R_b = 1$.

We have various energy levels as well as the spread of wavefunction near the boundary. It is clear all the bound state solutions are close to R_b. Using approximations: $K_\alpha(x) \approx \dfrac{\Gamma(\alpha)}{2} \left(\dfrac{2}{x}\right)^\alpha$ we can also establish the number of boundstates as: $N \simeq \kappa R_b$.

Laplacian on R^2 − elliptical disc

- Is the number of bound states related to the circumference of the ellipse? Answer is No.

- Consider Laplacian on R^2 − elliptical disc. The eigenvalue problem with Robin boundary conditions can be done in elliptical coordinates.

- x = c coshζ cosη, y = c sinhζ sinη. This is same as: z = c cosh w. For constant $\zeta = \zeta_0$ we get ellipse with major(minor) axis = c coshζ_0 (c sinhζ_0).

- The eigenvalue Eqn:$(\nabla^2 + \lambda)\phi = 0$ becomes under seperation of variables: $\phi = X(\zeta)Y(\eta)$.

$$Y'' + (A - 2q\cos 2\eta)Y = 0$$
$$X'' - (A - 2q\cosh 2\zeta)X = 0$$

where 4q = λc^2. Mathieu differential equations and we want the radial Mathieu functions of the third kind!

R^2 – elliptical disc

- Robin boundary conditions:

$$\frac{\kappa}{c} \partial_\zeta \phi \; + \; \phi|_{\zeta = \zeta_0}$$

- Numericals indicate the number of bound states is

$$N \; \propto \; \frac{c \, \sinh\zeta_0}{\kappa}$$

- It is the circumeference of the largest circle one can inscribe. Seems to be a result of Gromov's nonsqueezing theorem on symplectic manifolds.
- It states that one cannot embed a sphere into a cylinder through symplectic maps unless the radius of the sphere is $\leqslant$ radius of the cylinder.
- Can you squeeze a camel inside needle?
- Not if you are allowed only symplectic transformations. Uncertainty in classical statistical mechanics?

Gas of bosons in R^2 – D

- Does it lead instability due to negative energy solutions?
- No. At finite temperature and finite fixed density a condensate similar to Bose Einstein condensate is formed.
- On the otherhand QFT does not have fixed density and is stable only of $T > T_c$. Also we have to make sure not to include bound states in the 'in' or 'out' vacua.
- With $\lambda\phi^4$ interaction a space dependent potential is generated at one loop leading to low mass excitations at the boundary.

Fuzzy Disc - Brief review

- We define $\mathcal{A}_\theta(\mathrm{R}^2)$ as functions with quantised coordinates $\hat{x}$ and $\hat{y}$, such that: $[\hat{x}, \hat{y}] = \dfrac{i\theta}{2}$.

- a $\{a^\dagger\} = \hat{x} \pm i\hat{y}$ are the annihilation and creation operators serving the role of complex coordinates z.

- We define the map between functions on R^2 and operators on the Moyal plane, $f(z, \bar{z}) = \sum f^T_{mn} \bar{z}^m z^n$ can be associated with: $\hat{f} = \sum f_{mn} a^{\dagger m} a^n$. (We have used normal ordering).

- Inverse maps from operators on Moyal plane to functions can be given using coherent states. $a |z\rangle = z |z\rangle$. Inverse map from operators $\hat{f}$ to functions:

$$f(z, \bar{z}) = \langle z|\hat{f}|z\rangle$$

- One uses number operator basis $N |n\rangle = a^\dagger a|n\rangle = n|n\rangle$.

Review - contd..

- In this the operator can be written as $\hat{f} = \sum f_{mn}|m\rangle \langle n|$.
- We need the inner product between the bases:

$$\langle z|n\rangle = e^{-\frac{|z|^2}{2\theta}} \frac{\bar{z}^n}{\sqrt{n!\theta^n}}.$$

- This can be used to relate the operators in the two series.

$$f_{mn} = \sum_{0}^{\min(m,n)} f^T_{m-q,n-q} \frac{\sqrt{m!n!\theta^{m+n}}}{\theta^q q!}$$

- We can obtain a new expression for the function $f(z,\bar{z})$

$$f(z,\bar{z}) = \langle z|\hat{f}|z\rangle = e^{\frac{|z|^2}{\theta}} \sum f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}}$$

- We have set up the machinary required for Moyal plane from where the fuzzy disc should be carved out.

Fuzzy Disc

- Fuzzy Disc sub algebra (as explained by Lizzi) can be obtained by the projection operator $\hat{P}_\theta^N = \sum^N |n\rangle\langle n|$.

- This function for this operator can be computed as:

$$P_\theta^N(z, \bar{z}) = \langle z|\hat{P}_\theta^N|z\rangle = e^{\frac{-r^2}{\theta}} \sum^N \frac{r^{2n}}{n!\theta^n}$$

- The finite sum can be summed without difficulty as:

$$P_\theta^N = \frac{\Gamma(N+1, \frac{r^2}{\theta})}{\Gamma(N+1)}$$

- $\Gamma(N, r)$ is incomplete Gamma function. In the limit $N \to \infty$, $\theta \to 0$ with the requirement $N\theta = R^2$ gives functions (subclass) on disc.

- The projection gives truncated function:

$$f_\theta^N = e^{\frac{-r^2}{\theta}} \sum^N f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}}$$

Derivatives

- The derivative in Moyal plane $\partial_z\, f \;=\; \frac{1}{\theta}\langle z|[\hat{f}, a^\dagger]|z\rangle$ should be modified to:

$$\partial_z f_\theta^N \;=\; \frac{1}{\theta}\langle z|P_\theta^N[P_\theta^N \hat{f} P_\theta^N, a^\dagger]P_\theta^N|z\rangle$$

- The eigenvalue equation for the Laplacian can be written as:

$$-\Delta_N^2\, f_\theta^N \;=\; -\frac{4}{\theta^2}P_\theta^N[a, [P_\theta^N \hat{f} P_\theta^N, a\dagger]]P_\theta^N \;=\; \lambda^2 f_\theta^N$$

- If $k_{n,m}$ satisfy $J_n(kR) = 0$ then we can write down the eigensolutions of the 'Fuzzy' Laplacian. They are:

$$f_{n,m}^N = e^{in\phi}\left(\frac{k_{n,m}^2}{4}\right)^{n/2} e^{-Nr^2} \sum_n^N \sum_0^{j-n} (\frac{-k_{n,m}^2}{4N})^s \frac{N^{j-n}}{s!(n+s)!(j-s-n)!} r^{2j-n}$$

- In this framework Dirichlet boundary conditions are automatically imposed.

Robin boundary conditions

- Robin boundary conditions are: $(f + \kappa \partial_r f)|_R = 0$. This translates $(J_n(kr) + \kappa \partial_r J_n(kr))|_R = 0$.

- If $k_{n,m}$ staisfies the above equation then the eigenvalue is $(\frac{k_{n,m}}{R})^2$. These are still positive and go over to smoothly to Dirichlet condition.
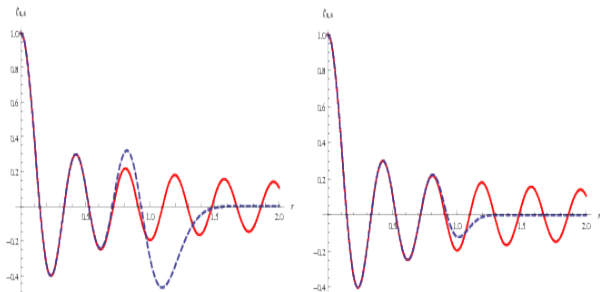
- The novel solutions are with $k^2 < 0$. They satisfy

$$(I_n(ikr) + \kappa \partial_r I_n(ikr))|_R = 0.$$

These are finite in number upto $n < n_{max}$.
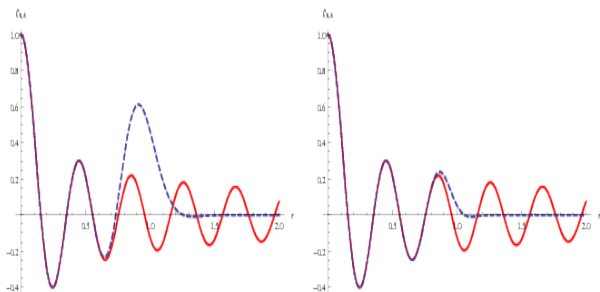
- The eigenvectors are:

$$I_n(ikr)e^{in(\phi+\pi/2)} = e^{in(\phi+\pi/2)}\left(\frac{ik_{|n|}r}{2}\right)^{|n|} \sum \frac{(ik_{|n|})^s\ r^{2s}}{2^{2s}s!(|n|+s)!}$$
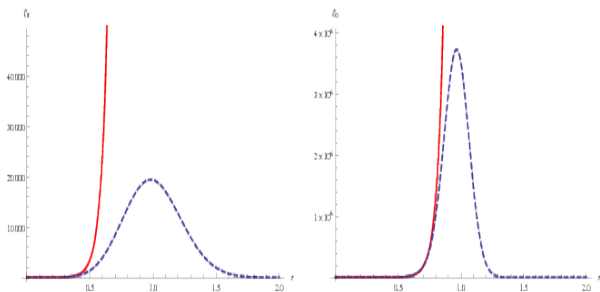
Figures

Comparison of commutative (red) and fuzzy (blue) with robin boundary conditions N=10, $\kappa = 5$, N=30, $\kappa = 5$ and positive energy

Figures

Comparison of commutative (red) and fuzzy (blue) with robinboundary conditions N=10, $\kappa = .1$, N=30, $\kappa= .1$ and positive energy

Figures

Comparison of commutative (red) and fuzzy (blue) with robinboundary
conditions N=10, $\kappa = .1$, N=30, $\kappa = .1$ and negative energy=edge states

Edge states

- We can truncate the series using the projectors and obtain 'eigen spectrum'. The edge bound states are the novel whose width is fized by the scale κ from the boundary.

- These are really the edge states in the fuzzy case unlike those obeying 'Dirichlet' which have nonzero value close to the boundary. This is because they are unstable and will move inwards to the 'bulk' under perturbation.

- To obtain the correct boundary in the fuzzy case one should apply a suitable infinite potential in the Moyal plane.

- Following Scholtz etal we will explain this.

Potential well in Moyal plane

- One goes back to Moyal plane. We need to carve out infinite potential well. We consider piecewise potential.

- Define

$$P_{N,M} = \sum_{N}^{N+M} |n\rangle\langle n|$$

  Then we can create 'well' by considering the potential

$$V = V_N P_{0,N} + V_{N,M} P_{N,M} + V_{M,\infty} P_{M,\infty}$$

- We get in the limit $N \to \infty$, $\theta \to 0$ such that $N\theta = R^2$ a disc if $V_{N,M} = \infty, V_N = 0$.

- We solve the equations in different regions and match the solutions at the 'boundaries'

Solutions

- Since the Schrodinger equation with constant potential is easy we will get:

$$\langle n| \quad \hat{\psi} \quad |n+m\rangle = c_1 \left(m, \frac{k^2 r^2}{4n}\right) \sqrt{m} n^{\frac{-m}{2}} L_n^m \left(\frac{-k^2 r^2}{4n}\right)$$

$$+ c_2 \left(m, \frac{k^2 r^2}{4n}\right) \sqrt{\frac{2\pi}{m!}} e^{-n} n^{\frac{2n+m+1}{2}} U \left(n+1, 1-m, \frac{-k^2 r^2}{4n}\right)$$

- Here L_n^m and U are the associate Laguerre Polynomials and the Kummer functions. They go to in the limit to Bessel functions of the appropriate kind.
- One needs to match the solutions at the boundaries for the problem.
- In the above potential the role of M indicates how the solutions from one region decays into other region. This measures the freedom of Robin boundary condition parameter κ in the commutative limit.

Solitons in NC space

- Noncommutative solitons in $2 + 1$ dimensional field theories were considered by Gopakumar, Minwalla, Strominger. They use the Hamiltonian (in rescaled variable)

$$H \; = \; \int d^2z(\phi(-\Delta^2)\phi + \theta V_*(\phi))$$

- In the infinite θ limit the potential term dominates and minimising that leads to soliton solutions.
- If $V_*(\phi) \; = \; -m^2\phi * \phi \; + \; \lambda\phi * \phi * \phi$, then $V' \; = \; -2m\phi \; + \; 3\lambda\phi * \phi$ which can be solved using projectors.
- In the finite θ case, derivative term along with the potential term gives for large θ soliton solutions.
- We also know there are no solitons in $\theta = 0$ commutative case.
- There exists a critical θ_c below which the NC solitons are unstable and we end up with commutative limits.

Solitons in Fuzzy Disc

- The situation continues in fuzzy disc with Dirichlet boundary conditions. There are bulk solitons for large θ limit and large radius R.

- There are also angular solitons in the fuzzy disc with Dirichlet boundary conditions.

- The situation changes with Robin boundary conditions.

- With this boundary condition the contribution of the Laplacian is no longer positive definite. This is because of the negative energy eigen modes localised close to the boundary.

- They are bounded below and finite in number. Now there are two length parameters θ and κ. The NC parameter contributes to the potential where as κ contributes to the Laplacian.

- We have in addition to the bulk solitons edge solitons localised close to the boundary.

Solitons in Fuzzy Disc

- These solitons survive even for low θ unlike the bulk ones since they depend mainly on the boundary properties.

- There is no criical θ_c for NC space which support solitons.

- They are low mass solitons since the θ play a small role.

- Riemannian manifold with boundary bring new features even in commutative theory due to the novel boundary conditions.

- Noncommutative deformations of BTZ blackhole leads to Noncommutative cylinder $R_+ \otimes S^1$ with the horizon at the boundary.

BTZ deformations

- It is defined by the relation: $[z, e^{i\phi}] = e^{i\phi}$. Here z corresponds to the 'time'.

- The quantum theory on NC cylinder can be related to that of Disc through conformal transformations in the commutative limit.

- The discretisation of the cylinder and field theory simulations show novel features.

- Novel stripe phases where translational invariance is broken can be seen.

- There are also stripes on the boundary which can be due to novel boundary conditions.

- Based on works in collaboration with Rakesh Tibrewala, V. Parameswaran Nair, Sanatan Digal, Kumar Gupta, Xavier Martin and Rahul Srivastava.

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THANK YOU!!

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