Finsler geometry with k-Poincaré symmetries

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(ArXiv: 1407.XXXX)

XXXIII Max Born Symposium - Wroclaw - Poland

Motivations - Beyond classical structure of spacetime

- Classical spacetime described as a (pseudo-)Riemannian manifold
- Quantum Gravity: spacetime needs some sort of quantization

- Modified (effective) spacetime probably requires also modification of particle dispersion relation ...

$$E^2 = m^2 + p^2 + \Delta(p, m, E_{Pl})$$ (flat spacetime)

- ... and an appropriate deformation of Poincaré symmetries in order to keep relativistic properties

→ look for effective spacetime description (holding close to Planck scale) that is compatible with MDR and relativistic

Free (classical) particle with k-Poincaré symmetries

1+1 dimension

- Algebra in bicrossproduct basis

(Poisson brackets: $\hbar \to 0$ limit of commutators)

$$\begin{aligned} \{P_0, P_1\} &= 0 \\ \{N, P_0\} &= P_1 \\ \{N, P_1\} &= P_0 - \ell P_0^2 - \frac{\ell}{2} P_1^2 \end{aligned}$$

- Casimir

$$C_\ell = P_0^2 - P_1^2 - \ell P_0 P_1^2$$

- Upon choosing the trivial symplectic structure

$$\begin{aligned} \{x^\mu, x^\nu\} &= 0 \\ \{x^\mu, p_\nu\} &= \delta_\mu^\nu \\ \{p_\mu, p_\nu\} &= 0 \end{aligned}$$

(Classical phase-space: coordinates related to k-Minkowski ones by a momentum-dependent redefinition)

one gets the representation:

$$P_0 = p_0, \qquad P_1 = p_1$$

$$N = x^0 p_1 + x^1 \left(p_0 - \ell p_0^2 - \frac{\ell}{2} p_1^2 \right)$$

Free (classical) particle with k-Poincaré symmetries

1+1 dimension

- Hamiltonian formalism (using Casimir as Hamiltonian)

$$\dot{p}_\mu = \{p_\mu, C_\ell\} = 0$$

$$\begin{aligned}\dot{x}^0 &= \{x^0, C_\ell\} = 2p_0 - \ell p_1^2 \\ \dot{x}^1 &= \{x^1, C_\ell\} = -2p_1(1 + \ell p_0)\end{aligned}$$

Amelino-Camelia, Matassa, Mercati, Rosati PRL 2010

→ Worldline:

$$x^1 - \bar{x}^1 = -\frac{\sqrt{p_0^2 - m^2}}{p_0}\left(1 + \ell\frac{2p_0^2 - m^2}{2p_0}\right)(x^0 - \bar{x}^0)$$

Rosati, Loret, Amelino-Camelia JPCS 2013
Amelino-Camelia, Barcaroli, GG, Loret CQG 2013

- Covariance under the action of symmetry generators:

$$(x^1)' = v(p_0') \cdot (x^0)' \Leftrightarrow x^1 = v(p_0) \cdot x^0$$

$$\Downarrow \quad f(x,p)' \simeq f(x,p) + \xi\{N, f(x,p)\}$$

$$\{N, x^1\} = v(p_0)\{N, x^0\} + \frac{\partial v(p_0)}{\partial p_0}\{N, p_0\}x^0 \Leftrightarrow x^1 = v(p_0)x^0$$

Geometrical interpretations

- By now it is quite well understood that modified symmetries are related to non-trivial properties of the momentum space geometry (curvature, torsion...) - see earlier talk by GAC

Born PRSLA 1938
Snyder PR 1947
(......)
Amelino-Camelia, Freidel, Kowalski-Glikman, Smolin PRD 2011, IJMPD 2011

momentum space of particles with k-Poincaré symmetries is (a portion of) de Sitter manifold

Kowalski-Glikman PLB 2002
Kowalski-Glikman, Nowak, CQG 2003

bicrossproduct basis corresponds to momentum space coordinates with metric:

$$\zeta^{\mu\nu}(p) = \begin{pmatrix} 1 & 0 \\ 0 & -(1+2\ell p_0) \end{pmatrix}$$

GG, Mercati, CQG 2013
Amelino-Camelia, Arzano, Kowalski-Glikman, Rosati, Trevisan CQG 2012

- for the corresponding description of spacetime there are several alternatives (noncommutative geometry, rainbow spacetime...) which are not easily generalized when there is curvature on both sides of phase space (noncommutative geom.) or rely on heuristic arguments (rainbow)

Finsler geometry

- Introduces a velocity-dependent generalization of Riemannian metric

$$g_{\mu\nu}(x,\dot{x}) \equiv \frac{1}{2}\frac{\partial F^2}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$$

$x \in M$ spacetime point
$\dot{x} \in T_x M$ tangent vector

defined starting from a norm $F(x,\dot{x}) = \sqrt{g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}$ over the tangent bundle instead than from an inner product

F satisfies usual norm properties:

$$F(x,\dot{x}) \neq 0 \text{ if } \dot{x} \neq 0$$

$$F(x,\lambda\dot{x}) = |\lambda| F(x,\dot{x}), \quad \lambda \in \mathbb{R}$$

for reviews : Rund 1959
Bao, Chern, Shen 2000

- Looks like a natural *consistent* framework to describe velocity/ momentum dependent metric

→ is this a good framework to describe a relativistic theory with deformed symmetries?

Finsler geometry of a particle with modified dispersion relation

- Start from the Lagrangian of a free relativistic particle with mass-shell condition $m^2 = \mathcal{M}(p)$

$$I = \int \left(\dot{x}^\mu p_\mu - \lambda\left(\mathcal{M}(p) - m^2\right)\right) d\tau$$

- Make use of Hamilton equations to find relation between velocities and momenta, invert and substitute $p_\mu \to p_\mu(\dot{x}, \lambda)$ then solve for the Lagrange multiplier

$$\longrightarrow \qquad I = \int \mathcal{L}\left(\dot{x}, \lambda(\dot{x})\right) d\tau$$

- The Lagrangian can be interpreted as a Finsler norm:

$$I = m \int F(x, \dot{x}) d\tau$$

so that the action is the straightforward generalization of the standard action for relativistic free particle

$$I_{std} = m \int \sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} d\tau$$

Girelli, Liberati, Sindoni PRD 2007

Finsler geometry of a particle with k-Poincaré symmetries

- Apply procedure just described to the case $m^2 = \mathcal{C}_\ell(p)$

$$I = \int \left(\dot{x}^\mu p_\mu - \lambda \left(p_0^2 - p_1^2 - \ell p_0 p_1^2 - m^2\right)\right) d\tau$$

- From Hamilton equations + minimization w.r.t Lagrange multiplier

$$\begin{aligned} p_0 &= m\frac{\dot{x}^0}{\sqrt{(\dot{x}_0)^2-(\dot{x}_1)^2}} - \frac{\ell}{2}m^2\frac{(\dot{x}^1)^2((\dot{x}_0)^2+(\dot{x}_1)^2)}{((\dot{x}_0)^2-(\dot{x}_1)^2)^2} \\ p_1 &= -m\frac{\dot{x}^1}{\sqrt{(\dot{x}_0)^2-(\dot{x}_1)^2}} + \frac{\ell}{2}m^2\frac{\dot{x}^1(\dot{x}_0)^3}{((\dot{x}_0)^2-(\dot{x}_1)^2)^2} \end{aligned}$$

- The Lagrangian as a function of velocities only is then

$$I = m\int \left(\sqrt{(\dot{x}_0)^2 - (\dot{x}_1)^2} + m\frac{\ell}{2}\frac{\dot{x}^0(\dot{x}^1)^2}{(\dot{x}_0)^2 - (\dot{x}_1)^2}\right) d\tau$$

$$\longrightarrow \qquad F(\dot{x}) = \sqrt{(\dot{x}_0)^2 - (\dot{x}_1)^2} + m\frac{\ell}{2}\frac{\dot{x}^0(\dot{x}^1)^2}{(\dot{x}_0)^2 - (\dot{x}_1)^2}$$

(flat spacetime - the norm is independent on coordinates)

Finsler spacetime metric

- Velocity-dependent spacetime metric $g_{\mu\nu} = \dfrac{1}{2}\dfrac{\partial F^2}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$:

$$g_{\mu\nu}(\dot{x}) = \begin{pmatrix} 1 + \frac{3}{2}\ell m \frac{\dot{x}^0(\dot{x}^1)^4}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{5/2}} & \ell\frac{m}{2}\frac{-4(\dot{x}^0)^2(\dot{x}^1)^3 + (\dot{x}^1)^5}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{5/2}} \\ \ell\frac{m}{2}\frac{-4(\dot{x}^0)^2(\dot{x}^1)^3 + (\dot{x}^1)^5}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{5/2}} & -1 + \frac{1}{2}\ell m(\dot{x}^0)^3 \frac{2(\dot{x}^0)^2 + (\dot{x}^1)^2}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{5/2}} \end{pmatrix}$$

in terms of momenta:

$$g_{\mu\nu}(p) = \begin{pmatrix} 1 + \frac{3}{2}\ell\frac{p_0 p_1^4}{m^4} & \frac{\ell}{2}\frac{4p_0^2 p_1^3 - p_1^5}{m^4} \\ \frac{\ell}{2}\frac{4p_0^2 p_1^3 - p_1^5}{m^4} & -1 + \frac{\ell}{2}p_0^3\frac{2p_0^2 + p_1^2}{m^4} \end{pmatrix}$$

- The inverse of this metric is found to have a simple relation with the particle dispersion relation:

$$g^{\mu\nu}(p)p_\mu p_\nu = p_0^2 - p_1^2 - \ell p_0 p_1^2 = \mathcal{C}_\ell$$

- In terms of this metric momenta are simply related to velocities:

$$p_\mu = m\frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta}}$$

Worldlines and symmetries

- The geodesic equation in Finsler geometry reads:

$$\ddot{x}^{\mu} + \Gamma^{\mu}_{\nu\rho}(x,\dot{x})\dot{x}^{\nu}\dot{x}^{\rho} = 0$$ (with affine parameterization F=1)

with generalized Christoffel symbols

$$\Gamma^{\mu}_{\nu\rho}(x,\dot{x}) = \frac{1}{2}g^{\mu\sigma}(x,\dot{x})\left[-\partial_{\sigma}g_{\nu\rho}(x,\dot{x}) + \partial_{\nu}g_{\rho\sigma}(x,\dot{x}) + \partial_{\rho}g_{\sigma\nu}(x,\dot{x})\right]$$

- Applied to our “k-Poincaré” case:

$$\Gamma^{\mu}_{\nu\rho}(x,\dot{x}) = 0$$ (flat geometry)

$$\longrightarrow \quad \ddot{x}^{\mu} = 0$$

(using affine parameterization)

$$\longrightarrow \quad x^1 - \bar{x}^1 = \left(\frac{\sqrt{(\dot{x}^0)^2 - 1}(1 + \frac{\ell}{2}m\dot{x}^0)}{\dot{x}^0}\right)(x^0 - \bar{x}^0)$$

when going to momenta this is same as the one derived earlier

Worldlines and symmetries

- Finsler geometry also gives a prescription for finding symmetries of the metric by use of a generalized Killing equation:

$$g_{\mu\rho}\,\partial_\nu\xi^\rho + g_{\nu\rho}\,\partial_\mu\xi^\rho + \frac{\partial g_{\mu\nu}}{\partial\dot{x}^\rho}\,\frac{\partial\xi^\rho}{\partial x^\sigma}\dot{x}^\sigma + \frac{\partial g_{\mu\nu}}{\partial x^\rho}\xi^\rho = 0$$

Rund 1959

- At zero order in *ℓ* one recovers standard Killing vectors:

$$\xi^\mu_{(0)} = \begin{pmatrix} a\,x^1 + d^0 \\ a\,x^0 + d^1 \end{pmatrix}$$

while the first-order correction for our case is

$$\xi^\mu_{(1)} = \begin{pmatrix} A^0 + d^0 m F_{[1]}(\dot{x}) + C x^1 + am(F_{[2]}(\dot{x})x^0 + F_{[3]}(\dot{x})x^1) \\ A^1 + d^1 m F_{[4]}(\dot{x}) + C x^0 + am(F_{[5]}(\dot{x})x^0 + F_{[6]}(\dot{x})x^1) \end{pmatrix}$$

$F_{[i]}(\dot{x})$ are defined functions of velocities - e.g. $F_{[1]}(\dot{x}) = \dfrac{(\dot{x}^1)^2((\dot{x}^0)^2 + (\dot{x}^1)^2)}{2\dot{x}^0((\dot{x}^0)^2 - (\dot{x}^1)^2)^{3/2}}$

$A^0,\ A^1,\ C$ are free parameters that can be arbitrary functions of velocities

Worldlines and symmetries

- To compare with k-Poincaré symmetries look at conserved charges

$$Q_F \equiv \xi^\mu \, p_\mu(\dot{x})$$

$$\longrightarrow \qquad \begin{aligned} Q_F^{(0)} &= d^0 p_0 + d^1 p_1 + a x^0 p_1 + a x^1 p_0 \\ Q_F^{(1)} &= A^0 p_0 + A^1 p_1 + C(p_0 x^1 + p_1 x^0) + \\ &\quad \frac{a\left(2p_0^3 p_1 x^0 + p_1^2 x^1\left(p_0^2+p_1^2\right)\right)}{2m^2} + \frac{d^0 p_1^2\left(p_0^2+p_1^2\right)}{2m^2} + \frac{d^1 p_0^3 p_1}{m^2} \end{aligned}$$

and compare with representation of k-Poincaré generators

- Asking that the charges reproduce the representation of k-Poincaré generators constraints the three free functions

$$A^1 = -d^1\, m \frac{(\dot{x}^0)^3}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{3/2}} + d^0\, m \frac{\dot{x}^1((\dot{x}^0)^2 + (\dot{x}^1)^2)}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{3/2}} + A^0 \frac{\dot{x}^0}{\dot{x}^1}$$

$$C = -a\, m \frac{(\dot{x}^0)^3}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{3/2}}$$

More on freedom in definition of symmetry generators

- Killing eqs results are consistent with the symmetries of k-Poincaré - in bicrossproduct basis
- Freedom provided by other choices of free functions in Finsler conserved charges is linked to the freedom of redefining the k-P boost generator (s.t. the Casimir is unchanged)

$$\mathcal{N}_{\mathcal{C}_\kappa - compatible} = p_1 x^0 + p_0 x^1 + \ell \left(A p_0 + B p_1 + \alpha p_0 p_1 x^0 + \gamma (p_0 p_1 x^1 + p_1^2 x^0) + (\alpha - 1) p_0^2 x^1 - \frac{1}{2} p_1^2 x^1 \right)$$

- On the other hand, changing basis in k-P, so that the form of the Casimir is modified, amounts to changing the corresponding spacetime Finsler geometry (different metric with different conserved charges)

$$p_0 \to p_0$$
$$p_1 \to p_1 \left(1 + \frac{\ell}{2}(p_1 - p_0)\right) \longleftrightarrow \quad \mathcal{C}_\ell^{(new)} = p_0^2 - p_1^2 - \ell p_1^3$$

$$\longrightarrow \quad g_{\mu\nu}^{(new)} = \begin{pmatrix} 1 - \ell m\, (\dot{x}^1)^3 \frac{(\dot{x}^0)^2 + \frac{1}{2}(\dot{x}^1)^2}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{\frac{5}{2}}} & \frac{3}{2}\ell m \frac{(\dot{x}^0)^3 (\dot{x}^1)^2}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{\frac{5}{2}}} \\ \frac{3}{2}\ell m \frac{(\dot{x}^0)^3 (\dot{x}^1)^2}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{\frac{5}{2}}} & -1 - \ell m\, \dot{x}^1 \frac{3(\dot{x}^0)^4 - \frac{5}{2}(\dot{x}^0)^2 (\dot{x}^1)^2 + (\dot{x}^1)^4}{((\dot{x}^0)^2 - (\dot{x}^1)^2)^{\frac{5}{2}}} \end{pmatrix}$$

Summary and outlook

- Finsler generalization of Riemannian geometry can be used to describe spacetime geometry seen by a particle with given (modified) dispersion relation
- When the dispersion relation is modified, the resulting Finsler metric is velocity dependent, but flat is the dispersion relation is a deformation of the special-relativistic one
- Proposal for a 'rainbow' metric associated to classical particles with k-Poincaré-inspired symmetries
- This provides a consistent framework to derive physical properties of the particle: propagation, symmetries
- can it be used to treat more complicated cases, when gravity is introduced?
- how to introduce interactions?