Noncommutative Geometry on $U(gl(n))$, its braided deformations, and applications

Dimitri Gurevich

with P.Saponov

Valenciennes University, France

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Plan

First, we want to define "partial derivatives" on the algebra $U(gl(n)_\hbar)$. This algebra is generated by elements n_i^j subject to

$$n_i^j\, n_k^l - n_k^l\, n_i^j - \hbar(\delta_k^j\, n_i^l - \delta_i^l\, n_k^j) = 0.$$

These partial derivatives should satisfy $\partial_{n_i^j}(n_k^l) = \delta_j^l\, \delta_k^i$ and a version of the Leibniz rule. The following Leibniz rule, presented via a coproduct, is compatible with the algebraic structure in $U(gl(n)_\hbar)$

$$\Delta(\partial_{n_i^j}) = \partial_{n_i^j} \otimes 1 + 1 \otimes \partial_{n_i^j} + \hbar \sum \partial_{n_i^k} \otimes \partial_{n_k^j}.$$

Similar "partial derivatives" can be defined for super-algebras $U(gl(m|n)_\hbar)$ and their "braided analogs" mentioned below.

Consider a vector space V over the ground field $\mathbb{K}=\mathbb{C}$. We call an invertible linear operator $R: V^{\otimes 2}\to V^{\otimes 2}$ *braiding* if it satisfies the so-called *braid relation*

$$R_{12}\,R_{23}\,R_{12}=R_{23}\,R_{12}\,R_{23},\quad R_{12}=R\otimes I,\ \ R_{23}=I\otimes R.$$

A braiding R is called *involutive symmetry* if $R^2=I$.
A braiding is called *Hecke symmetry* if it is subject to the Hecke condition

$$(q\,I-R)(q^{-1}\,I+R)=0,\ \ q\in\mathbb{K}.$$

It has two eigenvalues: q and $-q^{-1}$. The most known example comes from the QG $U_q(sl(n))$.
As for the braidings coming from the QG of other series B_n, C_n, D_n each of them has 3 eigenvalues and it is called BMW symmetry.

The simplest examples are as follows. By fixing a basis $\{x, y\} \in V$ and the corresponding basis $\{x\otimes x, x\otimes y, y\otimes x, y\otimes y\}$ in $V^{\otimes 2}$ we represent Hecke symmetries R by the matrices

$$\begin{pmatrix} q & 0 & 0 & 0\\ 0 & q-q^{-1} & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} q & 0 & 0 & 0\\ 0 & q-q^{-1} & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

They are deformations of a usual flip and a super-flip respectively. However, there is a lot of Hecke symmetries which are deformations neither of flips nor of super-flips.

What are associative algebras which can be associated with a given braiding? By assuming R to be a Hecke symmetry we can associate with it "symmetric" and "skew-symmetric" algebras as follows

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \;\; \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle$$

It is possible to show that if $R = R(q)$ is a deformation of the usual flip $P = R(1)$ (we call it "quasi-classical") then dimensions of homogeneous components of the both algebras are classical for a generic q. A proof of this claim is based on constructing the projectors of "symmetrization" and skew-symmetrization"

$$P_+^{(m)} : V^{\otimes m} \to Sym_R^{(m)}(V), \;\; P_-^{(m)} : V^{\otimes m} \to \bigwedge\nolimits_R^{(m)}(V).$$

via the projectors $P_\pm^{(2)}$.

It is easy to define Hilbert-Poincaré series $\mathcal{P}_\pm(t)$ corresponding to these algebras: $\mathcal{P}_-(t) = \sum_k dim \bigwedge_R^k(V)t^k$ and similarly, $\mathcal{P}_+(t)$. The following holds $\mathcal{P}_-(-t)\mathcal{P}_+(t) = 1$.

Theorem

The HP series $\mathcal{P}_-(t)$ (and hence $\mathcal{P}_+(t)$) is a rational function:

$$\mathcal{P}_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \ldots + a_r\, t^r}{1 - b_1\, t + \ldots + (-1)^s\, b_s\, t^s} = \frac{\prod_{i=1}^r (1 + x_i t)}{\prod_{j=1}^s (1 - y_j t)},$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

We call the couple $(r|s)$ bi-rank. In this sense all Hecke symmetries are similar to super-flips.

We are interested in in such braidings which can be extended to the space $V \otimes V^* \cong End(V)$. So we want to have an extension $R_{\mathrm{End}(V)} : End(V)^{\otimes 2} \to End(V)^{\otimes 2}$ with good properties. Such braidings R are called skew-invertible.
It is tempting to define "braided" analog of the algebra $U(sl(n)_\hbar)$ by putting

$$X \otimes Y - R_{\mathrm{End}(V)}(X \otimes Y) = X \circ Y - \circ R_{\mathrm{End}(V)}(X \otimes Y)$$

and to treat the r.h.s. as braided $[X, Y]$. However, this way is reasonable iff R is an involutive symmetry. It is just my construction of the early 80's.
In general (i.e. if R is Hecke), there is no Jacobi relation and no good deformation property for this algebra.

Another known algebra related to a braiding is the so-called RTT algebra defined by

$$R\,T_1\,T_2 = T_1\,T_2\,R,\ \ T_1 = T\otimes I,\ \ T_2 = I\otimes T$$

where $T = (t_i^j)$, $1\le i,j\le n$ and t_i^j are generators.
If R is a "quasi-classical" Hecke symmetry the RTT algebra is a deformation of that $Sym(gl(n))$. It means that dimensions of homogeneous components are classical for a generic q.
Also, we consider the so-called Reflection Equation (RE) algebra defined by

$$R\,L_1\,R\,L_1 - L_1\,R\,L_1\,R = 0,\ \ L_1 = L\otimes I.$$

Here $L_1 = L\otimes I$ and $L = (l_i^j)$. Similarly to the RTT algebra, if R is a quasi-classical Hecke symmetry the RE algebra is a deformation of $Sym(gl(n))$.

The most natural way of showing good deformation property of these algebras consists in constructing projectors of "symmetrization" and "skew-symmetrization" and to study their properties.

Note that the both algebras can be identified with $V \otimes V^*$. Namely, $t_i^j = x_i \otimes x^j$ and $l_i^j = x_i \otimes x^j$ where $\{x_i\}$ is a basis in V and $\{x^j\}$ is the dual basis but only in the latter case the pairing $x_i \otimes x^j \to \delta_i^j$ is a "categorical morphism" (in the case related to the QG it is covariant w.r.t. to its action.

In a sense it is possible to say that RE algebra "belongs" to the closed category generated by the space V.

Also, consider the so-called modified RE algebra

$$R\, L_1\, R\, L_1 - L_1\, R\, L_1\, R = \hbar(R\, L_1 - L_1\, R).$$

This is two parameter deformation of the algebra $Sym(gl(n))$ (if R comes from the QG) and consequently there exists the corresponding P.p. on $Sym(gl(n))$.
Note that the relations in the mRE algebra can be written as (we put $\hbar = 1$)

$$l_i^j \otimes l_k^l - Q(l_i^j \otimes l_k^l) = l_i^j \circ l_k^l - \circ Q(l_i^j \otimes l_k^l) = [l_i^j, l_k^l]$$

but Q differs from $End(V)$ above.
For this "braided" bracket there is an analog of the Jacobi identity. Besides, its enveloping algebra (which is nothing but the mRE algebra presented in another way) is a deformation of $U(gl(n))$.

This construction admits a sl-reduction. So, a braided analog of the Lie algebra $sl(n)$ can be also defined with good properties. As for other series, the corresponding braided Lie bracket can be readily defined but its enveloping algebra does not have "good deformation property" (Donin).
Also, the REA and corresponding mREA are isomorphic to each other for a generic q but for $q \to 1$ this isomorphism fails.

How to construct a representation category of the mREA ? By assuming $\hbar = 1$ we use the identification above. Namely, we apply an element $x_i x^j$ to an element $x_a \in V$ by pairing x^j and x_a. Thus, we get a representation of the mREA into $End(V)$. In order to represent the mREA into $End(V^*)$ we should first to transpose x_i and x^j by $R_{\mathrm{End}(V)}$ and to apply the pairing again. (Note that the operation conjugation in the classical case is just this transposing.) By extending these representations via the coproduct

$$\Delta(l_i^j) = l_i^j \otimes 1 + 1 \otimes l_i^j + (q - q^{-1}) \sum_k l_i^k \otimes l_k^j,$$

we get a representation category looking like that of $U(gl(n))$. Note that in this extension the operator $R_{End(V)}$ is also involved.

Other properties of the REA are

1. In the RE algebra (modified or not) there exists a big center generated by elements

$$Tr_R L^k, \; k = 0, 1, 2...$$

where Tr_R is a "braided analog" of the usual trace.

2. For the generating matrix L there is an analog of the Cayley-Hamilton identity. In the simplest case it is

$$L^2 - (q\, Tr_R(L) + q^{-1}\hbar)L + (\frac{q^2}{2_q}(q\,(Tr_R\, L)^2 - Tr_R(L^2)) +$$

$$\frac{q\hbar}{2_q} Tr_R(L))I = 0.$$

Note that the coefficients of this CH identity are central. So the roots of this polynomial defined by

$$\mu_1 + \mu_2 = (q\, Tr_R(L) + q^{-1}\hbar),$$

$$\mu_1 \cdot \mu_2 = (\frac{q^2}{2_q}(q\,(Tr_R\, L)^2 - Tr_R(L^2)) + \frac{q\hbar}{2_q} Tr_R(L))$$

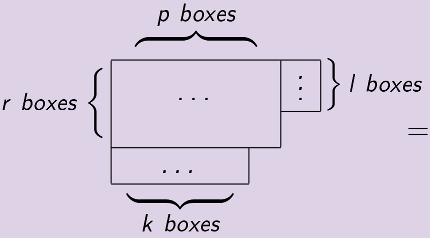
are elements of the central extension of the (modified) RE algebra. These roots play an important role in applications.

Theorem

In general, CH identity is (here $(p|r)$ is the bi-rank of R)

$$\sum_{i=0}^{p+r} L^{p+r-i} \sum_{k=\max\{0,i-r\}}^{\min\{i,p\}} (-1)^k\, q^{2k-i}\, s_{[p|r]^k_{i-k}}(L) = 0,$$

where s_λ is the Schur polynomial and



$$= \Big((p+1)^l, p^{(r-l)}, k\Big) =: [r|p]^l_k\,.$$

As for quantization of the mentioned P.p. restricted to a generic orbit, it is reasonable to describe it by a quotient

$$mREA/ < Tr_R L = \alpha_1, ..., Tr_R L^n = \alpha_n >$$

with appropriate α_i.

It is interesting question for what values of α_i this quotient is a regular braided variety? It is so if the eigenvalues μ_i meet the condition

$$\mu_i - q^{-2}\mu_j - q^{-1}\hbar \neq 0, \ \forall i, j.$$

For all reasons above the mRE algebra is often considered as algebra of vector fields similar to those arising from the Lie algebra $gl(n)$. But what is a "function space" where the mRE algebra acts? It was an RTT algebra in papers by Woronowicz, Isaev, Pyatove and others.

We replaced an RTT algebra by the corresponding mREA. By doing so we defined "partial derivatives" on this mREA and constructed a braided analog of the Weyl algebra. This "braided" Weyl algebra is defined by the following system

$$R\,N_1\,R\,N_1 - N_1\,R\,N_1\,R = \hbar\,(R\,N_1 - N_1\,R)$$

$$R^{-1}D_1\,R^{-1}D_1 = D_1\,R^{-1}D_1\,R^{-1}$$

$$D_1\,R\,N_1\,R - R\,N_1\,R^{-1}D_1 = R + \hbar\,D_1\,R\,.$$

The third line is called "permutation relations" between the mREA generated by entries n_i^j of the matrix N and the algebra generated by entries of the matrix D (they are considered to be analogs of the partial derivatives in n_i^j).
Fortunately, this system admits the $q \to 1$ limit. Thus, we get "partial derivatives" on the algebra $U(gl(n)_\hbar)$ mentioned above. Below we consider an example with $n = 2$.

Denote a, b, c, d the standard generators of the algebra $U(gl(2)_\hbar)$ such that

$$[a, b] = \hbar\, b,\ [a, c] = -\hbar\, c,\ [a, d] = 0, \ldots.., [d, c] = \hbar\, c.$$

Now, pass to generators of the compact form, namely, $U(u(2)_\hbar)$

$$t = \frac{1}{2}(a + d),\ \ x = \frac{i}{2}(b + c),\ \ y = \frac{1}{2}(c - b),\ \ z = \frac{i}{2}(a - d)$$

we get the standard $u(2)_\hbar$ table of commutators

$$[x,\ y] = \hbar\, z,\ \ [y,\ z] = \hbar\, x,\ \ [z,\ x] = \hbar\, y,\ \ t \text{ is central.}$$

Then the corresponding permutation relations become

$$[\partial_t, t] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_t, x] = -\frac{\hbar}{2}\partial_x,\ [\partial_t, y] = -\frac{\hbar}{2}\partial_y,\ [\partial_t, z] = -\frac{\hbar}{2}\partial_z,$$

$$[\partial_x, t] = \frac{\hbar}{2}\partial_x,\ [\partial_x, x] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_x, y] = \frac{\hbar}{2}\partial_z,\ [\partial_x, z] = -\frac{\hbar}{2}\partial_y,$$

$$[\partial_y, t] = \frac{\hbar}{2}\partial_y,\ [\partial_y, x] = -\frac{\hbar}{2}\partial_z,\ [\partial_y, y] = \frac{\hbar}{2}\partial_t + 1,\ [\partial_y, z] = \frac{\hbar}{2}\partial_x,$$

$$[\partial_z, t] = \frac{\hbar}{2}\partial_z,\ [\partial_z, x] = \frac{\hbar}{2}\partial_y,\ [\partial_z, y] = -\frac{\hbar}{2}\partial_x,\ [\partial_z, z] = \frac{\hbar}{2}\partial_t + 1.$$

Besides, the generators $\partial_t ..., \partial_z$ commute with each other and generate a commutative algebra $\mathcal{D}$. Thus, we get a Weyl algebra generated by two subalgebras $U(u(2)_\hbar)$ and $\mathcal{D}$ and the above permutation relations.

In order to convert the partial derivatives into operators we have to use the counit on $\mathcal{D}$

$$\varepsilon(\partial_t) = ... = \varepsilon(\partial_z) = 0,\ \varepsilon(1) = 1$$

extended in the multiplicative way. Then we define $\partial(a)$ by permutating ∂ and a and by applying ε to the right factor from $\mathcal{D}$. For instance, in virtue of the permutation relations we have

$$\partial_x\, yz = (y\partial_x + \frac{\hbar}{2}\partial_z)\, z = y(z\partial_x - \frac{\hbar}{2}\partial_y) + \frac{\hbar}{2}(z\partial_z + \frac{\hbar}{2}\partial_t + 1).$$

Now, by applying the counit we conclude that $\partial_x(yz) = \frac{\hbar}{2}$. This result turns into the classical one as $\hbar = 0$.

Observe that such partial derivatives are difference operators. Thus, we have

$$\partial_t(f(t)) = \frac{2}{\hbar}(f(t+\frac{\hbar}{2}) - f(t)),$$

$$\partial_t(f(x)) = \frac{1}{\hbar}(f(x - i\frac{\hbar}{2}) + f(x + i\frac{\hbar}{2}) - 2f(x)),$$

$$\partial_x(f(t)) = \partial_x(f(y)) = \partial_x(f(z)) = 0,$$

$$\partial_x(f(x)) = \frac{i}{\hbar}(f(x - i\frac{\hbar}{2}) - f(x + i\frac{\hbar}{2})$$

etc.

As for the de Rham operator d it can be defined on "functions" via

$$d(f) = dt\,\partial_t(f) + dx\,\partial_x(f) + dy\,\partial_y(f) + dz\,\partial_z(f).$$

In a similar way we can define de Rham operator d on differential forms. Namely, we put

$$d(\omega\, f) = \omega\, d(f) = \omega\, dt\,\partial_t(f) + \omega\, dx\,\partial_x(f) + \omega\, dy\,\partial_y(f) + \omega\, dz\,\partial_z(f)$$

where ω is a pure differential form ($d\,t, ...d\,t\,dx, ...$ and so on). The relations between the differentials $dt, ..., dz$ are assumed to be classical $dt\,dx = -dx\,dt$, i.e. these generators anticommute. This property together with the commutativity of the partial derivatives entails $d^2 = 0$, i.e. d is a differential indeed.

By introducing the permutation relations

$$du\otimes a = a\otimes du,\ \ \forall\ u\in\{t,\,x,\,y,\,z\},\ \forall\ a\in U(u(2)_\hbar)$$

we can introduce the structure of an associative algebra on the space $\Omega(U(u(2)_\hbar))$, but the de Rham operator d is not compatible with this structure via the classical Leibniz rule. For instance, we have $d(yz) = dx\,\frac{\hbar}{2} + dy\,z + dz\,y$.

On the algebra in question the coproduct mentioned in Introduction becomes

$$\Delta(\partial_t) = \partial_t \otimes 1 + 1 \otimes \partial_t + \frac{\hbar}{2}(\partial_t \otimes \partial_t - \partial_x \otimes \partial_x - \partial_y \otimes \partial_y - \partial_z \otimes \partial_z),$$

$$\Delta(\partial_x) = \partial_x \otimes 1 + 1 \otimes \partial_x + \frac{\hbar}{2}(\partial_t \otimes \partial_x + \partial_x \otimes \partial_t + \partial_y \otimes \partial_z - \partial_z \otimes \partial_y)...$$

Observe that we first found the permutation relations and afterwards Meljanac and Skoda (from Zagreb) presented the Leibniz rule via this coproduct.

Now, we want to extend the Weyl algebra. Since the algebra $U(u(2)_\hbar)$ has the Ore property, we can consider the field of fractions a/b. However, it is not clear what are permutation relations of the derivatives and elements $1/a$ and what is the action of the former elements on the latter ones.
In the classical case if V is a vector field and consequently, it is subject to the classical Leibniz rule, we have

$$0 = V(1) = V(aa^{-1}) = V(a)a^{-1} + aV(a^{-1}).$$

Thus, we get $V(a^{-1}) = -a^{-1}\, V(a)a^{-1}$.
In our present setting we are dealing with another coproduct (or equivalently new permutation relations).

We use the shifted derivative $\tilde\partial_t = \partial_t + \frac{2}{\hbar}$ instead of that ∂_t. Its advantage is following: by permuting the column $(\tilde\partial_t, \partial_x, \partial_y, \partial_z)^T$ (here T stand for the transposition) with an element $a \in U(u(2)_{\frac{\hbar}{2}})$ we have

$$(\tilde\partial_t, \partial_x, \partial_y, \partial_z)^T\, a = \Theta_a (\tilde\partial_t, \partial_x, \partial_y, \partial_z)^T$$

where Θ_a is a matrix with entries from $U(u(2)_\hbar)$.

By permuting the column $(\tilde\partial_t, \partial_x, \partial_y, \partial_z)^T$ with the element $1 = a\, a^{-1}$ we get that the product of the matrices Θ_a and $\Theta_{a^{-1}}$ is the identity matrix. So, in order to get the permutation relations between this column and the element a^{-1} we have to invert the matrix Θ_a.

Let us consider an example when such a spectral decomposition exists. Let $a = x$. Then in virtue of the permutation relations we have

$$\begin{pmatrix} \tilde\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} x = \begin{pmatrix} x & -\frac{\hbar}{2} & 0 & 0 \\ \frac{\hbar}{2} & x & 0 & 0 \\ 0 & 0 & x & -\frac{\hbar}{2} \\ 0 & 0 & \frac{\hbar}{2} & x \end{pmatrix} \begin{pmatrix} \tilde\partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \tag{1}$$

We do not know a general receipt for inverting these matrices for any element $a \in U(u(2)_\hbar)$. However, for some elements $a \in U(u(2)_\hbar)$ we have such a receipt. For instance, it is feasible if we assume the matrix Θ_a to have a spectral decomposition

$$\Theta_a = \Sigma_i \lambda_i P_i$$

where λ_i are elements of the algebra $U(u(2)_\hbar)$ and P_i are complementary idempotents, also composed from the elements of $U(u(2)_h)$ and commuting with all λ_j. Then we have

$$(\tilde{\partial}_t, \partial_x, \partial_y, \partial_z)^T f(a) = \Sigma_i f(\lambda_i) P_i \, (\tilde{\partial}_t, \partial_x, \partial_y, \partial_z)^T,$$

where f is any polynomial or even a series.

The matrix Θ_x entering the formula in the above example has the following spectral decomposition

$$\Theta_x = \lambda_1 P_1 + \lambda_2 P_2,$$

where $\lambda_1 = x - h$, $\lambda_2 = x + h$, $(h = \frac{\hbar}{2i})$ while the idempotent P_1 (resp., P_2) is composed from two identical blocs

$$\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \ (\text{resp.,} \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}). \qquad (2)$$

Thus, we can define the permutation relations between $(\tilde{\partial}_t, \partial_x, \partial_y, \partial_z)^T$ and x^{-1} by putting $\Theta_{x^{-1}} = \lambda_1^{-1} P_1 + \lambda_2^{-1} P_2$.

We want to extend the Weyl algebra more. Namely, we want to introduce an analog $r_\hbar$ of the radius $r = \sqrt{x^2 + y^2 + z^2}$ and to define permutation relations between the partial derivatives and the elements $f(r_\hbar)$ as well as the action of the partial derivatives on such elements.
We need them in order to consider a NC analog of the Schwarzschild model and similar ones.

The central role in defining this quantity is played by the Cayley-Hamilton (CH) identity for the "generating matrix" N of the algebra $U(u(2)_\hbar)$

$$N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} t - i\,z & -i\,x - y \\ -i\,x + y & t + i\,z \end{pmatrix}. \qquad (3)$$

(Note that the algebra $U(u(2)_\hbar)$ can be defined by
$P\,N_1\,P\,N_1 - N_1\,P\,N_1\,P = \hbar(P\,N_1 - N_1\,P)$.)
Then the following CH is valid

$$\chi(N) = N^2 - (2\,t + \frac{\hbar}{2})\,N + (t^2 + x^2 + y^2 + z^2 + \hbar\,t)\,I = 0. \quad (4)$$

Let μ_1 and μ_2 be the roots of this polynomial. They belong to an algebraic extension of the center of the algebra $U(u(2)_\hbar)$. Then we define the quantum radius to be $r_\hbar = \frac{\mu_1 - \mu_2}{2i}$. It is equal $r_\hbar = \sqrt{x^2 + y^2 + z^2 + h^2}$, where $h = \frac{\hbar}{2i}$.
By using a method similar to that discussed above we can define permutation relations between the partial derivatives and $r_\hbar$ and the action of the derivatives on it. We get

$$\partial_x (f(r_\hbar)) = \frac{f(r_\hbar + h) - f(r_\hbar - h)}{2h} \frac{x}{r_\hbar} = \frac{x}{r_\hbar} \partial_{r_\hbar} f(r_\hbar),$$

where

$$\partial_{r_\hbar}(f(r_\hbar)) = \frac{f(r_\hbar + h) - f(r_\hbar - h)}{2h}.$$

Now, apply this calculus to quantization (different from the canonical one) of differential operators. If a given differential operator has constant coefficients we consider the same operator on the algebra $U(u(2)_\hbar)$ as defined above. F.e. a "quantum version" of the Klein-Gordon operator is defined in the classical way

$$(\Box - m^2)\, f = 0, \;\; \Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$$

Here f is an element of the algebra $U(u(2)_\hbar)$ or its extension and m is the "mass of a NC particle". In a similar way we define NC analogs of other wave (Dirac, Maxwell...) operators.

Consider a NC analog of the Dirac monopole. It is a stationary spherically symmetric solution to the Maxwell system. Moreover, the vector of the electric field $\mathcal{E} = (E_1, E_2, E_3)$ is assumed to be trivial. Thus, for the vector of magnetic field we have

$$\mathrm{div}\, \mathcal{H} = 0, \mathrm{curl}\, \mathcal{H} = (0, 0, 0).$$

By looking for a solution of this system under the form $\mathcal{H} = f(r)(x, y, z)$ it is easy to find $f(r) = C/r^3$.

In our present setting the equations are the same. We are looking for a solution under the form $\mathcal{H} = f(r_\hbar)(x, y, z)$. We have

$$\frac{r_\hbar^2 - h^2}{r_\hbar}\frac{f(r_\hbar + h) - f(r_\hbar - h)}{2h} +$$

$$3\frac{f(r_\hbar + h)(r_\hbar + h) + f(r_\hbar - h)(r_\hbar - h)}{2r_\hbar} = 0.$$

We get $f(r_\hbar) = C(h)(\delta(r_\hbar + h) + \delta(r_\hbar - h))$.

What next?

1. We plan to get a higher dimensional counterparts of the derivative $\partial_{r_\hbar}$.

2. In terms of these counterparts it is possible to get a difference analog of the Calogero-Moser model and to write down NC analogs of some matrix models.

3. To construct an analog of the Hodge operator and apply this stuff to gravity.