Deforming $\mathcal{N} = 4$ Supersymmetric Quantum Mechanics

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Motivations and contents

Supersymmetric Quantum Mechanics (SQM) (E.Witten, 1983) is the simplest ($d = 1$) supersymmetric theory:

- Displays the salient features of higher-dimensional supersymmetric theories via the dimensional reduction;
- Provides superextensions of integrable models like Calogero-Moser systems, Landau-type models, etc;
- Extended SUSY in $d = 1$ ($\mathcal{N} \geq 2$) exhibits interesting specific features: dualities between various supermultiplets (J.S.Gates, Jr. & L.Rana, 1995, A.Pashnev & F.Toppan, 2001), nonlinear “cousins” of off-shell linear multiplets (E.I., S.Krivonos, O.Lechtenfeld, 2003, 2004), etc.

Symmetry group of the standard SQMs is $\mathcal{N}$ extended $d = 1$ “super Poincaré”

$$\{Q^A, Q^B\} = 2\delta^{AB} H\,, \quad [H, Q^A] = 0\,, \quad A, B = 1 \ldots \mathcal{N}\,. \tag{1}$$

Recently, there was a substantial interest in rigid supersymmetric theories based on curved analogs of the Poincaré supergroup in diverse dimensions (e.g.,T.Dumitrescu, G.Festuccia, N.Seiberg, 2011, 2012). There is the hope that their study will lead to a further progress in understanding, e.g., the generic gauge/gravity correspondence. Can we define analogous deformations of the above simplest $\mathcal{N} = 1, d = 1$ supersymmetry?

A way to define such generalized SQM models is suggested by $\mathcal{N}=2, d=1$ Poincaré superalgebra in the complex notation

$$Q=\frac{1}{\sqrt{2}}(Q^1+iQ^2)\,,\ \bar{Q}=\frac{1}{\sqrt{2}}(Q^1-iQ^2)\,,$$

$$\{Q,\bar{Q}\}=2H\,,\ Q^2=\bar{Q}^2=0,\ [H,Q]=[H,\bar{Q}]=0\,. \qquad (2)$$

$$[J,Q]=Q\,,\quad [J,\bar{Q}]=-\bar{Q}\,,\quad [H,J]=0\,. \qquad (3)$$

The relations (2) and (3) define the superalgebra $u(1|1)$, with H being the relevant central charge generator and J the automorphism $u(1)$ one.

This two-fold interpretation of $\mathcal{N}=2, d=1$ Poincaré superalgebra suggests two ways of extending it to higher-rank $d=1$ supersymmetries.

A. Standard extension:

$$(\mathcal{N}=2\,,\ d=1)\quad\Rightarrow\quad(\mathcal{N}>2\,,\ d=1\ \text{“super Poincaré”})\,,$$

B. Non-standard extension:

$$(\mathcal{N}=2\,,\ d=1)\equiv u(1|1)\ \subset\ su(2|1)\ \subset\ su(2|2)\ \subset\ \ldots\,.$$

In the links of the chain B, the closure of supercharges contains, besides H, also internal symmetry generators.

It is interesting to construct new SQM models associated with the sequence B, starting from the simplest $su(2|1)$ case. This is the subject of my talk.

- The widely recognized way of constructing invariant actions of supersymmetric theories, including SQM , is Superspace Approach.
- Our aim is to construct the worldline superfield realizations of $SU(2|1)$ and to show that most off-shell multiplets of $\mathcal{N} = 4, d = 1$ supersymmetry have the well-defined $SU(2|1)$ analogs.
- In particular, the SQM models known as “week supersymmetry models” (A.Smilga, 2004) are based on the $SU(2|1)$ multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$. They are easily reproduced from our superfield approach.
- $SU(2|1)$ has also invariant chiral subspaces which are natural carriers of the chiral multiplets $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. A interesting feature of the relevant component actions is the presence of the bosonic $d = 1$ Wess-Zumino terms along with the second-order kinetic terms.
- In fact, there is a one-parameter family of chiral $SU(2|1)$ superspaces, with physically non-equivalent $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplets defined on them (E.I., S.Sidorov, 1312.6821 [hep-th]).
- $SU(2|1)$ also admits a supercoset which is an analog of the harmonic analytic superspace of the standard $\mathcal{N} = 4, d = 1$ supersymmetry (E.I., O.Lechtenfeld, 2004). So one can define $SU(2|1)$ analogs of the analytic $\mathcal{N} = 4$ superfields $(\mathbf{4}, \mathbf{4}, \mathbf{0})$ and $(\mathbf{3}, \mathbf{4}, \mathbf{1})$.

$SU(2|1)$ superspace

- The (central-extended) superalgebra $su(2|1)$:

$$\{Q^i, \bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\delta^i_j H\,, \qquad \left[I^i_j, I^k_l\right] = \delta^k_j I^i_l - \delta^i_l I^k_j\,,$$
$$\left[I^i_j, \bar{Q}_l\right] = \frac{1}{2}\delta^i_j \bar{Q}_l - \delta^i_l \bar{Q}_j\,, \qquad \left[I^i_j, Q^k\right] = \delta^k_j Q^i - \frac{1}{2}\delta^i_j Q^k,$$
$$\left[F, \bar{Q}_l\right] = -\frac{1}{2}\bar{Q}_l\,, \qquad \left[F, Q^k\right] = \frac{1}{2}Q^k.$$

- The supercoset:

$$\frac{SU(2|1)}{SU(2)\times U(1)} \sim \frac{\{Q^i, \bar{Q}_j, H, I^i_j, F\}}{\{I^i_j, F\}}\,.$$

The superspace coordinates $\{t, \theta_i, \bar{\theta}^j\}$ are identified with the parameters associated with the coset generators. An element of this supercoset can be conveniently parametrized as

$$g = \exp\left(itH + i\tilde{\theta}_i Q^i - i\tilde{\bar{\theta}}^j \bar{Q}_j\right),$$
$$\tilde{\theta}_i = \left[1 - \frac{2m}{3}\left(\bar{\theta}\cdot\theta\right)\right]\theta_i\,.$$

- Transformation properties under $Q, \bar{Q}$

$$\delta\theta_i = \epsilon_i + 2m\,(\bar{\epsilon}\cdot\theta)\,\theta_i\,, \qquad \delta\bar{\theta}^j = \bar{\epsilon}^j - 2m\left(\epsilon\cdot\bar{\theta}\right)\bar{\theta}^j,$$
$$\delta t = i\left[\left(\epsilon\cdot\bar{\theta}\right) + (\bar{\epsilon}\cdot\theta)\right].$$

- Invariant integration measure

$$\mu = dtd^2\theta d^2\bar{\theta}(1 + 2m\,\bar{\theta}\cdot\theta)\,, \quad \delta\mu = 0\,.$$

- Generators

$$Q^i = -i\frac{\partial}{\partial\theta_i} + 2im\bar{\theta}^i\bar{\theta}^j\frac{\partial}{\partial\bar{\theta}^j} + \bar{\theta}^i\frac{\partial}{\partial t}\,,\ \bar{Q}_j = i\frac{\partial}{\partial\bar{\theta}^j} + 2im\theta_j\theta_k\frac{\partial}{\partial\theta_k} - \theta_j\frac{\partial}{\partial t}\,,$$
$$I^i_j = \left(\bar{\theta}^i\frac{\partial}{\partial\bar{\theta}^j} - \theta_j\frac{\partial}{\partial\theta_i}\right) - \frac{\delta^i_j}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),$$
$$F = \frac{1}{2}\left(\bar{\theta}^k\frac{\partial}{\partial\bar{\theta}^k} - \theta_k\frac{\partial}{\partial\theta_k}\right),\ H = i\partial_t.$$

- Left-covariant Cartan forms

$$g^{-1}dg = e^{-B}d\,e^{B} + i\,dt\,H = i\Delta\theta_i Q^i - i\Delta\bar{\theta}^j \bar{Q}_j + i\Delta h^j_i\, I^i_j + i\Delta\hat{h}\,F + i\Delta t\,H\,,$$

$$B := \left(i\tilde{\theta}_i Q^i - i\tilde{\bar{\theta}}^j \bar{Q}_j\right)$$

- Explicitly:

$$\Delta\theta_i = d\theta_i + m\left(d\theta_l\bar{\theta}^l\theta_i - d\theta_i\bar{\theta}^k\theta_k\right) + \frac{m^2}{4}d\theta_i\left(\bar{\theta}\cdot\theta\right)^2,$$

$$\Delta\bar{\theta}^j = d\bar{\theta}^j - m\left(d\bar{\theta}^l\theta_l\bar{\theta}^j - d\bar{\theta}^j\theta_k\bar{\theta}^k\right) + \frac{m^2}{4}d\bar{\theta}^j\left(\bar{\theta}\cdot\theta\right)^2,$$

$$\Delta t = dt + i\left(d\theta_i\bar{\theta}^i + d\bar{\theta}^i\theta_i\right)\left(1 - 2m\,\bar{\theta}\cdot\theta\right).$$

- Covariant derivatives

$$\mathcal{D}\Phi_A := d\Phi_A + \left[i\Delta h^j_i\,\hat{I}^i_j + i\Delta\hat{h}\,\hat{F}\right]_A^B\Phi_B \equiv \left[\Delta\theta_i\mathcal{D}^i - \Delta\bar{\theta}^j\bar{\mathcal{D}}_j + \Delta t\,\mathcal{D}_t\right]\Phi_A\,,$$

$$\mathcal{D}^i = \left[1 + m\left(\bar{\theta}\cdot\theta\right) - \frac{3m^2}{4}\left(\bar{\theta}\cdot\theta\right)^2\right]\frac{\partial}{\partial\theta_i} - m\bar{\theta}^i\theta_j\frac{\partial}{\partial\theta_j} - i\bar{\theta}^i\frac{\partial}{\partial t} + \ldots\,,$$

$$\bar{\mathcal{D}}_j = -\left[1 + m\left(\bar{\theta}\cdot\theta\right) - \frac{3m^2}{4}\left(\bar{\theta}\cdot\theta\right)^2\right]\frac{\partial}{\partial\bar{\theta}^j} + m\bar{\theta}^k\theta_j\frac{\partial}{\partial\bar{\theta}^k} + i\theta_j\frac{\partial}{\partial t} + \ldots\,.$$

Here "dots" stand for matrix $U(2)$ connection parts.

$(1,4,3)$ multiplet: invariant action

- The $(\mathbf{1},\mathbf{4},\mathbf{3})$ multiplet is described by the real neutral superfield $G(t,\theta,\bar\theta)$ satisfying

$$\varepsilon^{lj}\bar{\mathcal{D}}_l\,\bar{\mathcal{D}}_j G = \varepsilon_{lj}\mathcal{D}^l\,\mathcal{D}^j G = 0 \quad \Rightarrow$$

$$G = x - mx\,(\bar\theta\cdot\theta)\,[1-2m\,(\bar\theta\cdot\theta)] + \frac{\ddot{x}}{2}\,(\bar\theta\cdot\theta)^2 - i\,(\bar\theta\cdot\theta)\left(\theta_i\,\dot\psi^i + \bar\theta^j\,\dot{\bar\psi}_j\right)$$
$$+\left[1-2m\,(\bar\theta\cdot\theta)\right]\left(\theta_i\,\psi^i - \bar\theta^j\,\bar\psi_j\right) + \bar\theta^j\theta_i\,B^i_j\,,\quad B^k_k = 0\,.$$

- The irreducible set of off-shell fields is $x(t),\psi^i(t),\bar\psi_i(t),B^i_j(t)(B^k_k=0)$, In the limit $m=0$ it is reduced to the ordinary $(\mathbf{1},\mathbf{4},\mathbf{3})$ superfield.
- The ϵ transformation law of G,

$$\delta G = -\left(i\epsilon_i Q^i - i\bar\epsilon^j\bar{Q}_j\right)G,$$

implies

$$\delta x = (\bar\epsilon\cdot\bar\psi) - (\epsilon\cdot\psi)\,,\qquad \delta\psi^i = i\bar\epsilon^i\dot{x} - m\bar\epsilon^i x + \bar\epsilon^k B^i_k\,,$$
$$\delta B_{(ij)} = -2i\left[\epsilon_{(i}\dot\psi_{j)} + \bar\epsilon_{(i}\dot{\bar\psi}_{j)}\right] + 2m\left[\bar\epsilon_{(i}\bar\psi_{j)} - \epsilon_{(i}\psi_{j)}\right].$$

- Invariant action

$$\mathcal{L} = -\int d^2\theta\, d^2\bar{\theta}\, (1 + 2m\,\bar{\theta}\cdot\theta)\ f(G)\,, \quad S = \int dt\mathcal{L}\,.$$

- Doing θ integral and eliminating the auxiliary field,

$$B_{(ij)} = \frac{g'(x)}{g(x)}\,\psi_{(i}\bar{\psi}_{j)}\,, \quad g := f''\,,$$

we obtain the on-shell action

$$\begin{aligned}\mathcal{L} \quad = \quad & \dot{x}^2 g(x) + i\left(\bar{\psi}_i\dot{\psi}^i - \dot{\bar{\psi}}_i\psi^i\right) g(x) - \frac{1}{2}\left(\bar{\psi}_i\psi^i\right)^2 \left[g''(x) - \frac{3\,(g'(x))^2}{2g(x)}\right] \\ & -m^2x^2g(x) + 2m\bar{\psi}_i\psi^i g(x) + mx\bar{\psi}_i\psi^i g'(x).\end{aligned}$$

- This Lagrangian can be simplified by passing to new variables $y(x)$,

$$\dot{x}^2 g(x) = \frac{1}{2}\dot{y}^2 , \quad \Rightarrow y'(x) = \sqrt{2g(x)} ,$$

and $\zeta^i = \psi^i y'(x)$. We find

$$\begin{aligned} \mathcal{L} \quad = \quad & \frac{\dot{y}^2}{2} + \frac{i}{2}\left(\bar{\zeta}_i \dot{\zeta}^i - \dot{\bar{\zeta}}_i \zeta^i\right) - \frac{m^2}{2} V^2(y) + m\bar{\zeta}_i \zeta^i V'(y) \\ & - \frac{1}{2}\left(\bar{\zeta}_i \zeta^i\right)^2 \partial_y \left(\frac{V'(y) - 1}{V(y)}\right) . \end{aligned}$$

Here, $V(y) := xy'(x) = x(y)\frac{1}{x'(y)}$.

- Thus we have obtained the Lagrangian involving an arbitrary function $V(y)$. The on-shell supersymmetry transformations read

$$\begin{aligned} & \delta y = \bar{\epsilon}^k \bar{\zeta}_k - \epsilon_k \zeta^k , \\ & \delta \zeta^i = i\bar{\epsilon}^i \dot{y} - m\bar{\epsilon}^i V(y) - \left(\epsilon_k \zeta^k \zeta^i + \bar{\epsilon}^k \bar{\zeta}_k \zeta^i - \bar{\epsilon}^i \bar{\zeta}_k \zeta^k\right) \frac{V'(y) - 1}{V(y)} . \end{aligned}$$

These Lagrangian and the transformations are just those defining the SQM model with “weak” $\mathcal{N} = 4$ supersymmetry (A. Smilga, 2004).

$(1, 4, 3)$ multiplet: quantization

- We consider the simplest case with $f(x) = \frac{x^2}{4}$

$$\mathcal{L} = \frac{\dot{x}^2}{2} - \frac{m^2 x^2}{2} + \frac{i}{2}\left(\bar{\psi}_i \dot{\psi}^i - \dot{\bar{\psi}}_i \psi^i\right) + m\bar{\psi}_i \psi^i ,$$

  The action is invariant under the transformations

$$\delta x = \left(\bar{\epsilon} \cdot \bar{\psi}\right) - \left(\epsilon \cdot \psi\right), \qquad \delta \psi^i = i\bar{\epsilon}^i \dot{x} - m\bar{\epsilon}^i x \,. \tag{4}$$

- The conserved Noether charges and Hamiltonian read:

$$Q^i = \psi^i \left(p - imx\right), \qquad \bar{Q}_i = \bar{\psi}_i \left(p + imx\right),$$
$$F = \frac{1}{2}\psi^k \bar{\psi}_k, \qquad I^i_j = \psi^i \bar{\psi}_j - \frac{1}{2}\delta^i_j \psi^k \bar{\psi}_k .$$
$$H = \frac{p^2}{2} + \frac{m^2 x^2}{2} + m\psi^i \bar{\psi}_i \,.$$

  This H is $SU(2|1)$ extension of the harmonic oscillator Hamiltonian.

The Poisson brackets are imposed as

$$\{x, p\} = 1, \qquad \{\psi^i, \bar{\psi}_j\} = -i\delta^i_j\,.$$

We quantize in the standard way

$$[\hat{x}, \hat{p}] = i\,, \quad \{\hat{\psi}^i, \hat{\bar{\psi}}_j\} = \delta^i_j\,, \quad \hat{p} = -i\partial_x\,, \ \hat{\bar{\psi}}_j = \partial/\partial\hat{\psi}^j\,,$$

and represent the quantum Hamiltonian as

$$\hat{H} \quad = \quad \frac{1}{2}\left(\hat{p} + im\hat{x}\right)\left(\hat{p} - im\hat{x}\right) + m\hat{\psi}^i\hat{\bar{\psi}}_i\,.$$

This H and the remaining quantum generators

$$\hat{Q}^i = \hat{\psi}^i\left(\hat{p} - im\hat{x}\right), \qquad \hat{\bar{Q}}_i = \hat{\bar{\psi}}_i\left(\hat{p} + im\hat{x}\right),$$
$$\hat{F} = \frac{1}{2}\hat{\psi}^k\hat{\bar{\psi}}_k, \qquad \hat{I}^i_j = \hat{\psi}^i\hat{\bar{\psi}}_j - \frac{1}{2}\delta^i_j\hat{\psi}^k\hat{\bar{\psi}}_k\,.$$

can be directly checked to form the superalgebra $su(2|1)$.

Spectrum

- We construct the Hilbert space of wave functions in terms of the harmonic oscillator wave functions. The super wave-function $\Omega^{(\ell)}$ at the energy level $\ell, \ell \geq 2$, reveals a four-fold degeneracy

$$\Omega^{(\ell)} = a^{(\ell)}|\ell\rangle + b_i^{(\ell)}\,\psi^i|\ell-1\rangle + \frac{1}{2}\,c^{(\ell)}\,\varepsilon_{ij}\psi^i\psi^j|\ell-2\rangle, \qquad \ell \geq 2\,,$$

  where $|\ell\rangle, |\ell-1\rangle, |\ell-2\rangle$ are the harmonic oscillator functions.

- We treat the operators $\hat{p} \pm imx$ in $\hat{H}$ as the creation and annihilation operators and impose the standard conditions

$$\hat{\bar{\psi}}_k|\ell\rangle = 0, \qquad (\hat{p} - im\hat{x})\,|0\rangle = 0, \qquad (\hat{p} + im\hat{x})\,|\ell\rangle = |\ell+1\rangle.$$

  The spectrum of the Hamiltonian is then

$$\hat{H}\,\Omega^{(\ell)} = m\,\ell\,\Omega^{(\ell)}, \qquad m > 0.$$

- The ground state ($\ell = 0$) and the first excited states ($\ell = 1$) are special, they encompass non-equal numbers of bosonic and fermionic states:

$$\Omega^{(0)} = a^{(0)}\,|0\rangle\,, \qquad \Omega^{(1)} = a^{(1)}\,|1\rangle + b_i^{(1)}\,\psi^i|0\rangle\,.$$

$SU(2|1)$ representation content

- The ground state is annihilated by all $SU(2|1)$ generators including Q^i and $\bar{Q}_i$, so it is $SU(2|1)$ singlet.
- The states with $\ell = 1$ form the fundamental $(\mathbf{2}|\mathbf{1})$ representation of $SU(2|1)$. The action of the supercharges on them is

$$Q^i\,\psi^k|0\rangle = 0\,, \qquad \bar{Q}_i\,\psi^k|0\rangle = \delta_i^k\,|1\rangle\,,$$
$$Q^i\,|1\rangle = 2m\,\psi^i|0\rangle\,, \qquad \bar{Q}_i\,|1\rangle = 0\,.$$

- The states with $\ell > 1$ form the representations $(\mathbf{2}|\mathbf{2})$, with equal numbers of bosonic and fermionic states.
- It is instructive to see what values two $SU(2|1)$ Casimir operators C_2, C_3 take on all these states. The explicit form of these operators in terms of the $SU(2|1)$ generators is well known, so let me skip it.
- For our quantum-mechanical realization Casimirs are reduced to the following nice form

$$m^2 C_2 = \hat{H}\left(\hat{H} - m\right), \quad m^3 C_3 = \hat{H}\left(\hat{H} - m\right)\left(\hat{H} - \frac{m}{2}\right).$$

Thus they are fully specified by the energy spectrum of $\hat{H}$.

- The values of Casimir operators for the finite-dimensional $SU(2|1)$ representations are known to be

$$C_2 = (\beta^2 - \lambda^2), \quad C_3 = \beta(\beta^2 - \lambda^2) = \beta C_2.$$

These representations are characterized by some positive number λ ("highest weight") which can be half-integer or integer and a real number β related to the eigenvalues of the $U(1)$ generator F.

- Comparing this with the above expressions for Casimirs in terms of $\hat{H}$, we find that $\lambda = 1/2$ for any $\Omega^{(\ell)}$ and

$$C_2(\ell) = (\ell - 1)\,\ell, \quad C_3(\ell) = (\ell - 1/2)\,(\ell - 1)\,\ell, \quad \beta(\ell) = (\ell - 1/2)\,.$$

The ground state with $\ell = 0$ is atypical, because Casimir operators take zero values on it. On the states with $\ell = 1$ both Casimirs vanish as well, so these states also form an atypical $SU(2|1)$ representation. The fact that the fundamental representation of $SU(2|1)$ is atypical is well known. On the $\ell > 1$ states both Casimirs are non-zero, so these states belong to the typical $SU(2|1)$ representations characterized by equal numbers of the bosonic and fermionic states.

Chiral multiplet

- One can also define $SU(2|1)$ counterpart of the $\mathcal{N}=4, d=1$ chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$. This is due to the existence of the chiral coset

$$\frac{\{Q^i,\bar{Q}_j,H,I^i_k,F\}}{\{\bar{Q}_j,I^i_k,F\}}\sim(t_L,\theta_i)\,,\qquad t_L=t+\frac{i}{2m}\ln(1+2m\,\bar{\theta}\cdot\theta),$$
$$\delta\theta_i=\epsilon_i+2m\,(\bar{\epsilon}\cdot\theta)\theta_i\,,\qquad \delta t_L=2i\,(\bar{\epsilon}\cdot\theta)\,.$$

- The multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$ is described by the chiral superfield Φ

$$\bar{\mathcal{D}}_j\Phi=0,\qquad \hat{I}^i_j\Phi=0,\qquad \hat{F}\Phi=2\kappa\Phi\,, \tag{5}$$

where in general $\kappa\neq 0$.

- The solution of this constraint is

$$\Phi=\left[1+m\left(\bar{\theta}\cdot\theta\right)\right]^{-\kappa}\Phi_L(t_L,\theta),\quad \Phi_L(t_L,\theta)=z+\sqrt{2}\,\theta_i\xi^i+\varepsilon^{ij}\theta_i\theta_jF.$$

- The superfield Φ_L and its components transform as

$$\delta^*\Phi_L=4\kappa m\,(\bar{\epsilon}^j\theta_j)\,\Phi_L\;\Rightarrow$$
$$\delta z=-\sqrt{2}\,\epsilon_i\xi^i,\quad \delta\xi^i=\sqrt{2}\,i\bar{\epsilon}^i\nabla_t z-\sqrt{2}\,\varepsilon^{ik}\epsilon_kF,$$
$$\delta F=-\sqrt{2}\,\varepsilon_{ik}\bar{\epsilon}^k\left[m\xi^i+i\nabla_t\xi^i\right],\quad \nabla_t:=\partial_t+2i\kappa m\,.$$

Invariant Lagrangian

- General superfield Lagrangian is constructed as

$$\mathcal{L}_k = \frac{1}{4}\int d^2\theta\, d^2\bar{\theta}(1 + 2m\,\bar{\theta}\cdot\theta)\, f(\Phi, \Phi^\dagger).$$

- Its component form, after eliminating the auxiliary field by its equation of motion,

$$F = -\frac{1}{2}\varepsilon_{kl}\xi^k\xi^l\frac{g_z}{g}\,,$$

is as follows:

$$\begin{aligned}\mathcal{L} \;=\;& g\dot{\bar{z}}\dot{z} + 2i\kappa m\left(\dot{\bar{z}}z - \dot{z}\bar{z}\right)g - \frac{im}{2}\left(\dot{\bar{z}}f_{\bar{z}} - \dot{z}f_z\right) - \frac{i}{2}\left(\bar{\xi}\cdot\xi\right)\left(\dot{\bar{z}}g_{\bar{z}} - \dot{z}g_z\right)\\ &+\frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i - \dot{\bar{\xi}}_i\xi^i\right)g - m^2V - m\left(\bar{\xi}\cdot\xi\right)U + \frac{1}{2}\left(\bar{\xi}\cdot\xi\right)^2 R\,,\end{aligned}$$

where

$$\begin{aligned} V &= \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)f - \kappa^2\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)^2 f,\\ U &= \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)g - (1 - 2\kappa)\,g,\\ R &= g_{z\bar{z}} - \frac{g_z g_{\bar{z}}}{g}\,.\end{aligned}$$

- It is invariant under the transformations

$$\delta z = -\sqrt{2}\,\epsilon_i \xi^i, \qquad \delta \xi^i = \sqrt{2}\, i\bar{\epsilon}^i \nabla_t z + \sqrt{2}\,\epsilon_k \xi^k \xi^i \frac{g_z}{g}.$$

- Bosonic Lagrangian has the form

$$\mathcal{L} = g\dot{\bar{z}}\dot{z} + 2i\kappa m \left(\dot{\bar{z}}z - \dot{z}\bar{z}\right) g - \frac{im}{2}\left(\dot{\bar{z}}f_{\bar{z}} - \dot{z}f_z\right) - m^2 V\,,$$
$$V = \kappa\left(\bar{z}\partial_{\bar{z}} + z\partial_z\right) f - \kappa^2 \left(\bar{z}\partial_{\bar{z}} + z\partial_z\right)^2 f.$$

- Thus the standard $\mathcal{N} = 4, d = 1$ kinetic term is deformed to non-trivial Lagrangian with WZ-term, and potential term. The latter vanishes for $\kappa = 0$, however, the WZ term vanishes only in the limit $m = 0$.
- So, the basic novel point compared to the standard $\mathcal{N} = 4$ Kähler sigma model for the multiplet $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ is the necessary presence of the WZ term with the strength m, together with the Kähler kinetic term.

Quantum generators

- The Poisson brackets are imposed as:

$$\{z, p_z\}_{PB} = 1, \qquad \{\xi^i, \bar{\xi}_j\}_{PB} = -i\delta^i_j \, g^{-1} .$$

- It is convenient to make the substitution

$$\left(z, \xi^i\right) \longrightarrow \left(z, \eta^i\right), \qquad \eta^i = g^{\frac{1}{2}} \xi^i ,$$

$$\{z, p_z\}_{PB} = 1, \qquad \{\eta^i, \bar{\eta}_j\}_{PB} = -i\delta^i_j , \qquad \{p_z, \eta^i\}_{PB} = \{p_z, \bar{\eta}_j\}_{PB} = 0 .$$

- We quantize in the standard way

$$[\hat{z}, \hat{p}_z] = i , \quad \{\hat{\eta}^i, \hat{\bar{\eta}}_j\} = \delta^i_j , \quad [\hat{p}_z, \hat{\eta}^i] = [\hat{p}_z, \hat{\bar{\eta}}_j] = 0 ,$$

$$\hat{p}_z = -i\partial_z , \qquad \hat{\bar{\eta}}_j = \frac{\partial}{\partial \hat{\eta}^j} .$$

- To construct the quantum supercharges, we resort to the techniques developed by (A. Smilga, 1987). Its basic point is the Weyl-ordering of bosonic and fermionic operators in the classical Noether supercharges: the quantum Hamiltonian is then defined as the anticommutator of these ordered supercharges.
- Using this approach, we obtain

$$\hat{Q}^i = \sqrt{2}\,\hat{\eta}^i\, g^{-\frac{1}{2}}\nabla_z\,, \qquad \hat{\bar{Q}}_j = \sqrt{2}\,\hat{\bar{\eta}}_j\, g^{-\frac{1}{2}}\bar{\nabla}_{\bar{z}}\,,$$

$$\hat{F} = -2\kappa\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \left(2\kappa - \frac{1}{2}\right)\hat{\eta}^k\hat{\bar{\eta}}_k\,, \qquad \hat{I}^i_j = \hat{\eta}^i\hat{\bar{\eta}}_j - \frac{1}{2}\delta^i_j\hat{\eta}^k\hat{\bar{\eta}}_k\,.$$

- These operators satisfy the $su(2|1)$ superalgebra with the following quantum Hamiltonian

$$\hat{H} = \bar{\nabla}_{\bar{z}}\, g^{-1}\nabla_z - 2\kappa m\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) + m\,(1-2\kappa)\,\hat{\eta}^k\hat{\bar{\eta}}_k + \frac{1}{4}\,g^{-2}R\,\varepsilon_{kl}\varepsilon^{nj}\hat{\eta}^k\hat{\eta}^l\hat{\bar{\eta}}_n\hat{\bar{\eta}}_j.$$

- Here,

$$\nabla_z = -i\partial_z - \frac{i}{2}m\partial_z f + \frac{i}{2}g^{-1}\partial_z g\left(\hat{\eta}^k\hat{\bar{\eta}}_k - 1\right),$$

$$\bar{\nabla}_{\bar{z}} = -i\partial_{\bar{z}} + \frac{i}{2}m\partial_{\bar{z}} f + \frac{i}{2}g^{-1}\partial_{\bar{z}} g\left(\hat{\bar{\eta}}_k\hat{\eta}^k - 1\right).$$

Simplified model on a complex plane

- The model on a plane corresponds to the simplest Kähler potential

$$f\left(\Phi,\Phi^\dagger\right)=\Phi\Phi^\dagger \quad \Rightarrow$$

$$\mathcal{L}=\dot{\bar{z}}\dot{z}+im\left(2\kappa-\frac{1}{2}\right)\left(\dot{\bar{z}}z-\dot{z}\bar{z}\right)+\frac{i}{2}\left(\bar{\xi}_i\dot{\xi}^i-\dot{\bar{\xi}}_i\xi^i\right)$$
$$+2\kappa\left(2\kappa-1\right)m^2\bar{z}z+\left(1-2\kappa\right)m\left(\bar{\xi}\cdot\xi\right).$$

The Lagrangian is invariant under the transformations

$$\delta z=-\sqrt{2}\,\epsilon_i\xi^i,\qquad \delta\xi^i=\sqrt{2}\,i\bar{\epsilon}^i\dot{z}-2\sqrt{2}\,\kappa m\bar{\epsilon}^i z. \tag{7}$$

- The quantum Hamiltonian reads

$$\hat{H}=\bar{\nabla}_{\bar{z}}\nabla_z-2\kappa m\left(\hat{z}\partial_z-\hat{\bar{z}}\partial_{\bar{z}}\right)+m\left(1-2\kappa\right)\hat{\eta}^k\hat{\bar{\eta}}_k$$

and forms, together with the quantum operators

$$\hat{Q}^i=\sqrt{2}\,\hat{\eta}^i\nabla_z\,,\qquad \hat{\bar{Q}}_j=\sqrt{2}\,\hat{\bar{\eta}}_j\bar{\nabla}_{\bar{z}}\,,$$

$$\hat{F}=-2\kappa\left(\hat{z}\partial_z-\hat{\bar{z}}\partial_{\bar{z}}\right)-\left(2\kappa-\frac{1}{2}\right)\hat{\eta}^k\hat{\bar{\eta}}_k\,,\qquad \hat{I}^i_j=\hat{\eta}^i\hat{\bar{\eta}}_j-\frac{1}{2}\delta^i_j\hat{\eta}^k\hat{\bar{\eta}}_k\,,$$

the $su(2|1)$ superalgebra, with $\nabla_z=-i\partial_z-\frac{i}{2}m\bar{z}\,,\bar{\nabla}_{\bar{z}}=-i\partial_{\bar{z}}+\frac{i}{2}mz\,,$

$$\left[\nabla_z,\bar{\nabla}_{\bar{z}}\right]=m.$$

Wave functions and spectrum

- We will make use of the fact that there exists an extra $U(1)$ charge generator,

$$\hat{E} = -\left(\hat{z}\partial_z - \hat{\bar{z}}\partial_{\bar{z}}\right) - \hat{\eta}^k \hat{\bar{\eta}}_k \,,$$

which commutes with all $SU(2|1)$ generators, including H.

- Hence we can construct the relevant wave functions in terms of the set of bosonic eigenfunctions of this external generator

$$\Omega^{(\alpha)} = \bar{z}^{\alpha}\, A\,(z\bar{z})\,, \quad \hat{E}\,\Omega^{(\alpha)} = \alpha\,\Omega^{(\alpha)}\,, \tag{8}$$

with α being some positive real number.

- Requiring this set to simultaneously form the full set of the eigenfunctions of the bosonic part of the Hamiltonian (i.e. for the sector with zero fermionic charge) yields

$$\Omega^{(\alpha)} \to \Omega^{(\ell;\alpha)}\,, \quad \hat{H}\,\Omega^{(\ell;\alpha)} = m(\ell + 2\kappa\alpha)\,\Omega^{(\ell;\alpha)}$$
$$\Omega^{(\ell;\alpha)} = \bar{z}^{\alpha} e^{-\frac{mz\bar{z}}{2}} L_{\ell}^{(\alpha)}\,(m\,z\bar{z})\,,$$

where $L_{\ell}^{(\alpha)}$ are Laguerre polynomials and ℓ is Landau level.

- Acting by supercharges on $\Omega^{(\ell;\alpha)}$ and imposing the obvious physical condition,

$$\bar{\eta}_j \, \Omega^{(\ell;\alpha)} = 0 \;\Rightarrow\; \bar{Q}_j \, \Omega^{(\ell;\alpha)} = 0 \,,$$

  we obtain other eigenstates of $\hat{H}$ and $\hat{E}$.

- The full set of eigenfunctions obtained in this way reads:

$$\begin{aligned}
\Psi^{(\ell;\alpha)} &= \left[a^{(\ell;\alpha)} + b_i^{(\ell;\alpha)} \eta^i \, \nabla_z + \frac{1}{2} \, c^{(\ell;\alpha)} \, \varepsilon_{ij} \eta^i \eta^j \, \nabla_z^2 \right] \Omega^{(\ell;\alpha)}, \quad \ell \geq 2, \\
\Psi^{(1;\alpha)} &= a^{(1;\alpha)} \, \Omega^{(1;\alpha)} + b_i^{(1;\alpha)} \eta^i \, \nabla_z \, \Omega^{(1;\alpha)}, \\
\Psi^{(0;\alpha)} &= a^{(0;\alpha)} \, \Omega^{(0;\alpha)},
\end{aligned}$$

  where a, b, c are some numerical coefficients.

- We observe that the ground state ($\ell = 0$) and the first excited states ($\ell = 1$) are special, in the sense that they encompass non-equal numbers of bosonic and fermionic states. Indeed, $Q^i \, \Omega^{(0;\alpha)} = \bar{Q}_i \, \Omega^{(0;\alpha)} = 0$, i.e. $\Omega^{(0;\alpha)}$ is a singlet of $SU(2|1)$ for any α.

- The wave functions for $\ell = 1$ form the fundamental representation of $SU(2|1)$ (one bosonic and two fermionic states), while those for $\ell \geq 2$ form the typical $(2|2)$ representations.

Casimirs

- Casimir operators for the considered model can be expressed through the operators $\hat{H}$ and $\hat{E}$:

$$
\begin{aligned}
m^2 C_2 &= \left(\hat{H} - 2\kappa m\hat{E}\right)\left(\hat{H} - 2\kappa m\hat{E} - m\right),\\
m^3 C_3 &= \left(\hat{H} - 2\kappa m\hat{E}\right)\left(\hat{H} - 2\kappa m\hat{E} - m\right)\left(\hat{H} - 2\kappa m\hat{E} - \frac{m}{2}\right)
\end{aligned}
$$

- For the quantum states they do not depend on the additional parameter κ and in fact have the same form as for the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ model

$$
C_2(\ell) = (\ell - 1)\,\ell, \quad C_3(\ell) = (\ell - 1/2)\,(\ell - 1)\,\ell, \quad \beta(\ell) = (\ell - 1/2)
$$

- Thus they are vanishing for the wave functions with $\ell = 0, 1$, confirming the interpretation of the corresponding representations as atypical, and are non-vanishing on the wave functions with $\ell \geq 2$, implying them to form typical representations of $SU(2|1)$.

Generalized $SU(2|1)$ chirality

- One can choose another $SU(2|1)$ coset as the basic superspace

$$\frac{SU(2|1) \rtimes U(1)_{\text{ext}}}{SU(2) \times U(1)_{\text{ext}}} = \frac{SU(2|1)}{SU(2)} \sim \frac{\{Q^i, \bar{Q}_j, \tilde{H}, I^i_j\}}{\{I^i_j\}}\,.$$

The Hamiltonian is now the full internal $U(1)$ generator $\tilde{H} = H - mF$.

- Since there is no $U(1)$ generator in the stability subgroup, the covariant spinor derivatives $\mathcal{D}_i$ and $\bar{D}^i$ are $U(1)$-inert and one can define generalized chirality condition

$$(\cos\lambda\,\bar{\mathcal{D}}_i - \sin\lambda\,\mathcal{D}_i)\Phi = 0\,, \qquad (9)$$

λ being a new real parameter.

- The components of Φ are now transformed with the manifest t dependence

$$\delta z = -\sqrt{2}\cos\lambda\,(\epsilon\cdot\xi)\,e^{\frac{i}{2}mt} + \sqrt{2}\sin\lambda\,(\bar{\epsilon}\cdot\xi)\,e^{-\frac{i}{2}mt},$$
$$\delta\xi^i = \sqrt{2}\,\bar{\epsilon}^i\,[i\cos\lambda\,\dot{z} - \sin\lambda\,B]\,e^{-\frac{i}{2}mt} - \sqrt{2}\,\epsilon^i\,[i\sin\lambda\,\dot{z} + \cos\lambda\,B]\,e^{\frac{i}{2}mt},$$
$$\delta B = \sqrt{2}\cos\lambda[i(\bar{\epsilon}\cdot\dot{\xi}) + \frac{m}{2}(\bar{\epsilon}\cdot\xi)]e^{-\frac{i}{2}mt} + \sqrt{2}\sin\lambda[i(\epsilon\cdot\dot{\xi}) - \frac{m}{2}(\epsilon\cdot\xi)]e^{\frac{i}{2}mt}$$

- The most general $SU(2|1)$ invariant action of $\varphi^a(t_L,\hat{\theta}),\ a=1,\ldots N$, is specified by an arbitrary Kähler potential $f(\varphi^a,\bar{\varphi}^{\bar{a}})$:

$$S_{\rm kin}=\int dt\,\mathcal{L}_{\rm kin}=\frac{1}{4}\int d\mu\, f(\varphi^a,\bar{\varphi}^{\bar{a}})\,.$$

- After eliminating auxiliary fields, its bosonic core reads

$$\mathcal{L}^{\rm on}_{\rm kin}|=g_{\bar{a}b}\dot{\bar{z}}^{\bar{a}}\dot{z}^b-\frac{i}{2}m\cos 2\lambda(\dot{\bar{z}}^{\bar{a}}f_{\bar{a}}-\dot{z}^a f_a)-\frac{m^2}{4}g^{\bar{a}b}\sin^2 2\lambda\, f_{\bar{a}}f_b\,.$$

It is recognized as the Lagrangian of the Kähler oscillator (S. Bellucci, A. Nersessian, 2003, 2004) extended by a coupling to an external magnetic field.

- The supercharges can easily be constructed, in both the classical and quantum cases. They do not commute with the Hamiltonian $\tilde{H}$, but are still conserved due to their explicit t-dependence

$$\frac{d}{dt}Q^i=\partial_t Q^i+\{Q^i,\tilde{H}\}=0,\qquad \frac{d}{dt}\bar{Q}_j=\partial_t\bar{Q}_j+\{\bar{Q}_j,\tilde{H}\}=0\,.$$

Finally, it will be instructive to give the explicit form of the quantum supercharges and Hamiltonian for the simplest case

$$f(\varphi, \bar{\varphi}) = \varphi\bar{\varphi} = z\bar{z} + \ldots .$$

The classical Dirac brackets are quantized in the standard way,

$$[\hat{z}, \hat{p}_z] = i\,, \quad \{\hat{\eta}^i, \hat{\bar{\eta}}_j\} = \delta^i_j\,, \quad [\hat{p}_z, \hat{\eta}^i] = [\hat{p}_z, \hat{\bar{\eta}}_j] = 0\,, \quad \hat{p}_z = -i\partial_z\,, \;\; \hat{\bar{\eta}}_j = \frac{\partial}{\partial\hat{\eta}^j}\,.$$

The quantum supercharges, after passing to the picture in which they bear no explicit t dependence, are

$$\begin{aligned}
\hat{Q}^i &= \sqrt{2}\,[\cos\lambda\,\eta^i\pi(m) - \sin\lambda\,\bar{\eta}^i\bar{\pi}(-m)]\,,\\
\hat{\bar{Q}}_j &= \sqrt{2}\,[\cos\lambda\,\bar{\eta}_j\bar{\pi}(m) + \sin\lambda\,\eta_j\pi(-m)]\,,\\
\pi(m) &:= p_z - \frac{i}{2}m\bar{z}\,, \qquad \bar{\pi}(m) = p_{\bar{z}} + \frac{i}{2}mz\,. \qquad (10)
\end{aligned}$$

They satisfy the $su(2|1)$ superalgebra with the quantum Hamiltonian

$$\hat{\tilde{H}} = \Big[\bar{\pi}\,(m\cos 2\lambda)\,\pi\,(m\cos 2\lambda) + \frac{m^2}{4}\sin^2 2\lambda\, z\bar{z} + \frac{m}{2}\cos 2\lambda\,\eta^i\bar{\eta}_i\Big].$$

Summary and Outlook

- We considered a new type of $\mathcal{N} = 4$ supersymmetric mechanics which is based on the supergroup $SU(2|1)$. It is a deformation of the standard $\mathcal{N} = 4$ mechanics by a mass parameter m. The SQM models of this type are expected to be related to the rigid supersymmetric models on higher-dimensional curved superspaces.

- We constructed the superfield formalism on two different coset manifolds of $SU(2|1)$ treated as the real and chiral $SU(2|1), d = 1$ superspaces. The corresponding SQM models are based on the off-shell multiplets $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ and $(\mathbf{2}, \mathbf{4}, \mathbf{2})$. We found the existence of two non-equivalent types of the $SU(2|1)$ $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ multiplets.

- The $SU(2|1)$ SQM models reveal surprising features. For the $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ multiplet the kinetic term of the phyisical bosonic field is inevitably accompanied by the generalized oscillator-type mass term with m playing the role of mass. For the $(\mathbf{2}, \mathbf{4}, \mathbf{2})$ models, the kinetic term is accompanied by the $d = 1$ WZ term with the strength $\sim m$ and potential terms $\sim m$.

- In both cases the spaces of the quantum states reveal deviations from the standard rule of equality of the bosonic and fermionic states, in accordance with the existence of atypical $SU(2|1)$ representations.

- *Some further lines of development*:

  (a) Multi-particle extensions: to take a few superfields of one or different types, to construct the relevant off- and on-shell actions, to quantize, to identify the relevant target bosonic geometries (m-deformed?), etc.

  (b) To inquire whether other $\mathcal{N}=4, d=1$ multiplets (e.g. the multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$) have their $SU(2|1)$ counterparts and to construct the corresponding SQM models. In this connection, it would be useful to define some other coset $SU(2|1)$ superspaces. For instance, there exists the coset

$$\frac{\{Q^i, \bar{Q}_j, H, I^i_j, F\}}{\{Q^1, \bar{Q}_2, F, I^1_2, I^1_1 = -I^2_2\}} \sim \{Q^2, \bar{Q}_1, H, I^2_1\}\,, \tag{11}$$

  which is none other than the $SU(2|1)$ analog of the analytic harmonic $\mathcal{N}=4, d=1$ superspace (E.I., O.Lechtenfeld, 2003). The latter superspace is the carrier of the "root" $\mathcal{N}=4, d=1$ multiplet $(\mathbf{4},\mathbf{4},\mathbf{0})$) from which all other $\mathcal{N}=4, d=1$ multiplets can be deduced following the well defined procedure (F.Delduc, E.I., 2006, 2007). Thus the similar root multiplet can be defined in the $SU(2|1)$ case as well.

  (c) To generalize all this to the next in complexity case of the supergroup $SU(2|2)$. It involves 8 supercharges and so can be treated as a deformation of $\mathcal{N}=8, d=1$ supersymmetry (and of $\mathcal{N}=(4,4), d=2$ supersymmetry, in fact).

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THANKS FOR YOUR ATTENTION!