# On Relation Between Deformed Heisenberg Algebra And Finite Dimensional Lie Algebras

Anatol Nowicki

## Plan:

- *Motivation*

- *Three dimensional Lie algebras for positive and even deformation function*

- *Four dimesional Lie algebra (extended Heisenberg algebra) for positive deformation function*

- *Final remarks*

## Motivation

Let us recall some examples of deformed algebras:

(1) two dimensional deformed space-time algebra - (*Majid, Ruegg*)

$$[x_0\,,x]\;=\;i\,\lambda\,x\,,\qquad \lambda \text{ real deformation parameter}$$

(2) "Planck scale" quantum algebra (*Majid*)

$$[x\,,p]\;=\;i\,\left(1-e^{-\lambda\,x}\right)\,,\qquad (\hbar=c=1)$$

(3) deformed Heisenberg algebra with minimal length - (*Kempf*)

$$[x\,,p]\;=\;i\,\left(1+\lambda^2\,p^2\right)$$

(4) deformed two dimensional Borel algebras related with Jordanian twist (*Ogievetsky*)

$$[x\,,p]\;=\;\lambda\,p(p-1)\,,\qquad [\sigma\,,x]\;=\;\lambda\,(1-e^{\sigma})$$

where $\sigma=\ln p$.

comments:

(i) in example (1) we deal with two-dimensional Lie algebra $\mathfrak{aff}(\mathcal{R})$ of the group of all affine transformations of the real line

(ii) other examples are related to infinite-dimensional Lie algebras

remark: *In all presented two dimensional cases we deal with two variables $x\,,p$, a function $f(p)$ and commutation relations in the form*

$$[x\,,p] \;=\; i\,f(p)$$

## Linearization of deformed algebra:

We assume that

- $x\,,p$ and $f(p)$ can be represented as hermitean operators $\mathbf{X}\,,\mathbf{P}\,,\mathbf{F}$ in appropriate Hilbert space

- in "momentum" representation we have the following realizations

$$\mathbf{P} \;=\; p\,, \qquad \mathbf{F} \;=\; f(p)\,, \qquad \mathbf{X} \;=\; i\,f(p)\frac{d}{dp}$$

acting on square integrable functions $\phi(p) \in \mathcal{L}^2(-a\,,a;f)$ $(a \leq \infty)$ with the scalar product

$$\langle\psi\,,\phi\rangle \;=\; \int_{-a}^{a}\frac{dp}{f(p)}\psi^*(p)\phi(p)$$

remarks:

– because of positive definite scalar product we have

$$|\,\phi\,|^2 > \;0 \quad\Longrightarrow\quad f(p)\;>\;0$$

– the hermiticity of $X$ implies

$$\phi(-a) = \phi(a) \quad \text{or} \quad \phi(-a) = -\phi(a)$$

## Commutation relations

$$\begin{aligned} [\mathbf{X}\,,\mathbf{P}] &= i\,\mathbf{F}\,, & [\mathbf{P}\,,\mathbf{F}] &= 0 \\ [\mathbf{X}\,,\mathbf{F}] &= i\,f(p)\,f'(p) \;=\; i\,\mathbf{F}_1\,, & [\mathbf{P}\,,\mathbf{F}_1] &= 0 \end{aligned}$$

In order to get a finite low-dimensional Lie algebra we require that algebra of three operators $\mathbf{X}$, $\mathbf{P}$ and $\mathbf{F}$ is linear and to close this algebra we put

$$[\mathbf{X}\,,\mathbf{F}] \;=\; i\,f\,f' \;=\; i\,(\alpha+\beta p+\gamma f) \;=\; i\,(\alpha+\beta\,\mathbf{P}+\gamma\,\mathbf{F})$$

where $\alpha$, $\beta$ and $\gamma$ are real parameters.

## Three dimensional Lie algebras

We consider the case of deformation function $f(p)$ is positive and even function i.e. $f(-p) = f(p)$ then

$$f(p)f'(p) \;=\; \beta\,p$$

and positive function of deformation

$$f(p) \;=\; \sqrt{1+\beta p^2}$$

Lie algebra generated by $\mathbf{X}\,,\mathbf{P}\,,\mathbf{F}$ satisfy the relations

$$[\mathbf{X}\,,\mathbf{P}] \;=\; i\,\mathbf{F}\,, \quad [\mathbf{X}\,,\mathbf{F}] \;=\; i\,\beta\,\mathbf{P}\,, \quad [\mathbf{P}\,,\mathbf{F}] \;=\; 0$$

with second order Casimir operator

$$C_2 \;=\; \mathbf{P}^2 - \frac{1}{\beta}\,\mathbf{F}^2$$

Let us redefine the generators ($\beta = \pm\lambda^2$)

$$\mathbf{A}_3 \;=\; \frac{1}{\lambda}\,\mathbf{X}\,, \quad \mathbf{P}_\pm \;=\; \mathbf{F}\pm\lambda\,\mathbf{P} \;=\; \mathbf{A}_2\pm\mathbf{A}_1$$

*commutation relations*

$$\begin{aligned}
[\mathbf{A}_3\,,\mathbf{P}_+] &= i\,\mathrm{sign}(\beta)\,(\mathbf{A}_1 + \mathrm{sign}(\beta)\,\mathbf{A}_2)\ ,\\
[\mathbf{A}_3\,,\mathbf{P}_-] &= i\,\mathrm{sign}(\beta)\,(\mathbf{A}_1 - \mathrm{sign}(\beta)\,\mathbf{A}_2)
\end{aligned}$$

In this basis the Casimir operator take the form

$$C_2(\beta) \;=\; \mathbf{P}^2 - \frac{1}{\beta}\,\mathbf{F}^2 = \begin{cases} C_2(\lambda^2) \equiv \frac{1}{\lambda^2}\,\mathbf{P}_+\mathbf{P}_-\,; \quad \beta = \lambda^2 > 0, \\[2ex] C_2(-\lambda^2) \equiv \frac{1}{2\lambda^2}\left(\mathbf{P}_+^2 + \mathbf{P}_-^2\right)\ ; \beta = -\lambda^2 < 0\,. \end{cases}$$

Depending on sign of the deformation parameter $\beta$ we get two types of three-dimensional Lie algebras:

(a) the case $\beta > 0$ the algebra of inhomogeneous hyperbolic rotations of two-dimensional plane $\mathfrak{iso}(1,1)$ satisfying relations

$$[\mathbf{A}_3\,,\mathbf{P}_+] \;=\; i\,\mathbf{P}_+\,, \quad [\mathbf{A}_3\,,\mathbf{P}_-] \;=\; -i\,\mathbf{P}_-\,, \quad [\mathbf{P}_+\,,\mathbf{P}_-] \;=\; 0$$

explicit realization of generators in "momentum representation"

$$\mathbf{A}_3 \;=\; \frac{i}{\lambda}\sqrt{1+\beta\,p^2}\frac{d}{dp}\,, \quad \mathbf{P}_\pm = \sqrt{1+\lambda^2\,p^2} \pm \lambda\,p \;=\; e^{\pm \mathrm{arcsh}(\lambda \mathbf{P})}$$

acting on the square integrable functions $\phi \in \mathcal{L}^2(\mathfrak{R})$ with the scalar product

$$\langle \psi\,,\phi\rangle \;=\; \int_{-\infty}^{\infty} \frac{dp}{\sqrt{1+\lambda^2 p^2}}\,\psi^*(p)\,\phi(p)$$

(b) the case $\beta < 0$ the algebra of inhomogeneous rotations of two-dimensional Euclidean plane $\mathfrak{iso}(2)$ satisfying relations

$$[\mathbf{A}_3\,,\mathbf{P}_+] \;=\; i\,\mathbf{P}_-\,, \quad [\mathbf{A}_3\,,\mathbf{P}_-] \;=\; -i\,\mathbf{P}_+\,, \quad [\mathbf{P}_+\,,\mathbf{P}_-] \;=\; 0$$

explicit realization of generators

$$\mathbf{A}_3 \;=\; \frac{i}{\lambda}\sqrt{1-\beta\,p^2}\frac{d}{dp}\,, \quad \mathbf{P}_\pm = \sqrt{1-\lambda^2\,p^2} \pm \lambda\,p \;=\; e^{\pm \mathrm{arcsh}(\lambda \mathbf{P})}$$

Hermicity condition of generators implies that $-1 \le \lambda\,p \le 1$ and they acting on the square integrable functions $\phi \in \mathcal{L}^2(-1/\lambda\,,1/\lambda)$ with the scalar product

$$\langle \psi\,,\phi\rangle \;=\; \int_{-1/\lambda}^{1/\lambda} \frac{dp}{\sqrt{1-\lambda^2 p^2}}\,\psi^*(p)\,\phi(p)$$

## Four dimensional Lie algebra

We assume that deformation function $f(p) = 1 + w(p)$ is positive then

$$f(p)f'(p) \;=\; (1 + w(p))\, w'(p) \;=\; w(p)$$

and positive function of deformation is given by

$$f(p) \;=\; 1 + W\left(e^{p+a}\right) \;=\; 1 + w(p)$$

with arbitrary real constant $a \in \mathcal{R}$ and function $W(z)$ is the *Lambert function*. The function $W(z)$ is positive and increasing function (concave) on positive semi-axis $z \in \mathcal{R}_+$.

Lie algebra is generated by

$$\mathbf{P} \;=\; p\,, \quad \mathbf{F} \;=\; w(p)\,, \quad \mathbf{X} \;=\; i\,(1 + w(p))\frac{d}{dp}$$

and satisfy the commutation relations

$$[\mathbf{X}\,,\mathbf{P}] \;=\; i\,(\mathbf{I} + \mathbf{F})\,, \quad [\mathbf{X}\,,\mathbf{F}] \;=\; i\,\mathbf{F}\,, \quad [\mathbf{P}\,,\mathbf{F}] \;=\; 0$$

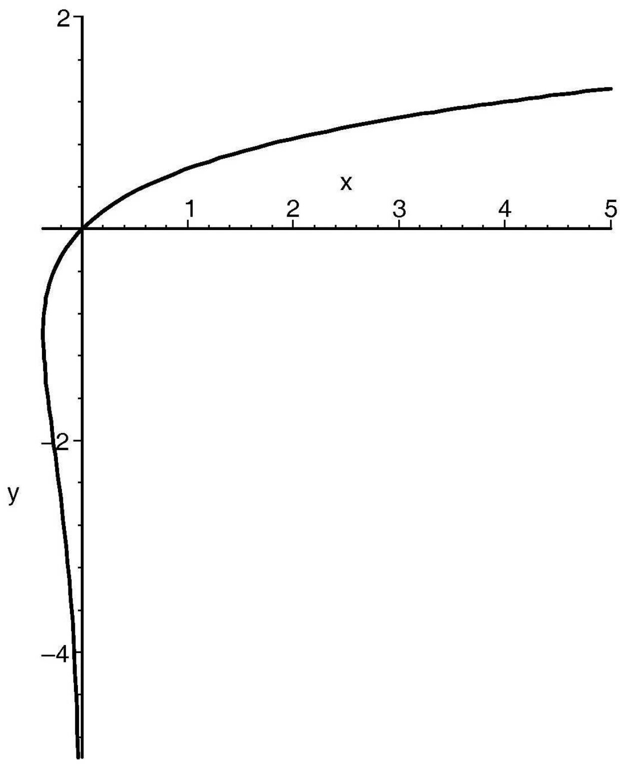
equivalently, introducing generator $\mathbf{\Pi} = \mathbf{P} - \mathbf{F}$

$$[\mathbf{X}\,,\mathbf{\Pi}] \;=\; i\,\mathbf{I}\,, \quad [\mathbf{X}\,,\mathbf{F}] \;=\; i\,\mathbf{F}\,, \quad [\mathbf{\Pi}\,,\mathbf{F}] \;=\; 0$$

remarks:
(i) operators $\mathbf{X}\,,\mathbf{\Pi}$ generate three dimensional Heisenberg algebra $(\mathbf{X}\,,\mathbf{\Pi}\,,Z=\mathbf{I})$

(ii) operators $\mathbf{X}\,,\mathbf{F}$ generate two-dimensional algebra $\mathfrak{aff}(\mathcal{R})$

**some comments on the Lambert function $W(z)$:**




The Lambert function $W(z)$ is one-to-one on positive real axis $0 \leq z < \infty$ and the inverse function of $z \rightarrow ze^{z}$ i.e.

$$W(ze^{z}) = z$$

and the following relations hold

$$W(z)\, e^{W(z)} \;=\; z\,, \quad W'(z) \equiv \frac{dW}{dz} \;=\; \frac{e^{-W(z)}}{1+W(z)}$$

$$W''(z) \;=\; -\left(W'(z)\right)^2 \left(\frac{2+W(z)}{1+W(z)}\right)$$

The last relation tell us that because the function $W(z)$ is non-negative on the positive real line, therefore $W''(z) < 0$ and $W(z)\,,\, z \in \mathcal{R}_+$ is concave.

The first two formulae imply that

$$z\ (1+W(z))\,\frac{dW(z)}{dz} \;=\; W(z)$$

Assuming that $z = e^{p+a}\,,\, a \in \mathcal{R}$ and $w(p) = W(e^{p+a})$ we get

$$(1+w(p))\,\frac{dw(p)}{dp} \;=\; w(p)$$

## explicit differential realization of generators

$$\mathbf{P} \;=\; p\,, \quad \mathbf{F} \;=\; W(e^{p+a})\,, \quad \mathbf{X} \;=\; i\,\left(1+W(e^{p+a})\right)\frac{d}{dp}$$

acting on the square integrable functions $\phi \in \mathcal{L}^2(\mathcal{R})$ with the scalar product

$$\langle \psi\,, \phi \rangle \;=\; \int_{-\infty}^{\infty} \frac{dp}{1+W(e^{p+a})}\,\psi^*(p)\,\phi(p)$$

or changing the variables $p \to \xi$

$$\xi \;=\; p - W\left(e^{p+a}\right)\,, \quad p \;=\; \xi + e^{\xi}$$

we get an equivalent differential realization of generators

$$\mathbf{X} \;=\; i\,\frac{d}{d\xi}\,, \quad \mathbf{P} \;=\; \xi + e^{\xi}\,, \quad \mathbf{F} \;=\; 1 + e^{\xi}$$

## explicit matrix realization of generators

Four dimensional algebra $\mathfrak{H}^{ext}$ (*extended Heisenberg algebra*) generated by $X\,,\Pi\,,Z\,,F$ with non-vanishing commutation relations

$$[\mathbf{X}\,,\mathbf{\Pi}] \;=\; \mathbf{Z}\,, \qquad [\mathbf{X}\,,\mathbf{F}] \;=\; \mathbf{F}$$

can be realized by $5\times 5$ non-hermitian matrices

$$V(x\,,y\,,z\,,v) \;=\; x\,X + \pi\,\Pi + z\,Z + \varphi\,F \;=\; \begin{bmatrix} 0 & x & z & 0 & 0 \\ 0 & 0 & \pi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & \varphi \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $x, \pi, z, \varphi \in \mathcal{R}$ and

$$\left[V(x,\pi,z,\varphi)\,,V(x',\pi',z',\varphi')\right] \;=\; (x\pi' - x'\pi)\,Z + (x\varphi' - x'\varphi)\,F$$

conclusion: algebra $\mathfrak{H}^{ext}$ is *solvable*.

## Final remarks

We conclude that by linearization of deformation function we get the following

low dimensional Lie algebras

| $2-dimensional$ | $3-dimensional$ | $4-dimensional$ |
|---|---|---|
| $[X,P]=P$ | $[X,P]=F$ <br> $[X,F]=P$ | $[X,\Pi]=Z$ <br> $[X,F]=F$ |
| $f(p) \;=\; p$ | $f(p)=\sqrt{1+\beta\,p^2}$ | $f(p)=1+W(e^{p+a})$ |
| $\mathfrak{aff}(\mathcal{R})\sim B(2)$ | $\mathfrak{iso}(2)\sim\mathfrak{so}(2)\ni\mathfrak{t}^2$ <br> $\mathfrak{iso}(1,1)\sim\mathfrak{so}(1,1)\ni\mathfrak{t}^2$ | $\mathfrak{H}^{ext}$ |

where $B(2)$ is two dimensional Borel Lie algebra.

Partially based on the article:

A.N. and V.M. Tkachuk, J. Phys. A **47** (2014) 025207