Semiduality and compatible r-matrices for 3d gravity

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based on work with B. J. Schroers

Outline

Why $3d$?

Riemann-tensor $\frac{D^2(D^2-1)}{12}$ components in D dimensions:

Why 3*d*?

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- 4D: 20 (10 Weyl and 10 Ricci)
- 3D: 6 (Ricci)

Why $3d$?

Einstein equations (without cosmological constant)

$$R_{ab} - \frac{1}{2} R g_{ab} = -8\pi G T_{ab}$$

has a flat solution for $T_{ab} = 0$.

Why 3*d*?

Physically 3*d* spacetime has no local degrees of freedom:

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Physically 3*d* spacetime has no local degrees of freedom:

- No gravitational waves in the classical theory
- No gravitons in the quantum theory

Why $3d$?

- $R_{abcd} = 0 \implies$ any point in the spacetime M has a neighborhood U_i that is isometric to Minkowski space $(V^{2,1}, \eta)$
- U_i can be extended globally and the geometry is trivial

Why 3d?

- $R_{abcd} = 0 \implies$ any point in the spacetime M has a neighborhood U_i that is isometric to Minkowski space $(V^{2,1}, \eta)$

- U_i can be extended globally and the geometry is trivial

- But if M contains non-contractible curves, this extension is nontrivial

Why $3d$?

Thus $3d$ gravity is relatively simple

Why 3*d*?

Thus 3*d* gravity is relatively simple

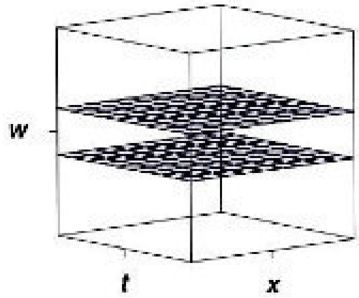
Testing ground for the role of NCG in quantum gravity

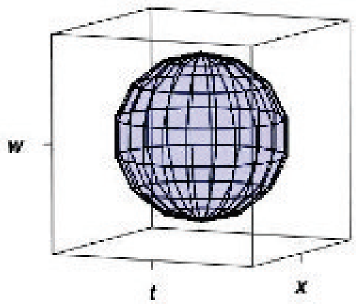
Outline

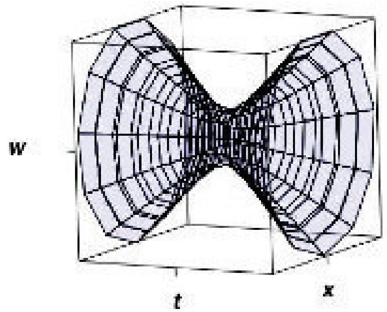
Model spacetimes

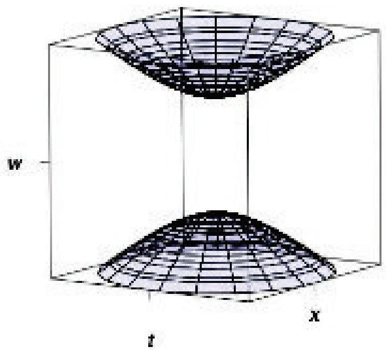
Switching on the cosmological constant Λ

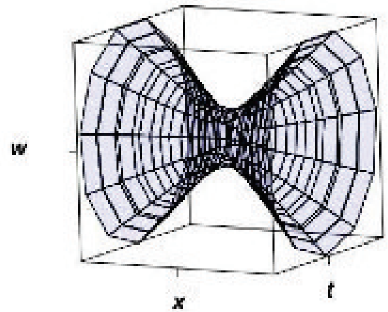
Model spacetimes











Isometry groups of 3d gravity

Isometry groups of the local model spacetimes play a fundamental role in 3d gravity:

Isometry groups of 3d gravity

Isometry groups of the local model spacetimes play a fundamental role in 3d gravity:

- Construction of globally non-trivial solutions of the Einstein equations on a general 3-manifold;
- In the Chern-Simons formulation of 3d gravity, they play the role of gauge groups.

Isometry groups of 3d gravity at a glance

Λ	Euclidean sig.($c^2 < 0$)	Lorentzian sig.($c^2 > 0$)
$\Lambda = 0$	$ISO(3) = SU(2) \ltimes \mathbb{R}^3$	$ISO(2,1) = SU(1,1) \ltimes \mathbb{R}^3$
$\Lambda > 0$	$SO(4) \cong \frac{(SU(2) \times SU(2))}{\mathbb{Z}_2}$	$SO(3,1) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$
$\Lambda < 0$	$SO(3,1) \cong \frac{SL(2,\mathbb{C})}{\mathbb{Z}_2}$	$SO(2,2) \cong \frac{(SL(2,\mathbb{R}) \times SL(2,\mathbb{R}))}{\mathbb{Z}_2}$

Lie algebras local isometry groups

The Lie algebras, denoted by $\mathfrak{g}_\lambda$, are the six-dimensional Lie algebra with generators J_a and P_a, $a = 0, 1, 2$ with Lie brackets

$$[J_a, J_a] = \varepsilon_{abc} J^c, \quad [J_a, P_b] = \varepsilon_{abc} P^c \qquad [P_a, P_b] = \lambda \varepsilon_{abc} J^c.$$

where

$$\lambda = -c^2 \Lambda.$$

Outline

Generalised Chern-Simons action for 3*d* gravity

A CS theory on a 3*d* manifold requires:

- a gauge group
- Ad-invariant, non-degenerate, symmetric bilinear form on the Lie algebra of the gauge group

Generalised Chern-Simons action for $3d$ gravity

Consider a $3d$ spacetime manifold M of topology $\mathbb{R} \times S$.

Generalised Chern-Simons action for $3d$ gravity

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The gauge field is locally a 1-form $A \in \mathfrak{g}_\lambda$

$$A = \omega_a J^a + e_a P^a,$$

where

- $\omega = \omega^a J_a$ is the spin connection on the frame bundle
- the 1-form e_a is a dreibein (provided it is invertible).

Generalised Chern-Simons action for 3d gravity

The curvature of this connection is given by

$$F = dA + \frac{1}{2}[A \wedge A] = R + C + T,$$

which contains

- the Riemann curvature

$$R = d\omega + \frac{1}{2}[\omega \wedge \omega]$$

- a cosmological term

$$C = \frac{\lambda}{2}\epsilon^{abc} e_a \wedge e_b J_c$$

- the torsion

$$T = (de^c + \epsilon^{abc}\omega_a \wedge e_b)P_c.$$

Generalised Chern-Simons action for $3d$ gravity

The CS action for A is then defined by

$$I_{\alpha\beta}(A) = \int_M (A \wedge dA)_{\alpha\beta} + \frac{1}{3}(A \wedge [A, A])_{\alpha\beta}$$

where

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where

$$(J_a, J_b)_{\alpha\beta} = \beta\eta_{ab}, \ (J_a, P_b)_{\alpha\beta} = \alpha\eta_{ab}, \ (P_a, P_b)_{\alpha\beta} = \lambda\beta\eta_{ab}.$$

(E. Witten, C. Meusburger, B. J Schroers)

Generalised Chern-Simons action for $3d$ gravity

After integrating by parts and dropping the boundary term, the action becomes

$$I_\tau(A) = \alpha \int_M \left(2e^a \wedge R_a + \frac{\lambda}{3}\epsilon_{abc} e^a \wedge e^b \wedge e^c \right)$$

$$+\beta \int_M \left(\omega^a \wedge d\omega_a + \frac{1}{3}\epsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c + \lambda e^a \wedge T_a \right),$$

Generalised Chern-Simons action for $3d$ gravity

We identify

$$\alpha = \frac{1}{16\pi G}, \qquad \beta = \text{Immirzi parameter}$$

to see the explicit dependence of the CS action on Λ, c, G and the Immirzi parameter.

Outline

Classical $r-$matrices

For any Lie algebra $\mathfrak{g}$,

- let

$$r = r^{ab} X_a \otimes Y_b \in \mathfrak{g} \otimes \mathfrak{g}$$

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- Set

$$r_{12} = r^{ab} X_a \otimes Y_b \otimes 1$$

$$r_{13} = r^{ab} X_a \otimes 1 \otimes Y_b$$

$$r_{23} = r^{ab} 1 \otimes X_a \otimes Y_b$$

in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$.

Classical $r-$matrices

We define the classical Yang-Baxter map by

$$\mathrm{CYB} : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^{\otimes 3}, \quad r \mapsto [[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}].$$

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The equation

$$[[r, r]] = 0$$

is called the classical Yang-Baxter equation(CYBE).

Classical $r-$matrices

- Any solution of the CYBE in $\mathfrak{g} \otimes \mathfrak{g}$ is called a classical $r-$matrix.
- If
$$[[r, r]] \neq 0$$
but an invariant element of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ then r is said to satisfy the modified classical Yang-Baxter equation(MCYBE).

Relating Chern-Simons action and Hopf algebras

- Classical $r-$matrices provide bridge between a Chern-Simons theory and Hopf algebras.

Relating Chern-Simons action and Hopf algebras

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A classical $r-$matrix is said to be compatible with a CS action if:

Relating Chern-Simons action and Hopf algebras

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A classical $r-$matrix is said to be compatible with a CS action if:

- it satisfies the CYBE
- its symmetric part is equal to the Casimir associated to the $\mathrm{Ad}-$invariant, non-degenerate, symmetric bilinear form used in the CS action.

(V. V. Fock, A. A Rosly)

Relating Chern-Simons action and Hopf algebras

The Casimir associated $(\cdot\,,\,\cdot)_{\alpha\beta}$

$$K_{\alpha\beta} = \frac{1}{\alpha^2 - \lambda\beta^2}(\alpha(P_a \otimes J^a + J^a \otimes P_a) - \beta(P_a \otimes P^a + \lambda J_a \otimes J^a).$$

Quantum picture

- Poisson brackets of the extended phase space is given in terms the classical $r-$matrix.
- The quantisation of such classical systems implies quantum systems whose symmetries are given by Hopf algebras.

Quantum picture

Resulting quantum picture:

Quantum picture

Resulting quantum picture:

- a deformation of the model spacetimes into non-commutative spaces
- a replacement of the local isometry groups with 'quantum isometry groups' (QIGs)

Quantum isometry groups in $3d$ quantum gravity

Λ	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$D(U(\mathfrak{su}(2)))$	$D(U(\mathfrak{su}(1,1)))$
$\Lambda > 0$	$D(U_q(\mathfrak{su}(2)))$, q root of unity	$D(U_q(\mathfrak{su}(1,1)))\ q \in \mathbb{R}$
$\Lambda < 0$	$D(U_q(\mathfrak{su}(2)))$, $q \in \mathbb{R}$	$D(U_q(\mathfrak{sl}(2,\mathbb{R})))$, $q \in U(1)$

$$q = e^{-\frac{\hbar G \sqrt{\Lambda}}{c}},$$

Relating Chern-Simons action and Hopf algebras

Uniqueness of associated $r-$matrices ???

Relating Chern-Simons action and Hopf algebras

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Relating Chern-Simons action and Hopf algebras

Associated quantum groups include a family of

- bicrossproduct quantum groups
- quantum doubles.

(B. Schroers , S Majid, C Meusburger)

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Outline

- Why 3*d* gravity?
- Isometry groups of 3d gravity
- Generalised Chern-Simons action for 3*d* gravity
- Relating Chern-Simons action and Hopf algebras
- Most general $r-$matrix for the general CS action
- Semiduality and compatible $r-$matrices
- Conclusion

Most general $r-$matrix for the general CS action

The trick of generalised complexification

$$
\begin{aligned}
r &= (\mathrm{id} \otimes A + \theta \otimes B - B \otimes \theta + \theta \otimes \theta\ C) J^a \otimes J_a \\
&= J^a \otimes A(J_a) + P^a \otimes B(J_a) - B(J_a) \otimes P^a + P^a \otimes C(P_a) \\
&= A_{ba} J^a \otimes J^b + B_{ba} P^a \otimes J^b - B_{ba} J^b \otimes P^a + C_{ba} P^a \otimes P^b,
\end{aligned}
$$

where

$$P_a = \theta J_a, \quad \theta^2 = -\lambda$$

and

$$A_{ab} = -A_{ba} \quad C_{ab} = -C_{ba}.$$

Most general $r-$matrix for the general CS action

We are trying to solve

$$[[r, r]] = \Omega$$

where Ω is the most general invariant element in $(\mathfrak{g}_\lambda)^3$; in terms of real parameters $\tilde{\alpha}, \tilde{\beta}$:

$$\begin{aligned}\Omega = \tilde{\alpha}\epsilon_{abc}(\lambda J^a \otimes J^b \otimes J^c + J^a \otimes P^b \otimes P^c \\ + P^a \otimes J^b \otimes P^c + P^a \otimes P^b \otimes J^c) \\ + \tilde{\beta}\epsilon_{abc}(P^a \otimes P^b \otimes P^c + \lambda P^a \otimes J^b \otimes J^c \\ + \lambda J^a \otimes J^b \otimes P^c + \lambda J^a \otimes P^b \otimes J^c).\end{aligned}$$

Most general $r-$matrix for the general CS action

Or

$$\Omega = (\tilde{\alpha}(\lambda + \mathrm{id} \otimes \theta \otimes \theta + \theta \otimes \mathrm{id} \otimes \theta + \theta \otimes \theta \otimes \mathrm{id})$$
$$+ \tilde{\beta}(\theta \otimes \theta \otimes \theta + \lambda\theta \otimes \mathrm{id} \otimes \mathrm{id}$$
$$+ \lambda \mathrm{id} \otimes \mathrm{id} \otimes \theta + \lambda \mathrm{id} \otimes \theta \otimes \mathrm{id}))\epsilon_{abc} J^a \otimes J^b \otimes J^c.$$

Conditions on possible solutions

$$\begin{aligned}
\frac{1}{2}(A^2) + \frac{\lambda}{2}\Big((B)^2 - (B^2)\Big) &= -\tilde{\alpha}\lambda, \\
(CB) &= -\tilde{\beta}, \\
(B - (B))(B + B^t) + \frac{1}{2}\Big((B)^2 - (B^2)\Big)\,\mathrm{id} - CA \\
+\lambda(C^2 - \frac{1}{2}(C^2)\mathrm{id}) &= \tilde{\alpha}\,\mathrm{id}, \\
-A(B + B^t) - (B^t - (B))\,A - (AB)\mathrm{id} \\
+\lambda(B^t C - (B)C) &= \lambda\tilde{\beta}\,\mathrm{id}
\end{aligned}$$

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- a given QIG may arise for more than one action.
- Moreover, pairs of QIGs may be related by semiduality.

Semiduality and compatible $r-$matrices

Given

$$K_\alpha = 16\pi G(J_a \otimes P^a + P_a \otimes J^a)$$

Semiduality and compatible $r-$matrices

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$$K_\alpha = 16\pi G(J_a \otimes P^a + P_a \otimes J^a)$$

Cos const $\lambda = -c^2\Lambda$	Sig	r	QIG	Semidual of QIG
$\lambda = 0$	E	r_{D_0}	$U(\mathfrak{so}(4))$	$D\left(U(\mathfrak{su}(2))\right)$
$\lambda = 0$	L	r_{D_0} r_{B_0}	$U(\mathfrak{so}(2,2))$	$D(U(\mathfrak{sl}(2,\mathbb{R})))$ $\mathbb{C}[AN(2)] \blacktriangleright\!\!\triangleleft_s U(\mathfrak{sl}(2,\mathbb{R}))$

$$r_{D_0}^{as} = \frac{1}{2}(P_a \otimes J^a - J^a \otimes P_a)$$

$$r_{B_0}^{as} = \frac{1}{2}\epsilon_{abc} m^a (P^b \otimes J^c + J^b \otimes P^c), \ \mathbf{m}^2 = -1$$

Outline

Conclusion

- Semidualisation provides a way of classifying quantisation ambiguities in 3d gravity

THANK YOU!!!!