Covariant Quantum Mechanics description of particles in κ-Minkowski/κ-Poincaré

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Giovanni Amelino-Camelia, Valerio Astuti, G. R.,
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Introduction

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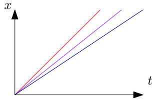
$$\rightarrow \quad v = dE/dp \simeq 1 - \frac{1}{M_p} E$$



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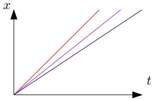

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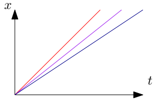

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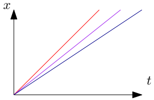

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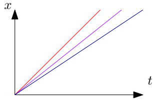

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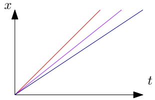

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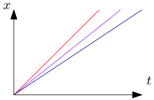

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  e.g. $\Delta t \propto 1/M_p \; \Delta E \; D(z)$ GRBs $\sim 10^{-2}$sec
- Quantum effects: fuzziness supposedly it grows with distance

Ng+van Dam, Lieu+Hillman,...

κ-Poincaré algebra and κ-Minkowski spacetime (1+1D)

Lukierski,Ruegg,Majid,...

κ-Minkowski $\mathcal{X}_\kappa$

$$[x_1, x_0] = \frac{i}{\kappa} x_1$$

$$\Delta(x_\mu) = x_\mu \otimes \mathbb{1} + \mathbb{1} \otimes x_\mu$$

$$S(x_\mu) = -x_\mu \qquad \epsilon(x_\mu) = 0$$

κ-Poincaré Hopf algebra $\mathcal{P}_\kappa$

$$[P_1, P_0] = 0 \qquad [N, P_0] = iP_1$$

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$$T = \mathbb{1} + \mathbf{d} \qquad \mathbf{d} = -ia_\mu^\kappa P^\mu$$

$$\mathbf{d} f(x) g(x) = (\mathbf{d} f(x))\, g(x) + f(x) \mathbf{d} g(x)$$

Leibniz

Covariant Quantum Mechanics

Perspective:
space and time are non-commutative operators acting on some Hilbert space

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No room for time operator in ordinary QM.

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- operators are represented in terms 2 copies of Heisenberg algebra (1+1D)

$$\begin{array}{ll} [\hat{\pi}_0, \hat{q}_0] = i & [\hat{\pi}_0, \hat{q}_1] = 0 \\ [\hat{\pi}_1, \hat{q}_0] = 0 & [\hat{\pi}_1, \hat{q}_1] = -i \end{array}$$

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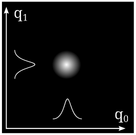
- spacetime points → spacetime regions gaussian states



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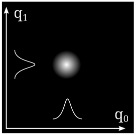

Physical Hilbert space after imposing the Hamiltonian constraint
→ physical degrees of freedom → worldlines

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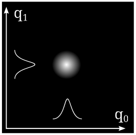

Physical Hilbert space after imposing the Hamiltonian constraint
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representing κ-Minkowski/κ-Poincaré in terms of Kinematical operators we can evaluate their expectation values on the gaussian states.

Kinematical Hilbert space, representations

$$[\hat{\pi}_0, \hat{q}_0] = i \quad [\hat{\pi}_0, \hat{q}_1] = 0$$
$$[\hat{\pi}_1, \hat{q}_0] = 0 \quad [\hat{\pi}_1, \hat{q}_1] = -i$$

$$\hat{x}_0 = \hat{q}_0$$
$$\hat{x}_1 = \hat{q}_1 e^{\frac{\hat{\pi}_0}{\kappa}}$$

$$P_0 \triangleright f(\hat{x}_0, \hat{x}_1) \longleftrightarrow [\hat{\pi}_0, f(\hat{q}_0, \hat{q}_1 e^{\frac{\hat{\pi}_0}{\kappa}})]$$
$$P_1 \triangleright f(\hat{x}_0, \hat{x}_1) \longleftrightarrow e^{-\frac{\hat{\pi}_0}{\kappa}} [\hat{\pi}_1, f(\hat{q}_0, \hat{q}_1 e^{\frac{\hat{\pi}_0}{\kappa}})]$$

$$\hat{a}_0^\kappa = a_0$$
$$\hat{a}_1^\kappa = a_1 e^{\frac{\hat{\pi}_0}{\kappa}}$$

$$N \triangleright f(\hat{x}) \equiv e^{-\frac{\hat{\pi}_0}{\kappa}} [\hat{\eta}, f(\hat{x})] \qquad \hat{\eta} \equiv \left(\kappa \frac{e^{2\frac{\hat{\pi}_0}{\kappa}} - 1}{2} + \frac{\hat{\pi}_1^2}{2\kappa} \right) \hat{q}_1 - \hat{\pi}_1 \hat{q}_0$$

Kinematical Hilbert space, fuzzy points

Scalar product $\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle = \int \mathcal{D}(\pi_\mu) \psi^\star(\pi_\mu) O(\pi_\mu) \psi(\pi_\mu)$

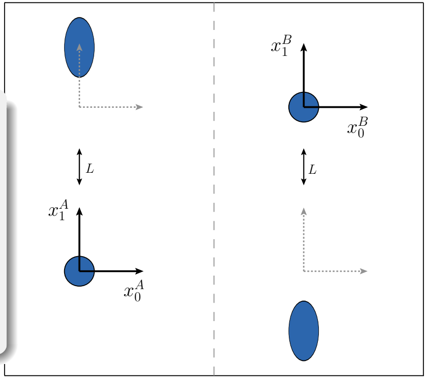
Invariant measure $\mathcal{D}(\pi_\mu) = d\pi_0 d\pi_1 e^{-\frac{\pi_0}{\kappa}}$

$$\langle T \triangleright \hat{x}_0 \rangle = \overline{q}_0 - a_0 - \frac{i}{2\kappa}$$

$$\delta \left(T \triangleright \hat{x}_0 \right) = \frac{1}{2\sigma_0}$$

$$\langle T \triangleright \hat{x}_1 \rangle = \left(\overline{q}_1 - a_1 \right) e^{\frac{\overline{\pi}_0}{\kappa}} e^{-\frac{\sigma_0^2}{2\kappa^2}}$$

$$\delta(T \triangleright \hat{x}_1) = e^{\frac{\overline{\pi}_0}{\kappa}} \left[\frac{1}{4\sigma_1^2} + (\overline{q}_1 - a_1)^2 \left(1 - e^{-\frac{\sigma_0^2}{\kappa^2}} \right) \right]^{1/2}$$



Alice Bob

Physical Hilbert space

Hamiltonian constraint $$\mathcal{H} = (2\kappa)^2 \sinh^2\left(\frac{\pi_0}{2\kappa}\right) - e^{-\frac{\pi_0}{\kappa}} \pi_1^2$$

$$\langle\psi|\phi\rangle_{\mathcal{H}} = \int e^{-\frac{\pi_0}{\kappa}} d\pi_1 d\pi_0 \delta\left(\mathcal{H}\right) \Theta(\pi_0) \psi^*(\pi) \phi(\pi)$$

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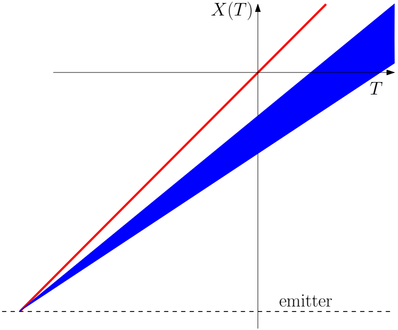
Localization operator

$$\hat{X}(T) = e^{\frac{\hat{\pi}_0}{\kappa}} \left(\hat{q}_1 - \hat{\mathcal{V}}\hat{q}_0 - \frac{1}{2}[\hat{q}_0, \hat{\mathcal{V}}] + \hat{\mathcal{V}}T\right)$$

$$\mathcal{V} \equiv \frac{\partial\mathcal{H}/\partial\pi^1}{\partial\mathcal{H}/\partial\pi^0} \qquad \left[\hat{\mathcal{H}}, \hat{X}\right] = 0 \qquad \hat{X}^\dagger = \hat{X}$$

$$\hat{\mathcal{V}} \equiv \frac{e^{-\frac{\pi_0}{\kappa}} \pi_1}{\kappa \sinh\left(\ell\pi_0\right) + \frac{1}{2\kappa} e^{-\frac{\pi_0}{\kappa}} \pi_1^2} \quad \xrightarrow[m=0]{\mathcal{H}=0} \quad 1 - \frac{E}{\kappa}$$

Physical Hilbert space, fuzzy worldlines



$$\langle \hat{\mathcal{V}} \rangle_{\mathcal{H}} \simeq 1 - \frac{1}{\kappa} \langle E \rangle$$

$$\delta X_{[\kappa]} \approx \left(\frac{\sigma^2 \langle E \rangle}{2\kappa} + \frac{D^2}{\kappa^2 \sigma^2} \right)^{1/2} \sim \frac{D}{\kappa \sigma}$$

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