A new class of integrable 4D systems related to contact geometry

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Integrable systems

- nonlinear
- tractable:
  - infinitely many explicit exact solutions
  - rich symmetry algebras
  - infinitely many conservation laws
- arise as compatibility conditions of overdetermined linear systems (Lax pairs)

Integrable systems in 3D and dispersionless limit

2D integrable system often can be promoted to 3D via central extension:

$$L_1\psi=\lambda\psi,\quad \frac{\partial\psi}{\partial t}=L_2\psi \overset{\lambda\to\partial_y}{\longrightarrow} \frac{\partial\psi}{\partial y}=L_1\psi,\quad \frac{\partial\psi}{\partial t}=L_2\psi,$$

where $L_i=\sum\limits_{j=0}^{n_i} a_{ij}(x,y,t)\partial_x^j$, $\psi=\psi(x,y,t)$.

Dispersionless (quasiclassical) limit: $x':=\hbar x$, $y':=\hbar y$, $t':=\hbar t$, $\psi:=\exp(iS/\hbar)$; $\hbar\to 0$ and omitting primes yields a nonlinear Lax pair of a general form

$$S_y=F(S_x,\boldsymbol{u}),\quad S_t=G(S_x,\boldsymbol{u}),$$

where $\boldsymbol{u}=\boldsymbol{u}(x,y,t)$ is the vector of unknown functions.

Example: the Kadomtsev–Petviashvili equation

The system

$$u_t + 6uu_x + u_{xxx} + 3v_y = 0,$$
$$u_y - v_x = 0$$

is equivalent to the compatibility condition for the Lax pair

$$\begin{aligned} \psi_y &= -i(\partial_x^2 + u)\psi, \\ \psi_t &= (-4\partial_x^3 - 6u\partial_x - 3u_x + 3iv)\psi. \end{aligned} \tag{1}$$

The KP equation proper arises upon eliminating v:

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0.$$

It is a (2+1)-dimensional generalization of the celebrated KdV equation $u_t + 6uu_x + u_{xxx} = 0$.

$x' := \hbar x$, $y' := \hbar y$, $t' := \hbar t$, $\psi := \exp(iS/\hbar)$ for (1), yields, after $\hbar \to 0$ and omitting the primes,

$$S_y = -S_x^2 + u, \quad S_t = 4S_x^3 - 6uS_x + 3v. \tag{2}$$

The compatibility condition for (2) is the dispersionless KP (dKP) equation

$$u_y = v_x, \quad u_t = 3v_y - 6uu_x, \tag{3}$$

which also has a linear nonisospectral Lax pair

$$\chi_y = \mathcal{X}_f(\chi), \quad \chi_y = \mathcal{X}_g(\chi),$$

where $\mathcal{X}_h = h_p\partial_x - h_x\partial_p$ is the Hamiltonian vector field with the Hamiltonian h, $f = -p^2 + u$, $g = 4p^3 - 6up + 3v$.

- (anti-)self-dual Yang–Mills equations
- (anti-)self-dual vacuum Einstein equations (the Plebański equation)
- equations for (anti-)self-dual conformal structure in 4D

A possible reason for scarcity of known 4D integrable systems: applying the central extension twice yields nothing interesting.

Main common feature of the above examples:

the systems in question are *dispersionless*.

Dispersionless systems: general definition

A *dispersionless system* in d independent variables $x^1, \ldots, x^d$ and N dependent variables $u^1, \ldots, u^N$ is a first-order homogeneous quasilinear system

$$A_1(\boldsymbol{u})\boldsymbol{u}_{x^1} + A_1(\boldsymbol{u})\boldsymbol{u}_{x^2} + \cdots + A_d(\boldsymbol{u})\boldsymbol{u}_{x^d} = 0, \qquad (4)$$

where A_i are $M \times N$ matrices, $M \geqslant N$, $\boldsymbol{u} \equiv (u^1, \ldots, u^N)^T$.

Our goal:

to present a systematic construction of 4D integrable dispersionless systems using contact geometry.

A dispersionless class of quasilinear second-order equations

A quasilinear second-order equation of the form

$$\sum_{i=1}^{d}\sum_{j=i}^{d} f_{ij}(v_{x^1},\dots,v_{x^d})v_{x^i x^j}=0$$

can be rewritten in the form (4) if $N := d$ and $u^j := v_{x^j}$:

$$\begin{cases} \sum_{i=1}^{d}\sum_{j=i}^{d} f_{ij}(u^1,\dots,u^d)u^i_{x^j}=0, \\ (u^i)_{x^j}=(u^j)_{x^i}, \quad i=1,\dots,d,\ j=i+1,\dots,d. \end{cases}$$

In this case $M > N$ for $d \geqslant 3$.

The self-dual Yang–Mills equations on a matrix Lie group

$$(J_{y^+}J^{-1})_{y^-} + (J_{z^+}J^{-1})_{z^-} = 0,$$

which can be rewritten in dispersionless form as

$$J_{z^+}J^{-1} - W_{y^-} = 0, \quad J_{y^+}J^{-1} + W_{z^-} = 0;$$

$$A_{y^+} = A_{z^+} = 0, \quad A_{y^-} = J_{y^-}J^{-1}, \quad A_{z^-} = J_{z^-}J^{-1}.$$

Here $y^\pm = \frac{1}{\sqrt{2}}(x^1 \pm \mathrm{i}x^2)$, $z^\pm = \frac{1}{\sqrt{2}}(x^0 \pm \mathrm{i}x^3)$,

where x^i are Euclidean coordinates and $\mathrm{i} = \sqrt{-1}$.

The matrix Lax pair reads

$$(\partial_{y^+} + \lambda(\partial_{z^-} - A_{z^-}))\psi = 0, \quad (\partial_{z^+} - \lambda(\partial_{y^-} - A_{y^-}))\psi = 0.$$

Dispersionless integrable systems in 2D

- *C*-integrable (linearizable) by the generalized hodograph method using symmetries (S.P. Tsarev) later extended to the translation-noninvariant case (M. Grundland, M. Sheftel, P. Winternitz)

- *S*-integrable:
  - * nonlinear Lax pairs on Poisson algebras (M. Golenishcheva-Kutuzova and A. Reyman)
  - * R-matrix approach for Poisson algebras (L.C. Li; M. Błaszak and B. Szablikowski (including central extension to 3D))

S-integrable 3D dispersionless integrable systems

- nonlinear Lax pairs using Hamilton–Jacobi equations (J. Gibbons, Y. Kodama, I.M. Krichever, V.E. Zakharov)
- classification results using hydrodynamic reductions (E.V. Ferapontov, K.R. Khusnutdinova, V.S. Novikov, A.V. Odesskii, M.V. Pavlov, Z.Popowicz, V.V. Sokolov, N. Stoilov, S.P. Tsarev et al.)
- exact solutions & related structures (incl. τ-functions etc. à la R. Hirota & E. Date, M. Jimbo, T. Miwa et al.) (L.M. Alonso, L. Bogdanov, J.-H. Chang, M. Dunajski, J. Gibbons, P. Grinevich, Y. Kodama, B. Konopelchenko, I.M. Krichever, S.V. Manakov, M. Mañas, E. Previato, P.M. Santini, K. Takasaki, T. Takebe, L.-P. Teo, D. Wu, V.E. Zakharov et al.)

Integrable systems: 3D vs 4D

	Dispersive	Dispersionless
3D	systematic construction (central extension)	systematic construction (Hamiltonian vect. fields; central extension)
4D	exceptional	exceptional

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Integrable systems: 3D vs 4D

	Dispersive	**Dispersionless**
3D	systematic construction (central extension)	systematic construction (Hamiltonian vect. fields; central extension)
4D	exceptional	systematic construction (contact vect. fields)

Contact Lax pairs for new 4D integrable systems

Let $\boldsymbol{u} = \boldsymbol{u}(x, y, z, t)$, $\boldsymbol{u} = (u^1, \ldots, u^N)^T$, and

$$X_h \stackrel{\mathrm{def}}{=} h_p \partial_x + (p h_z - h_x)\partial_p + (h - p h_p)\partial_z$$

be a *contact vector field* with the Hamiltonian $h = h(p, \boldsymbol{u})$;

p is the *variable spectral parameter*; $\boldsymbol{u}_p \equiv 0$.

Claim

The compatibility condition $(\chi_y)_t = (\chi_t)_y$, *where* $\chi = \chi(x, y, z, t, p)$, *for the* contact Lax pair

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi) \tag{5}$$

yields, for suitable $f(p, \boldsymbol{u})$ *and* $g(p, \boldsymbol{u})$, *broad new classes of 4D dispersionless integrable systems.*

$$(\chi_y)_t = (\chi_t)_y$$
$$\Updownarrow$$
$$f_t - g_y + \{f, g\} = 0, \qquad (6)$$

where $\{,\}$ is the *contact bracket*

$$\{f, g\} \stackrel{\rm df}{=} f_p g_x - g_p f_x - p\,(f_p g_z - g_p f_z) + f g_z - g f_z.$$

$$(6) \Leftrightarrow \sum_{j=1}^{N} \Big[f_{u^j} u^j_t - g_{u^j} u^j_y + (f_p g_{u^j} - g_p f_{u^j}) u^j_x$$
$$+ ((f - p f_p) g_{u^j} - (g - p g_p) f_{u^j}) u^j_z \Big] = 0,$$
$$\Updownarrow$$
$$A_0(\boldsymbol{u})\boldsymbol{u}_t + A_1(\boldsymbol{u})\boldsymbol{u}_x + A_2(\boldsymbol{u})\boldsymbol{u}_y + A_3(\boldsymbol{u})\boldsymbol{u}_z = 0, \qquad (7)$$

where A_i are $M \times N$ matrices, $M \geqslant N$.

Conservation laws from the contact Lax pairs

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi)$$
$$\Downarrow$$
$$(X_f(\chi))_t = (X_g(\chi))_y. \tag{8}$$
$\Downarrow$ (formal expansion of χ in p)

an infinite hierarchy of (nonlocal) conservation laws
for the associated integrable system

Contact Lax pairs: nonisospectrality

The contact Lax pair $\chi_y = X_f(\chi)$, $\chi_t = X_g(\chi)$ involves χ_p, so it belongs to the class of *nonisospectral* (cf. Burtsev Zakharov Mikhailov 1987 etc.) Lax pairs written in terms of vector fields,

$$\begin{aligned} \chi_y &= K_1(p, \boldsymbol{u})\chi_x + K_2(p, \boldsymbol{u})\chi_z + K_3(p, \boldsymbol{u})\chi_p, \\ \chi_t &= L_1(p, \boldsymbol{u})\chi_x + L_2(p, \boldsymbol{u})\chi_z + L_3(p, \boldsymbol{u})\chi_p, \end{aligned}$$

and thus it is amenable to an appropriate version of the inverse scattering transform (cf. Manakov & Santini 2014 etc.) making it possible to construct exact solutions for the associated integrable systems.

Reduction to the 3D case

$\boldsymbol{u}_z = 0,\ \chi_z = 0 \Rightarrow \chi_y = X_f(\chi),\ \chi_t = X_g(\chi)$ becomes

$$\chi_y = \mathcal{X}_f(\chi), \quad \chi_t = \mathcal{X}_g(\chi), \tag{9}$$

where $\mathcal{X}_h \stackrel{\text{def}}{=} h_p\partial_x - h_x\partial_p$ is a *Hamiltonian vector field* with the Hamiltonian $h(p, \boldsymbol{u})$.

The *nonlinear Lax pair*

$$S_y = f(S_x, \boldsymbol{u}), \quad S_t = g(S_x, \boldsymbol{u}) \tag{10}$$

implies (9) and plays an important role in the study of associated integrable systems.

Is there an analog of (10) for (5)?

The contact Lax pair (5), i.e.,

$$\chi_y = X_f(\chi), \quad \chi_t = X_g(\chi),$$

is compatible if and only if so is a nonlinear Lax pair

$$S_y = S_z f(S_x/S_z, \boldsymbol{u}), \quad S_t = S_z g(S_x/S_z, \boldsymbol{u}), \qquad (11)$$

where $S = S(x, y, z, t)$.

This is established using the identification $p = S_x/S_z$.

If $\boldsymbol{u}_z = 0$ and $S_z = 1$ then we recover (10), i.e.,

$$S_y = f(S_x, \boldsymbol{u}), \quad S_t = g(S_x, \boldsymbol{u}).$$

A simple example with discrete symmetry

Let $\boldsymbol{u} = (u, v, w, r)^T$; $f = u + vp^{-1}$ and $g = wp + rp^2$ involve disjoint sets of dependent variables, u, v and w, r. The nonlinear Lax pair

$$S_y = uS_z^2/S_x + vS_z, \quad S_t = wS_x + rS_x^2/S_z \tag{12}$$

is invariant under the simultaneous swap:

$$y \leftrightarrow t,\ x \leftrightarrow z,\ u \leftrightarrow r,\ v \leftrightarrow w.$$

$(S_y)_t = (S_t)_y \Rightarrow$ a system with the same swap symmetry:

$$\begin{aligned} u_t &= 2rv_x - 2vw_z + vr_x + wu_x, \quad & v_t &= vw_x + wv_x,\\ w_y &= 2vr_z + uw_z - 2ru_x + rv_z, \quad & r_y &= ur_z + ru_z. \end{aligned} \tag{13}$$

Potential form of system (13)

$u := b_y/b_z,\ v := a_x,\ w := a_t/a_x,\ r := b_z$: we used the potentials for the conservation laws $v_t = (vw)_x$, $r_y = (ur)_z$

$\Rightarrow$ the system invariant under the simultaneous swap

$$y \leftrightarrow t,\ x \leftrightarrow z,\ a \leftrightarrow b:$$

$$a_{yt} = 2a_x^2 b_{zz} - 2a_x b_{xy} + \frac{a_t}{a_x} a_{xy} + \frac{b_y}{b_z} a_{zt} + \frac{2a_x b_y}{b_z} b_{xz} + \frac{(a_x^2 b_z^2 - a_t b_y)}{a_x b_z} a_{xz},$$

$$b_{yt} = 2b_z^2 a_{xx} - 2b_z a_{zt} + \frac{b_y}{b_z} b_{zt} + \frac{a_t}{a_x} b_{xy} + \frac{2a_t b_z}{a_x} a_{xz} + \frac{(a_x^2 b_z^2 - a_t b_y)}{a_x b_z} b_{xz}.$$

Contact Lax pair

$$\chi_y = \left(pu_z - u_x + v_z - \frac{v_x}{p}\right)\chi_p - \frac{v}{p^2}\chi_x + \left(u + \frac{2v}{p}\right)\chi_z,$$

$$\chi_t = p\left(r_z p^2 + (w_z - r_x)p + w_x\right)\chi_p + (2pr + w)\chi_x - rp^2\chi_z.$$

Substituting there a formal expansion $\chi = \sum_{i=0}^{\infty} \chi_i p^i$ yields

$$\begin{aligned}(\chi_j)_x &= v^{-1}\big((j-2)u_z\chi_{j-2} - (j-1)(u_x - v_z)\chi_{j-1} - jv_x\chi_j \\ &\quad -(\chi_{j-2})_y + u(\chi_{j-2})_z + 2v(\chi_{j-1})_z\big),\\ (\chi_j)_t &= (j-2)r_z\chi_{j-2} + (j-1)(w_z - r_x)\chi_{j-1} + jw_x\chi_j \\ &\quad +2r(\chi_{j-1})_x + w(\chi_j)_x - r(\chi_{j-2})_z,\end{aligned}$$

where $j = 0, 1, 2, \ldots,$ and $\chi_j \stackrel{\text{def}}{=} 0$ for $j < 0$.

Conservation laws

We have $\chi_0 = \chi_0(y, z)$, and the compatibility conditions

$$((\chi_j)_x)_t = ((\chi_j)_t)_x, \quad j = 1, 2, \dots,$$

yield an infinite hierarchy of nonlocal conservation laws

$$D_t \left(v^{-1} \left((j-2)u_z\chi_{j-2} - (j-1)(u_x - v_z)\chi_{j-1} - jv_x\chi_j \right.\right.$$
$$\left.\left. -(\chi_{j-2})_y + u(\chi_{j-2})_z + 2v(\chi_{j-1})_z\right)\right)$$
$$= D_x \left((j-2)s_z\chi_{j-2} + (j-1)(w_z - r_x)\chi_{j-1} + jw_x\chi_j \right.$$
$$\left. +2r(\chi_{j-1})_x + w(\chi_j)_x - r(\chi_{j-2})_z\right), \; j = 1, 2, \dots$$

for our system

$$u_t = 2rv_x - 2vw_z + vr_x + wu_x, \;\; v_t = vw_x + wv_x,$$
$$w_y = 2vr_z + uw_z - 2ru_x + rv_z, \;\; r_y = ur_z + ru_z.$$

f and g polynomial in p: nonlinear Lax pair

Let m and n be natural numbers, $N = m + n + 1$, $\boldsymbol{u} = (u_0, \ldots, u_n, v_0, \ldots, v_{m-1})^T$;

$$f = p^{n+1} + \sum_{i=0}^{n} u_i p^i, \quad g = p^{m+1} + \frac{m}{n} u_n p^m + \sum_{j=0}^{m-1} v_j p^j$$

involve almost disjoint sets of dependent variables (cf. e.g. M. Pavlov & Z. Popowicz 2009 for similar 3D examples);

$$S_y = S_z \left(\left(\frac{S_x}{S_z} \right)^{n+1} + \sum_{i=0}^{n} u_i \left(\frac{S_x}{S_z} \right)^i \right),$$

$$S_t = S_z \left(\left(\frac{S_x}{S_z} \right)^{m+1} + \frac{m}{n} u_n \left(\frac{S_x}{S_z} \right)^m + \sum_{j=0}^{m-1} v_j \left(\frac{S_x}{S_z} \right)^j \right).$$

f and g polynomial in p: the system

$$(u_k)_t - (v_k)_y + m\,(u_{k-m-1})_z - n\,(v_{k-n-1})_z$$
$$+(n+1)\,(v_{k-n})_x - (m+1)\,(u_{k-m})_x$$
$$+\sum_{i=\max(0,k-m)}^{\min(n,k)} ((k-i-1)v_{k-i}\,(u_i)_z - (i-1)u_i\,(v_{k-i})_z)$$
$$-\sum_{i=\max(0,k+1-m)}^{\min(n,k+1)} ((k+1-i)v_{k+1-i}\,(u_i)_x - iu_i\,(v_{k+1-i})_x) = 0.$$

Here $k = 0, \dots, n+m$, $u_i \stackrel{\text{def}}{=} 0$ for $i > n$ and $i < 0$, $v_j \stackrel{\text{def}}{=} 0$ for $j > m$ and $j < 0$; $v_m \stackrel{\text{def}}{=} (m/n)u_n$.

This system is evolutionary: it can be solved for the z-derivatives $(u_i)_z$ and $(v_j)_z$ for all i and j.

Let $m = 2$, $n = 1$, $u_0 = u$, $u_1 = w$, $v_0 = v$, $v_1 = r$, so

$$f = p^2 + wp + u, \quad g = p^3 + 2wp^2 + rp + v,$$

which yields a four-component system

$$\begin{aligned}
&u_t - vu_z - ru_x + uv_z + wv_x - v_y = 0,\\
&2u_z + w_x + 2ww_z - r_z = 0,\\
&2r_x - 3u_x - 2w_y + 2wu_z - v_z - 2ww_x + 2uw_z = 0,\\
&w_t - r_y + 2v_x - 4wu_x + wr_x - rw_x - vw_z + ur_z = 0
\end{aligned}$$

that can be solved w.r.t. u_z, v_z, w_z, r_z (cf. above).

Claim

The above system is a 4D integrable generalization of the dispersionless Kadomtsev–Petviashvili (dKP) equation.

Reduction to the 3D dKP equation

u, v, w, r are independent of z, $w := 0$ and $r := 3u/2$:

$$4u_t - 4v_y - 6uu_x = 0, \quad 4v_x - 3u_y = 0.$$

Eliminating v yields, up to a rescaling, the dKP equation

$$(4u_t - 6uu_x)_x - 3u_{yy} = 0,$$

a.k.a. the 3D Khokhlov–Zabolotskaya equation, which has applications in fluid dynamics and nonlinear acoustics and is the *dispersionless limit* of the celebrated KP equation

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0.$$

f and g rational in p: nonlinear Lax pair

Let m and n be arbitrary natural numbers, $N = 2(m+n)$, $\boldsymbol{u} = (a_1, \ldots, a_m, u_1, \ldots, u_m, b_1, \ldots, b_n, v_1, \ldots, v_n)^T$,

$$f = \sum_{i=1}^{m} \frac{a_i}{(p - u_i)}, \quad g = \sum_{j=1}^{n} \frac{b_j}{(p - v_j)}.$$

The nonlinear Lax pair reads

$$S_y = \sum_{i=1}^{m} \frac{a_i S_z^2}{(S_x - u_i S_z)}, \quad S_t = \sum_{j=1}^{n} \frac{b_j S_z^2}{(S_x - v_j S_z)}.$$

The condition $(S_y)_t = (S_t)_y$ then yields a complicated non-evolutionary system, which reduces to a 3D system found by V.E. Zakharov in 1994 if $\boldsymbol{u}_z = 0$.

f and g rational in p: the system

$$(u_i)_t + \sum_{j=1}^{n} \left\{ \left(\frac{b_j}{v_j - u_i} \right)_x - \left(\frac{b_j u_i}{v_j - u_i} \right)_z - \frac{2b_j (u_i)_z}{v_j - u_i} \right\} = 0, \quad i = 1, \ldots, m,$$

$$(v_j)_y + \sum_{i=1}^{m} \left\{ - \left(\frac{a_i}{v_j - u_i} \right)_x + \left(\frac{a_i v_j}{v_j - u_i} \right)_z + \frac{2a_i (v_j)_z}{v_j - u_i} \right\} = 0, \quad j = 1, \ldots, n,$$

$$(a_i)_t + \sum_{j=1}^{n} \left\{ \left(\frac{a_i b_j}{(v_j - u_i)^2} \right)_x + \left(\frac{a_i b_j (v_j - 2u_i)}{(v_j - u_i)^2} \right)_z + \frac{3a_i (b_j)_z}{v_j - u_i} + \frac{3a_i b_j (v_j)_z}{(v_j - u_i)^2} \right\} = 0, \quad i = 1, \ldots, m,$$

$$(b_j)_y + \sum_{i=1}^{m} \left\{ \left(\frac{a_i b_j}{(v_j - u_i)^2} \right)_x + \left(\frac{a_i b_j (v_j - 2u_i)}{(v_j - u_i)^2} \right)_z + \frac{3a_i (b_j)_z}{v_j - u_i} + \frac{3a_i b_j (v_j)_z}{(v_j - u_i)^2} \right\} = 0, \quad j = 1, \ldots, n.$$

Open problems

- How, and under which conditions, can one construct dispersive deformations of our systems?
- How to discretize the systems in question and their Lax pairs while preserving integrability?
- Is it possible to construct integrable supersymmetric extensions for the systems under study?
- Can we find integrable generalizations of our systems to noncommutative spacetime?
- Is it possible to quantize the systems in question while preserving integrability?

Conclusions

- We have constructed in a systematic fashion new 4D integrable dispersionless systems using contact Lax pairs
- These new integrable systems can be solved using the inverse scattering transform or other methods
- Integrable 4D dispersionless systems are rare but not exceptional

For details please see
arXiv:1401.2122

Conclusions

- We have constructed in a systematic fashion new 4D integrable dispersionless systems using contact Lax pairs
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Dziękuję za uwagę

Appearances can be deceiving

Passing to a new dependent variable $z = z(x, y, t, S)$ and making S an independent variable in the system

$$S_y = S_z F(S_x/S_z, \boldsymbol{u}), \quad S_t = S_z G(S_x/S_z, \boldsymbol{u}), \qquad (*)$$

turns this system into

$$z_y = -F(-z_x, \boldsymbol{u}), \quad z_t = -G(-z_x, \boldsymbol{u})$$

so it may appear that we are back in the class of the 3D systems

$$S_x = K(S_x, \boldsymbol{u}), \quad S_t = L(S_x, \boldsymbol{u}),$$

with S replaced by z, and thus the systems (7) associated with $(*)$ are actually 3D systems in disguise rather than genuinely 4D.

Apply the above approach to

$$f = p^2 + wp + u, \quad g = p^3 + 2wp^2 + rp + v,$$

when the nonlinear Lax pair reads

$$S_y = (S_x^2/S_z) + uS_x + S_z,$$
$$S_t = (S_x^3/S_z^2) + 2w(S_x^2/S_z) + rS_x + wS_z,$$

and the associated four-component system

$$\begin{aligned}
&u_t - vu_z - ru_x + uv_z + wv_x - v_y = 0,\\
&2u_z + w_x + 2ww_z - r_z = 0,\\
&2r_x - 3u_x - 2w_y + 2wu_z - v_z - 2ww_x + 2uw_z = 0,\\
&w_t - r_y + 2v_x - 4wu_x + wr_x - rw_x - vw_z + ur_z = 0.
\end{aligned} \qquad (**)$$

Deceiving appearances demystified II

The transformation interchanging z and S yields ($s \equiv S$)

$$z_y = -u + wz_x - z_x^2, \quad z_t = z_x^3 - 2wz_x^2 + rz_x - v,$$

but $(**)$ goes into

$$-r_s + 2ww_s + w_x z_s - z_x w_s + 2u_s = 0,$$

$$-2w_s z_x^2 + (4ww_s + 3u_s - 2r_s)z_x$$
$$+(-2w_y - 2ww_x + 2r_x - 3u_x)z_s - v_s + 2wu_s = 0,$$

$$-w_s z_x^3 + (2ww_s - r_s)z_x^2 + (-2v_s + 4wu_s)z_x$$
$$+(-4wu_x - r_y + 2v_x - rw_x + w_t + wr_x)z_s = 0,$$

$$-u_s z_x^3 + (-v_s + 2wu_s)z_x^2 + (-ru_x - v_y + u_t + wv_x)z_s = 0$$

i.e. it does not become 3D.

Moreover, it involves z and its derivatives while it should not.