Euclidean κ-Minkowski and the Spectral Dimension

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Outline:

1. Realizations of the momentum space
   - Old Lorentzian version
   - New Euclidean version

2. Spectral dimension of spacetime
   - Effective dimensionality of space
   - Computed results for the dimension

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The abelian nilpotent group

Generators of the $an(n)$ algebra satisfy the commutation relations

$$[X_0, X_a] = \frac{i}{\kappa} X_a\,, \qquad [X_a, X_b] = 0\,, \qquad a, b = 1, \ldots, n\,, \qquad (1)$$

with $\kappa \in \mathbb{R}_+$, and are considered positions in the κ-Minkowski space-time. The ordered exponentials of the algebra elements

$$g = e^{-ik^a X_a} e^{ik_0 X_0}\,, \quad k_0, k_a \in \mathbb{R} \qquad (2)$$

form the $AN(n)$ group, interpreted as the corresponding momentum space. The matrix representation adopted in all the following is

$$X_0 = -\frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}\,, \qquad X_a = \frac{i}{\kappa}\begin{pmatrix} 0 & \mathbf{e}_a^T & 0 \\ \mathbf{e}_a & \mathbf{0} & \mathbf{e}_a \\ 0 & -\mathbf{e}_a^T & 0 \end{pmatrix}\,, \qquad (3)$$

where the n-vector $\mathbf{e}_a = (0, \ldots, 0, 1_a, 0, \ldots, 0)$ and $X_a^3 = 0$.

Lorentzian de Sitter mapping

Acting with g on a spacelike vector $(0,\ldots,0,\kappa)$ one obtains $g \triangleright (0,\ldots,0,\kappa) = (p_0, \{p_a\}, p_{-1})$, where

$$p_0 = \kappa \sinh\left(\frac{k_0}{\kappa}\right) + \frac{1}{2\kappa} e^{k_0/\kappa} k_a k^a \,,$$
$$p_a = e^{k_0/\kappa} k_a \,,$$
$$p_{-1} = \kappa \cosh\left(\frac{k_0}{\kappa}\right) - \frac{1}{2\kappa} e^{k_0/\kappa} k_a k^a \,. \qquad (4)$$

The coordinates obey $-p_0^2 + p_a p^a + p_{-1}^2 = \kappa^2$ and $p_0 + p_{-1} > 0$. For the other half of de Sitter g is replaced with

$$g \cdot \mathcal{N} \equiv g \cdot \begin{pmatrix} -1 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbb{1} & \mathbf{0} \\ 0 & \mathbf{0} & -1 \end{pmatrix} . \qquad (5)$$

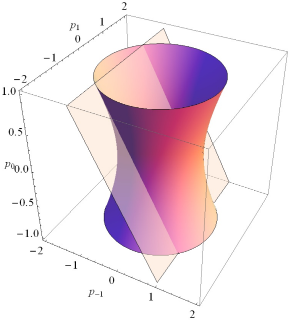

Figure: Lorentzian space of momenta

Lorentzian classical limit

Indeed, de Sitter space is equivalent to the quotient of groups $SO(n+1,1)/SO(n,1)$ and there exists a Iwasawa decomposition

$$SO(n+1,1) = AN(n)SO(n,1) \cup AN(n)\mathcal{N}SO(n,1)\,. \tag{6}$$

Furthermore, in the classical limit

$$\lim_{\kappa\to\infty} p_0 = k_0\,, \qquad \lim_{\kappa\to\infty} p_a = k_a\,, \qquad \lim_{\kappa\to\infty} p_{-1} = \infty\,. \tag{7}$$

Therefore p_{-1} can be considered a redundant coordinate, constrained by the hyperboloid condition $p_{-1}^2 = \kappa^2 + p_0^2 - p_a p^a$.

Euclidean anti-de Sitter mapping

Acting with g on a timelike vector $(\kappa, 0, \ldots, 0)$ one finds $g \triangleright (\kappa, 0, \ldots, 0) = (p_{-1}, \{p_a\}, p_0)$, where

$$p_0 = \kappa \sinh\left(\frac{k_0}{\kappa}\right) - \frac{1}{2\kappa} e^{k_0/\kappa} k_a k^a \,,$$
$$p_a = e^{k_0/\kappa} k_a \,,$$
$$p_{-1} = \kappa \cosh\left(\frac{k_0}{\kappa}\right) + \frac{1}{2\kappa} e^{k_0/\kappa} k_a k^a \,. \qquad (8)$$

The coordinates obey $p_0^2 + p_a p^a - p_{-1}^2 = -\kappa^2$ and $p_{-1} > 0$. For the other half of Euclidean anti-de Sitter g is replaced with

$$g \cdot \mathcal{N} \equiv g \cdot \begin{pmatrix} -1 & \mathbf{0} & 0 \\ \mathbf{0} & \mathbb{1} & \mathbf{0} \\ 0 & \mathbf{0} & -1 \end{pmatrix} . \qquad (9)$$

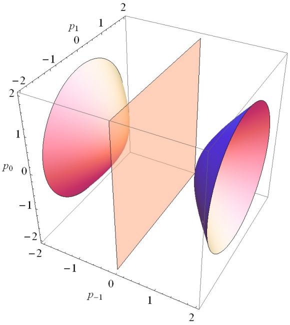

Figure: Euclidean space of momenta

Euclidean classical limit

Euclidean anti-de Sitter space is equivalent to the quotient of groups $SO(n+1,1)/SO(n+1)$ and there exists another Iwasawa decomposition

$$SO(n+1,1) = AN(n)SO(n+1) \cup AN(n)\mathcal{N}SO(n+1)\,. \qquad (10)$$

Again, in the classical limit

$$\lim_{\kappa\to\infty} p_0 = k_0\,, \qquad \lim_{\kappa\to\infty} p_a = k_a\,, \qquad \lim_{\kappa\to\infty} p_{-1} = \infty\,. \qquad (11)$$

And p_{-1} can be considered a redundant coordinate, constrained by the hyperboloid condition $p_{-1}^2 = \kappa^2 - p_0^2 - p_a p^a$.

Relation between both cases

The Euclidean case can be obtained through the Wick rotation of the Lorentzian case. One can make it in both coordinate systems at the same time. First, using $\kappa \mapsto i\kappa$, $k_0 \mapsto ik_0$ we obtain

$$p_0 = i\left(\kappa \sinh\left(\frac{k_0}{\kappa}\right) - \frac{1}{2\kappa} e^{k_0/\kappa} k_a k^a\right) ,$$
$$p_a = e^{k_0/\kappa} k_a ,$$
$$p_{-1} = i\left(\kappa \cosh\left(\frac{k_0}{\kappa}\right) + \frac{1}{2\kappa} e^{k_0/\kappa} k_a k^a\right) , \tag{12}$$

satysfying $-p_0^2 + p_a p^a + p_{-1}^2 = -\kappa^2$. Then, taking $p_0 \mapsto ip_0$, $p_{-1} \mapsto ip_{-1}$ we arrive at the final result with $p_0^2 + p_a p^a - p_{-1}^2 = -\kappa^2$.

Concept of the spectral dimension

On a given Riemannian manifold (M, h) one considers the (fictitious) diffusion process

$$\frac{\partial}{\partial\sigma} K(x, x_0; \sigma) = \Delta_h K(x, x_0; \sigma)\,, \quad K(x, x_0; 0) = \frac{\delta(x - x_0)}{\sqrt{\det h(x)}}\,. \tag{13}$$

It may be characterized by the average return probability

$$\mathcal{P}(\sigma) = \frac{1}{V} \int d^d x \, \sqrt{\det h(x)}\, K(x, x; \sigma)\,, \tag{14}$$

where V denotes the volume of M. Then the spectral dimension is extracted via the formula

$$d_S(\sigma) = -2\, \frac{d \log \mathcal{P}(\sigma)}{d \log \sigma}\,. \tag{15}$$

Why the Euclidean momentum space

Firstly, solution of the heat equation can be given in the form

$$K(x, x_0; \sigma) = \int d^d p \, e^{-\sigma C(p)} e^{ip(x-x_0)} \,, \tag{16}$$

where $C(p)$ is a momentum space version of the Laplacian. Secondly, to study the spectral dimension of a pseudo-Riemannian manifold one has to use its Riemannian counterpart. Thus we calculate the return probability from the Euclidean realization of $AN(n)$, which yields

$$\mathcal{P}(\sigma) = \int d^{n+2}p \, \delta\left(p_{-1}^2 - (p_0^2 + p_a p^a + \kappa^2)\right) \times \theta(p_{-1} - \kappa) \, e^{-\sigma C(p_0, \{p_a\})} = \int d^{n+1}p \, \frac{1}{2\sqrt{p_0^2 + p_a p^a + \kappa^2}} \, e^{-\sigma C(p_0, \{p_a\})} \,. \tag{17}$$

Spectral dim for the bicovariant Laplacian Casimir

According to the bicovariant differential calculus the Laplacian is

$$C_1(k_0,\{k_a\}) = 4\kappa^2 \sinh^2\left(\frac{k_0}{2\kappa}\right) + e^{k_0/\kappa} k_a k^a + \frac{1}{4\kappa^2}\left(4\kappa^2 \sinh^2\left(\frac{k_0}{2\kappa}\right) + e^{k_0/\kappa} k_a k^a\right)^2 \equiv p_0^2 + p_a p^a = C_1(p_0,\{p_a\})\,. \quad (18)$$

Then in 3+1 dim the spectral dimension

$$d_S(\sigma) = \frac{2\kappa\sqrt{\sigma}(2\kappa^2\sigma - 3) - \sqrt{\pi}e^{\kappa^2\sigma}(4\kappa^4\sigma^2 - 4\kappa^2\sigma + 3)\mathrm{erfc}(\kappa\sqrt{\sigma})}{-2\kappa\sqrt{\sigma} + \sqrt{\pi}e^{\kappa^2\sigma}(2\kappa^2\sigma - 1)\mathrm{erfc}(\kappa\sqrt{\sigma})}\,, \quad (19)$$

where $\mathrm{erfc}(.)$ is the complementary error function. In turn, in 2+1 dim

$$d_S(\sigma) = 2 + \frac{\kappa^2\sigma U(\frac{3}{2}, 1, \kappa^2\sigma)}{U(\frac{1}{2}, 0, \kappa^2\sigma)}\,, \quad (20)$$

where $U(.,.,.)$ is a Tricomi confluent hypergeometric function.

Spectral dim for the bicovariant Laplacian – cont.

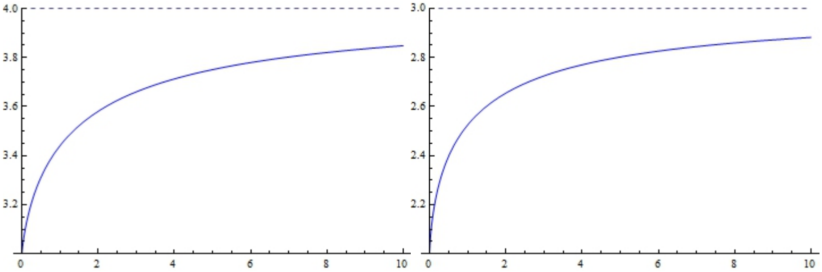

Figure: Spectral dimension $d_S(\sigma)$ for C_1 in 3+1 dim (left) and in 2+1 dim (right)

Both in 3+1 and 2+1 dim we observe dimensional reduction in small scales $\sigma\kappa^2 \approx 0$, respectively:

$$\lim_{\sigma\to 0} d_S = 3, 2\,, \qquad \lim_{\sigma\to +\infty} d_S = 4, 3\,. \tag{21}$$

Spectral dim for the bicrossproduct Laplacian

Another Laplacian, determined by the Casimir of the κ-Poincaré algebra in the bicrossproduct basis has the form

$$C_0(k_0, \{k_a\}) = 4\kappa^2 \sinh^2\left(\frac{k_0}{2\kappa}\right) + e^{k_0/\kappa} k_a k^a \equiv$$
$$2\kappa\left(\sqrt{p_0^2 + p_a p^a + \kappa^2} - \kappa\right) = C_0(p_0, \{p_a\})\,. \qquad (22)$$

(And we have a relation $C_1 = C_0 + \frac{1}{4\kappa^2} C_0^2$.) In this case in 3+1 dim the spectral dimension

$$d_S(\sigma) = \frac{8\kappa^2\sigma + 6}{2\kappa^2\sigma + 1}\,, \qquad (23)$$

while in 2+1 dim I found only a numerical solution.

Spectral dim for the bicrossproduct Laplacian – cont.

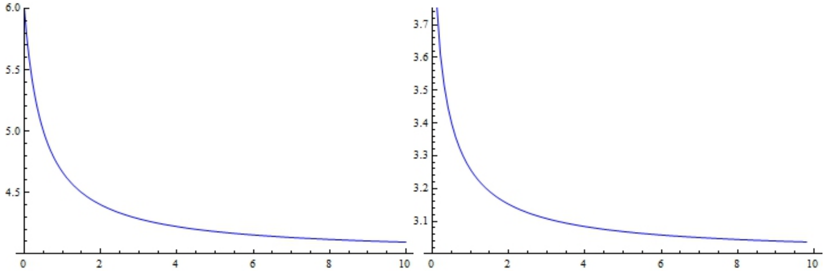

Figure: Spectral dimension $d_S(\sigma)$ for C_0 in 3+1 dim (left) and in 2+1 dim (right)

Here in small scales in 3+1 and 2+1 dim we find a superdiffusive behaviour, respectively:

$$\lim_{\sigma \to 0} d_S = 6, 4\,, \qquad \lim_{\sigma \to +\infty} d_S = 4, 3\,. \tag{24}$$

Spectral dim for the relative-locality Laplacian

In the relative locality approach to κ-Poincaré, Euclidean version of the Laplacian is given by the (squared) distance along geodesics in Euclidean anti-de Sitter space, i.e. spatial geodesics in one dimension higher anti-de Sitter space and thus

$$C_d(p_0,\{p_a\}) = \kappa^2 \mathrm{arccosh}^2\left(\frac{1}{\kappa}\sqrt{p_0^2 + p_a p^a + \kappa^2}\right) = \kappa^2 \mathrm{arccosh}^2\left(\cosh\left(\frac{k_0}{\kappa}\right) + \frac{1}{2\kappa^2} e^{k_0/\kappa} k_a k^a\right) = C_d(k_0,\{k_a\})\,. \tag{25}$$

Here both in 3+1 and 2+1 dim I obtained only numerical results.

Spectral dim for the relative-locality Laplacian – cont.

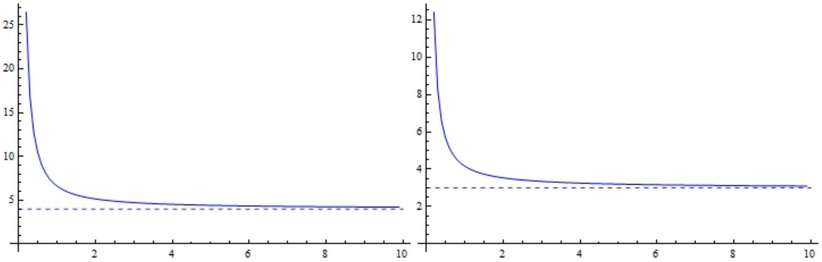

Figure: Spectral dimension $d_S(\sigma)$ for C_d in 3+1 dim (left) and in 2+1 dim (right)

In 3+1 and 2+1 dim in small scales the spectral dimension diverges, respectively:

$$\lim_{\sigma \to 0} d_S \approx +\infty\,, \qquad \lim_{\sigma \to +\infty} d_S \approx 4, 3\,. \tag{26}$$

Summary

Conclusions and open questions

- The simple recipe for an Euclidean version of the κ momentum space was introduced.
- The spectral dimension for three candidate Laplacians in 3+1 and 2+1 dim was calculated.
- The results can be compared with those for other models of spacetime in small scales.
- Is there a correct, physical Laplacian or do we have several distinct theories?
- Is the spectral dimension always a good measure of dimensionality?

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