Quantum symmetries and description of composed systems

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MAX BORN
11.12.1882 – 5.1.1970
$pq - qp = \frac{h}{2\pi i}$
HEDWIG BORN
GEB. EHRENBERG
14.12.1891–29.2.1972

Abstract

In quantum mechanics a state of a system is described by a complex valued function on configuration space. Consequently the Hilbert space assigned to a system composed of two parts is the tensor product of Hilbert spaces assigned to each part. The same holds for the algebra of observables: the C^*-algebra corresponding to a composed system is the tensor product of C^*-algebras corresponding to subsystems. This structure is well compatible with the action of classical groups.

This is not the case for quantum groups. If a global symmetry is described by a quantum group then we are forced to deform the concept tensor product replacing it by a more sophisticated concept of monoidal structure.

Abstract - cont.

The aim of the talk is

- to introduce the concept of monoidal structure on the category of C^*-algebras subject to an action of a fixed quantum group,
- to formulate natural conditions the good monoidal structure should obey and
- to discuss the existence and uniqueness of good monoidal structure.

Notation

Let X and Y be a norm closed subsets of a C^* algebra. We set

$$XY = \left\{ xy : \begin{array}{l} x \in X \\ y \in Y \end{array} \right\}^{\mathrm{CLS}},$$

where CLS stands for norm Closed Linear Span.

Category of C* algebras

Let C^* be a category whose objects are separable C^* algebras. If $X, Y \in C^*$ then by definition $\mathrm{Mor}(X, Y)$ is the set of all *-algebra homomorphisms φ acting from X into $M(Y)$ such that $\varphi(X)Y = Y$.

Any $\varphi \in \mathrm{Mor}(X, Y)$ admits a unique extension to a unital *-algebra homomorphism acting from $M(X)$ into $M(Y)$. Composition of morphisms is defined as composition of their extensions.

In what follows

$$\varphi : X \longrightarrow Y$$

means that $\varphi \in \mathrm{Mor}(X, Y)$. It does not imply that $\varphi(X) \subset Y$.

The concept of Crossed Product Algebra

Let X, Y, Z be C*-algebras, $\alpha \in \mathrm{Mor}(X, Z)$ and $\beta \in \mathrm{Mor}(Y, Z)$.
We say that Z is a crossed product of X and Y if

$$\alpha(X)\beta(Y) = Z.$$

Example:

$$Z = X \otimes Y$$
$$\alpha(x) = x \otimes I_Y$$
$$\beta(y) = I_X \otimes y$$

In this presentation for any C*-algebras X and Y, $X \otimes Y$ always denote the minimal (spatial) tensor product.

Crossed Product Algebra in practice

Let X, Y be separable C* algebras, H be a Hilbert space, $\alpha \in \mathrm{Rep}(X, H)$ and $\beta \in \mathrm{Rep}(Y, H)$. Then

$$\alpha(X)\beta(Y) = \beta(Y)\alpha(X)$$

if and only if

$$\alpha(X)\beta(Y) \text{ is a C}^* \text{ algebra.}$$

Moreover in this case $\alpha \in \mathrm{Mor}(X, Z)$ and $\beta \in \mathrm{Mor}(Y, Z)$, where $Z = \alpha(X)\beta(Y)$. Therefore Z is a crossed product of X and Y.

Locally compact quantum groups

Let G be a locally compact quantum group. This is a locally compact quantum space G endowed with a continuous associative mapping $G \times G \longrightarrow G$ (group rule) subject to certain axioms.

In practice we work with the C*-algebra $A = \mathcal{C}_\infty(G)$ endowed with a morphism $\Delta \in \mathrm{Mor}(A, A \otimes A)$ corresponding to the group rule on G. Shorthand notation:

$$G = (A, \Delta).$$

Strictly speaking one has to distinguish locally compact quantum group G from the corresponding Hopf C*-algebra (A, Δ).

$$\left\{ \begin{array}{c} \textit{actions} \\ \textit{of } G \end{array} \right\} = \left\{ \begin{array}{c} \textit{coactions} \\ \textit{of } (A, \Delta) \end{array} \right\}$$

Locally compact quantum groups

The present work does not use the full power of the Kustermans and Vaes theory of locally compact quantum groups. Instead we use the theory of manageable multiplicative unitaries. For us locally compact quantum groups are objects coming from multiplicative unitary operators. In particular we do not use the Haar weights.

Duality

Locally compact quantum groups appear in dual pairs:

$$G = (A, \Delta)$$
$$\widehat{G} = (\widehat{A}, \widehat{\Delta})$$

The duality is described by a bicharacter V. This is a unitary element of $\mathrm{M}(\widehat{A} \otimes A)$ such that

$$(\mathrm{id} \otimes \Delta)V = V_{12}V_{13},$$
$$(\widehat{\Delta} \otimes \mathrm{id})V = V_{23}V_{13}.$$

Quasitriangular quantum groups

Let R be a unitary element of $\mathrm{M}(\widehat{A} \otimes \widehat{A})$. We say that R is an universal R-matrix if

$$\left.\begin{aligned} (\mathrm{id} \otimes \widehat{\Delta})R &= R_{12}R_{13}, \\ (\widehat{\Delta} \otimes \mathrm{id})R &= R_{23}R_{13}, \\ R_{12}V_{13}V_{23} &= V_{23}V_{13}R_{12}. \end{aligned}\right\} \tag{1}$$

Definition 1

A locally compact quantum group $G = (A, \Delta)$ is called quasitriangular if there exists a unitary universal R-matrix in $\mathrm{M}(\widehat{A} \otimes \widehat{A})$.

C*-algebras subject to an action of G

Let X be a C*-algebra and $\rho \in \mathrm{Mor}(X, X \otimes A)$. We say that ρ is an action of G on X if

1. $$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\rho"'] & X \otimes A \arrow[d, "\rho\otimes\mathrm{id}"] \\ X \otimes A \arrow[r, "\mathrm{id}\otimes\Delta"'] & X \otimes A \otimes A \end{tikzcd} \tag{2}$$

is a commutative daigram.

2. $\ker \rho = \{0\}$,
3. $\rho(X)(I \otimes A) = X \otimes A$ (Podleś condition).

Remark

$$\left(\begin{array}{c} \text{Podleś} \\ \text{condition} \end{array} \right) \Longrightarrow \Big(\rho \in \mathrm{Mor}(X, X \otimes A) \Big)$$

Morphisms, functors and natural mappings

We shall use the language of the theory of categories. Notions of object, morphism, functor and natural mapping will appear. We work mainly with category C^*_G introduced in the next slide. The class of objects of the category will be denoted by the same symbol C^*_G and the set of morphisms acting from X into Y ($X, Y \in \mathrm{C}^*_G$) will be denoted by $\mathrm{Mor}_G(X, Y)$.

Functor	from	to
Proj_1 Proj_2 $\otimes$ $\boxtimes$	$\mathrm{C}^*_G \times \mathrm{C}^*_G$	C^*_G
$\otimes A$	C^*_G	C^*_G

Natural mapping	from	to
α	Proj_1	$\boxtimes$
β	Proj_2	$\boxtimes$
ρ	$\mathrm{id}_{\mathrm{C}^*_G}$	$\otimes A$

Category C^*_G

Objects are C^*-algebras with actions of G. For any $X \in C^*_G$, the action of G on X will be denoted by ρ^X. Morphisms in C^*_G are C^*-morphisms intertwining the actions of G:

Let X, Y be C^*-algebras with actions of G. We say that a morphism $\gamma \in \mathrm{Mor}(X, Y)$ intertwins the actions of G if

$$\begin{tikzcd} X \arrow[r, "\rho^X"] \arrow[d, "\gamma"'] & X \otimes A \arrow[d, "\gamma\otimes\mathrm{id}"] \\ Y \arrow[r, "\rho^Y"'] & Y \otimes A \end{tikzcd}$$

is a commutative diagram.

The set of all such morphisms will be denoted by $\mathrm{Mor}_G(X, Y)$.

Examples

Any C*-algebra X with the trivial action

$$\rho^X(x) = x \otimes I_A \in \mathrm{M}(X \otimes A)$$

is an object of C^*_G.

The field of complex numbers $\mathbb{C}$ is a C*-algebra. This is the initial object of category C*: For any C*-algebra X the mapping

$$1_X : \mathbb{C} \ni \lambda \longmapsto \lambda I_X \in \mathrm{M}(X)$$

is the only element of $\mathrm{Mor}(\mathbb{C}, X)$. Let $\rho^{\mathbb{C}} = 1_{\mathbb{C}\otimes A}$. Clearly $\rho^{\mathbb{C}}$ is a trivial action of G on $\mathbb{C}$ and $\mathbb{C} \in \mathrm{C}^*_G$.

For any $X \in \mathrm{C}^*_G$ we have:

$$I_X \in \mathrm{M}(X), \quad \mathrm{id}_X \in \mathrm{Mor}_G(X, X), \quad 1_X \in \mathrm{Mor}_G(\mathbb{C}, X).$$

$A = \mathcal{C}_\infty(G)$ with the action

$$\rho^A(a) = \Delta(a) \in \mathrm{M}(A \otimes A)$$

is an object of C^*_G. This is a distinguished object.

Let X be a C^*-algebra with any action of G. Then $X \otimes A$ with the action

$$\rho^{X\otimes A}(x \otimes a) = x \otimes \Delta(a) \in \mathrm{M}((X \otimes A) \otimes A)$$

is an object of C^*_G. The reader should notice that the action of G on $X \otimes A$ is induced by the action of G on A. The action ρ^X is ignored. However the commutative diagram (2) shows that ρ^X intertwines the actions of G on X and $X \otimes A$:

$$\rho^X \in \mathrm{Mor}_G(X, X \otimes A).$$

One may consider two functors: $\mathrm{id}_{C^*_G}$ and $\otimes A$ (tensoring objects by A and morphisms by id_A) acting within the category C^*_G. Then ρ become a natural mapping from $\mathrm{id}_{C^*_G}$ into $\otimes A$

Let $X, Y \in \mathrm{C}^*_G$. Then $X \otimes Y$ with the action

$$\rho^{X\otimes Y}(x \otimes y) = x \otimes \rho^Y(y) \in \mathrm{M}((X \otimes Y) \otimes A)$$

is an object of C^*_G. Again the action ρ^X is ignored: the action of G on $X \otimes Y$ is induced by the action of G on Y. With the standard tensor product of morphisms, $\otimes$ becomes a associative covariant functor acting from $\mathrm{C}^*_G \times \mathrm{C}^*_G$ into C^*_G.

Main topic

We are interested in monoidal structures on the category C^*_G. By definition a monoidal structure on C^*_G is an associative covariant functor $\boxtimes$ acting from $\mathrm{C}^*_G \times \mathrm{C}^*_G$ into C^*_G having $\mathbb{C}$ as neutral object:

$$
\begin{aligned}
X \boxtimes \mathbb{C} &= X = \mathbb{C} \boxtimes X, \\
X' \boxtimes \mathbb{C} &= X' = \mathbb{C} \boxtimes X', \\
\varphi \boxtimes \mathrm{id}_{\mathbb{C}} &= \varphi = \mathrm{id}_{\mathbb{C}} \boxtimes \varphi
\end{aligned}
$$

for any $X, X' \in \mathrm{C}^*_G$ and $\varphi \in \mathrm{Mor}(X, X')$.

Main result

Category C^*_G admits a monoidal structure (with certain natural properties) if and only if G is quasi-triangular. More than that: monoidal structures are in one to one correspondence with unitary universal R-matrices.

Remark: $\otimes$ is not monoidal

In general (when G is non-trivial i.e: is not a one-element group) the associative functor $\otimes$ does not define a monoidal structure on C^*_G. This is because $\mathbb{C}$ is not a neutral object for $\otimes$. Indeed, for any $X \in \mathrm{C}^*_G$ we have:

$$\begin{aligned} \mathbb{C} \otimes X &= X \\ X \otimes \mathbb{C} &= X_{tr}, \end{aligned}$$

where X_{tr} is the C^*-algebra X equipped with the trivial action of G. If G is not trivial then $A_{\mathrm{tr}} \neq A$.

Natural mappings α and β

Let $\boxtimes$ be a monoidal structure on C^*_G. For any $X, Y \in \mathrm{C}^*_G$ we set

$$\begin{aligned}\alpha^{XY} &= \mathrm{id}_X \boxtimes 1_Y,\\ \beta^{XY} &= 1_X \boxtimes \mathrm{id}_Y\,.\end{aligned}$$

Then

$$\begin{aligned}\alpha^{XY} &\in \mathrm{Mor}_G(X, X \boxtimes Y),\\ \beta^{XY} &\in \mathrm{Mor}_G(Y, X \boxtimes Y).\end{aligned}$$

In particular $\alpha^{X\mathbb{C}} \in \mathrm{Mor}(X, X)$ and $\beta^{\mathbb{C}Y} \in \mathrm{Mor}(Y, Y)$. Clearly

$$\begin{aligned}\alpha^{X\mathbb{C}} &= \mathrm{id}_X,\\ \beta^{\mathbb{C}Y} &= \mathrm{id}_Y\,.\end{aligned}$$

One can easily verify that for any $X, X', Y, Y' \in \mathrm{C}^*_G$, $\varphi \in \mathrm{Mor}_G(X, X')$ and $\psi \in \mathrm{Mor}_G(Y, Y')$ the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha^{XY}"] \arrow[d, "\varphi"'] & X \boxtimes Y \arrow[d, "\varphi\boxtimes\psi"] & Y \arrow[l, "\beta^{XY}"'] \arrow[d, "\psi"] \\ X' \arrow[r, "\alpha^{X'Y'}"'] & X' \boxtimes Y' & Y' \arrow[l, "\beta^{X'Y'}"] \end{tikzcd} \tag{3}$$

is commutative. It shows that α and β are natural mappings from Proj_1 and Proj_2 into $\boxtimes$.

Monoidal structure on C^*_G

Let $G = (A, \Delta)$ be a quasitriangular locally compact quantum group. In a forthcoming paper [R. Meyer, S. Roy and SLW] we introduced a monoidal structure $\boxtimes$ on the category C^*_G. It has the following properties:

Property 1

For any $X, Y \in C^*_G$, $X \boxtimes Y$ is a crossed product of X and Y:

$$X \boxtimes Y = \alpha^{XY}(X)\beta^{XY}(Y)$$

Property 2

The $\boxtimes$-product of injective morphisms is injective.

Property 3

$\boxtimes$ reduces to $\otimes$, when the action of G on one of the involved C*-algebras is trivial. More precisely: If $X, Y \in \mathrm{C}^*_G$ and if one of the actions ρ^X and ρ^Y is trivial then $X \boxtimes Y = X \otimes Y$ as C*-algebras. Moreover

If ρ^X is trivial then

$$\rho^{X \boxtimes Y}(x \otimes y) = x_1 \rho^Y(y)_{23}$$

If ρ^Y is trivial then

$$\rho^{X \boxtimes Y}(x \otimes y) = \rho^X(x)_{13} y_2$$

Similarly for morphisms:
If $\varphi \in \mathrm{Mor}_G(X, X')$, $\psi \in \mathrm{Mor}_G(Y, Y')$ and if the actions ρ^X and $\rho^{X'}$ are trivial then

$$\varphi \boxtimes \psi = \varphi \otimes \psi.$$

The last formula holds also under assumption of triviality of ρ^Y and $\rho^{Y'}$.

From R-matrix to monoidal structure

The category C^*_G contains a distinguished object A with $\rho^A = \Delta$. Let $V \in \mathrm{M}(\widehat{A} \otimes A)$ be the bicharacter describing the duality between $\widehat{G}$ and G. To make our formulae simpler we shall use the following shorthand notation:

$$V_{1\alpha} = \left[\left(\mathrm{id} \otimes \alpha^{AA}\right) V\right]_{13},$$
$$V_{2\beta} = \left[\left(\mathrm{id} \otimes \beta^{AA}\right) V\right]_{23}.$$

Clearly $V_{1\alpha}, V_{2\beta} \in \mathrm{M}(\widehat{A} \otimes \widehat{A} \otimes (A \boxtimes A))$. With this notation we have:

Theorem 2 (R. Meyer, S. Roy and SLW)

Let $G = (A, \Delta)$ be a quasitriangular locally compact quantum group with an universal unitary R-matrix $R \in \mathrm{M}(\widehat{A} \otimes \widehat{A})$. Then there exists a monoidal structure $\boxtimes$ on C^*_G having Properties 1,2 and 3 and such that

$$V_{1\alpha} V_{2\beta} = V_{2\beta} V_{1\alpha} R_{12}.$$

We have the following:

Theorem 3

Let $G = (A, \Delta)$ be a locally compact quantum group and $\boxtimes$ be a monoidal structure on C^*_G having Properties 1,2 and 3. Then there exists (unique) universal unitary R-matrix $R \in \mathrm{M}(\widehat{A} \otimes \widehat{A})$ such that

$$V_{1\alpha} V_{2\beta} = V_{2\beta} V_{1\alpha} R_{12}.$$

Plan of the proof of Thm 3

Let

$$\widetilde{R} = V_{1\alpha}^* V_{2\beta}^* V_{1\alpha} V_{2\beta}.$$

Then $\widetilde{R} \in \mathrm{M}(\widehat{A} \otimes \widehat{A} \otimes (A \boxtimes A))$. To prove Thm 3 we have to show that the $(A \boxtimes A)$ - leg of $\widetilde{R}$ is trivial. In other words we have to show that $\widetilde{R} = R_{12}$, where $R \in \mathrm{M}(\widehat{A} \otimes \widehat{A})$. Next we have to prove that R satisfies the relations (1) characteristic for unitary R-matrix. The latter is a matter of easy computations.

The proof is based on the following two propositions:

Let $X \in \mathrm{C}^*_G$ and $x \in \mathrm{M}(X)$. We say that x is G-invariant if $\rho^X(x) = x \otimes I_A$.

Proposition 4

Let $X, Y \in \mathrm{C}^*_G$, $x \in \mathrm{M}(X)$ and $y \in \mathrm{M}(Y)$. Assume that one of the elements x, y is G-invariant. Then

$$\alpha^{XY}(x)\beta^{XY}(y) = \beta^{XY}(y)\alpha^{XY}(x).$$

Proposition 5

Let $X, Y, Z, T \in \mathrm{C}^*_G$ *and* $u \in \mathrm{M}(X \boxtimes Z)$ *and* $v \in \mathrm{M}(Y \boxtimes T)$. *Assume that*

$$(\mathrm{id}_X \boxtimes 1_Y \boxtimes \mathrm{id}_Z \boxtimes 1_T)(u) = (1_X \boxtimes \mathrm{id}_Y \boxtimes 1_Z \boxtimes \mathrm{id}_T)(v).$$

Then $u = \lambda \mathrm{I}_{X \boxtimes Z}$ *and* $v = \lambda \mathrm{I}_{Y \boxtimes T}$, *where* $\lambda \in \mathbb{C}$.

Notation

We shall deal with the $\boxtimes$-products of two and four copies of the distinguished object A:

$$\begin{aligned} A^{\boxtimes 2} &= A \boxtimes A \\ A^{\boxtimes 4} &= A^{\boxtimes 2} \boxtimes A^{\boxtimes 2} \end{aligned}$$

To make our formulae shorter we shall write α and β instead of α^{AA} and β^{AA} and $\widetilde{\alpha}$ and $\widetilde{\beta}$ instead of $\alpha^{A^{\boxtimes 2}A^{\boxtimes 2}}$ and $\beta^{A^{\boxtimes 2}A^{\boxtimes 2}}$. Then

$$\begin{aligned} \alpha, \beta &\in \mathrm{Mor}_G(A, A^{\boxtimes 2}), \\ \widetilde{\alpha}, \widetilde{\beta} &\in \mathrm{Mor}_G(A^{\boxtimes 2}, A^{\boxtimes 4}). \end{aligned}$$

Composing these morphisms we obtain four morphisms from A into $A^{\boxtimes 4}$. Using the formulae expressing α and β as $\boxtimes$-products of id_A and 1_A one can easily verify that

$$\begin{aligned} (\alpha \boxtimes \alpha)\alpha &= \widetilde{\alpha}\alpha, & (\beta \boxtimes \beta)\alpha &= \widetilde{\alpha}\beta, \\ (\alpha \boxtimes \alpha)\beta &= \widetilde{\beta}\alpha, & (\beta \boxtimes \beta)\beta &= \widetilde{\beta}\beta. \end{aligned}$$

The following eight unitaries belonging to $\mathrm{M}(\widehat{A} \otimes \widehat{A} \otimes A^{\boxtimes 4})$ will be involved in our computations: For $i \in \{1, 2\}$ and $r, s \in \{\alpha, \beta\}$ we set:

$$V_{i,\widetilde{r}s} = \left\{ (\mathrm{id}_{\widehat{A}} \otimes \widetilde{r} \circ s) V \right\}_{i3}.$$

With this notation

$$\begin{aligned} \left(\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\widehat{A}} \otimes (\alpha \boxtimes \alpha)\right) \widetilde{R} &= V^*_{1,\widetilde{\alpha}\alpha} V^*_{2,\widetilde{\beta}\alpha} V_{1,\widetilde{\alpha}\alpha} V_{2,\widetilde{\beta}\alpha}, \\ \left(\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\widehat{A}} \otimes (\beta \boxtimes \beta)\right) \widetilde{R} &= V^*_{1,\widetilde{\alpha}\beta} V^*_{2,\widetilde{\beta}\beta} V_{1,\widetilde{\alpha}\beta} V_{2,\widetilde{\beta}\beta}. \end{aligned} \tag{4}$$

G invariance of $V_{1\alpha}V_{1\beta}^*$

We compute:

$$\left(\mathrm{id}_{\widehat{A}} \otimes \rho^{A\boxtimes A}\right) V_{1\alpha} = \left(\mathrm{id}_{\widehat{A}} \otimes \alpha \otimes \mathrm{id}_A\right) (\mathrm{id}_{\widehat{A}} \otimes \rho^A) V$$
$$= \left(\mathrm{id}_{\widehat{A}} \otimes \alpha \otimes \mathrm{id}_A\right) V_{12} V_{13} = V_{1\alpha} V_{13}.$$

Similarly

$$\left(\mathrm{id}_{\widehat{A}} \otimes \rho^{A\boxtimes A}\right) V_{1\beta} = V_{1\beta} V_{13}.$$

Therefore

$$\left(\mathrm{id}_{\widehat{A}} \otimes \rho^{A\boxtimes A}\right) \left(V_{1\alpha} V_{1\beta}^*\right) = V_{1\alpha} V_{1\beta}^* \otimes I_A \,.$$

It shows that the ‘second leg’ of $V_{1\alpha}V_{1\beta}^*$ is G-invariant.

The main steps of the proof

Proposition 4 shows now that $V_{2,\widetilde{\beta}\alpha} V^*_{2,\widetilde{\beta}\beta}$ commutes with $V_{1,\widetilde{\alpha}\beta}$ and that $V_{1,\widetilde{\alpha}\alpha} V^*_{1,\widetilde{\alpha}\beta}$ commutes with $V_{2,\widetilde{\beta}\alpha}$. Using this information one can easily show that the unitaries appearing on the right hand side of relations (4) are equal. Therefore

$$\left(\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\widehat{A}} \otimes (\alpha \boxtimes \alpha)\right) \widetilde{R} = \left(\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\widehat{A}} \otimes (\beta \boxtimes \beta)\right) \widetilde{R}.$$

Notice that

$$\begin{aligned} \alpha \boxtimes \alpha &= \mathrm{id}_A \boxtimes 1_A \boxtimes \mathrm{id}_A \boxtimes 1_A, \\ \beta \boxtimes \beta &= 1_A \boxtimes \mathrm{id}_A \boxtimes 1_A \boxtimes \mathrm{id}_A\,. \end{aligned}$$

Proposition 5 shows now that the 'last leg' of $\widetilde{R}$ is trivial: $\widetilde{R} = R_{12}$, where $R \in \mathrm{M}(\widehat{A} \otimes \widehat{A})$. To end the proof we have to show that R satisfies (1).

R is a unitary R-matrix

We already know that

$$V_{1\alpha} V_{2\beta} = V_{2\beta} V_{1\alpha} R_{12}, \tag{5}$$

where R is a unitary element of $\mathrm{M}(\widehat{A} \otimes \widehat{A})$. Applying $\widehat{\Delta}$ to the first and second leg we get:

$$V_{2\alpha} V_{1\alpha} V_{3\beta} = V_{3\beta} V_{2\alpha} V_{1\alpha} \left\{ (\widehat{\Delta} \otimes \mathrm{id}_{\widehat{A}}) R \right\}_{123},$$

$$V_{1\alpha} V_{3\beta} V_{2\beta} = V_{3\beta} V_{2\beta} V_{1\alpha} \left\{ (\mathrm{id}_{\widehat{A}} \otimes \widehat{\Delta}) R \right\}_{123}.$$

On the other hand we have:

$$\begin{aligned}
V_{2\alpha} V_{1\alpha} V_{3\beta} &= V_{2\alpha} V_{3\beta} V_{1\alpha} R_{13} \\
&= V_{3\beta} V_{2\alpha} R_{23} V_{1\alpha} R_{13} = V_{3\beta} V_{2\alpha} V_{1\alpha} R_{23} R_{13}, \\
V_{1\alpha} V_{3\beta} V_{2\beta} &= V_{3\beta} V_{1\alpha} R_{13} V_{2\beta} \\
&= V_{3\beta} V_{1\alpha} V_{2\beta} R_{13} = V_{3\beta} V_{2\beta} V_{1\alpha} R_{12} R_{13}
\end{aligned}$$

It shows that

$$(\widehat{\Delta} \otimes \mathrm{id}_{\widehat{A}})R = R_{23}R_{13},$$

$$(\mathrm{id}_{\widehat{A}} \otimes \widehat{\Delta})R = R_{12}R_{13}.$$

To prove the third relation of (1) we apply $\mathrm{id}_{\widehat{A}} \otimes \mathrm{id}_{\widehat{A}} \otimes \rho^{A \boxtimes A}$ to the both sides of (5):

$$\left\{(\mathrm{id}_{\widehat{A}} \otimes \rho^{A})V\right\}_{1\alpha 3}\left\{(\mathrm{id}_{\widehat{A}} \otimes \rho^{A})V\right\}_{2\beta 3}$$
$$= \left\{(\mathrm{id}_{\widehat{A}} \otimes \rho^{A})V\right\}_{2\beta 3}\left\{(\mathrm{id}_{\widehat{A}} \otimes \rho^{A})V\right\}_{1\alpha 3} R_{12},$$

$$V_{1\alpha}V_{13}V_{2\beta}V_{23} = V_{2\beta}V_{23}V_{1\alpha}V_{13}R_{12},$$

$$V_{1\alpha}V_{2\beta}V_{13}V_{23} = V_{2\beta}V_{1\alpha}V_{23}V_{13}R_{12},$$

$$R_{12}V_{13}V_{23} = V_{23}V_{13}R_{12},$$

$\square$

Comparing monoidal structures

Let $\boxtimes, \boxtimes'$ be monoidal structures on C^*_G and $\Phi : \boxtimes \longrightarrow \boxtimes'$ be a natural mapping. It means that for any pair of objects $X, Y \in \mathrm{C}^*_G$ we have morphism $\Phi^{XY} \in \mathrm{Mor}_G(X \boxtimes Y, X \boxtimes' Y)$ and that for any pair of morphisms $r \in \mathrm{Mor}_G(X, Z)$ and $s \in \mathrm{Mor}_G(Y, T)$ the diagram

$$\begin{tikzcd} X \boxtimes Y \arrow[r, "r \boxtimes s"] \arrow[d, "\Phi^{XY}"'] & Z \boxtimes T \arrow[d, "\Phi^{ZT}"] \\ X \boxtimes' Y \arrow[r, "r \boxtimes' s"'] & Z \boxtimes' T \end{tikzcd} \tag{6}$$

is commutative. We know that $\mathbb{C} \boxtimes X = \mathbb{C} \boxtimes' X = X = X \boxtimes \mathbb{C} = X \boxtimes' \mathbb{C}$. Therefore $\Phi^{X\mathbb{C}}, \Phi^{\mathbb{C}X} \in \mathrm{Mor}_G(X, X)$. We say that Φ is normalized if $\Phi^{X\mathbb{C}} = \mathrm{id}_X = \Phi^{\mathbb{C}X}$ for any $X \in \mathrm{C}^*_G$.

Uniqueness of monoidal structure

Theorem 6

Let $\boxtimes$ and $\boxtimes'$ be monoidal structures on C^*_G corresponding to the same R-matrix. Then there exists one and only one normalized natural mapping $\Phi : \boxtimes \longrightarrow \boxtimes'$. The morphism

$$\Phi^{XY} \in \mathrm{Mor}_G(X \boxtimes Y, X \boxtimes' Y)$$

is an isomorphism.