Vacuum fluctuations in theories with non-trivial momentum space

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How many ‘dimensions’ at the Planck scale?

- spectral dimension

- Hausdorff dimension of momentum space

- quantum vacuum dimension

Spectral dimension as a probe of geometry

Heat diffusion on a Riemannian manifold:

$$\frac{\partial}{\partial s} K(\xi_0, \xi, s) + \Delta K(\xi_0, \xi, s) = 0$$

Return probability density:

$$P(s) = \frac{1}{V} \int d\xi \sqrt{|g|} K(\xi, \xi, s)$$

$$K = < \xi | e^{-s\Delta} | \xi_0 > \quad \longrightarrow \quad P(s) = \frac{1}{V} \sum_j e^{-\lambda_j s}$$

(sum over eigenvalues of the Laplacian)

related to the geometrical properties of the manifold

$$P(s) = \frac{1}{V(4\pi s)^{d/2}} \sum_n a_n s^n$$

(heat trace expansion)

$$a_0 = \int \sqrt{|g|}, \quad a_1 \sim \int \sqrt{|g|} R, \quad a_2 \sim \int \sqrt{|g|} \left[5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2(R_{\mu\nu\rho\sigma})^2 \right], \ \ldots$$

Spectral dimension as a probe of geometry

Spectral dimension: $$d_S(s) = -2\frac{d\ln P(s)}{d\ln s}$$

in smooth Riemannian manifold

- flat space: $$P(s) = (4\pi s)^{-d/2} \longrightarrow d_S(s) \equiv d$$
- in general: (Riem. manif.) $$d_S(s) = d - 2\frac{\sum_{n=1} n\, a_n s^n}{\sum_{n=0} a_n s^n} \longrightarrow \begin{array}{l} d_S(s \to 0) = d \\ d_S(s \to \infty) = 0 \end{array}$$

Spectral dimension in Quantum Gravity

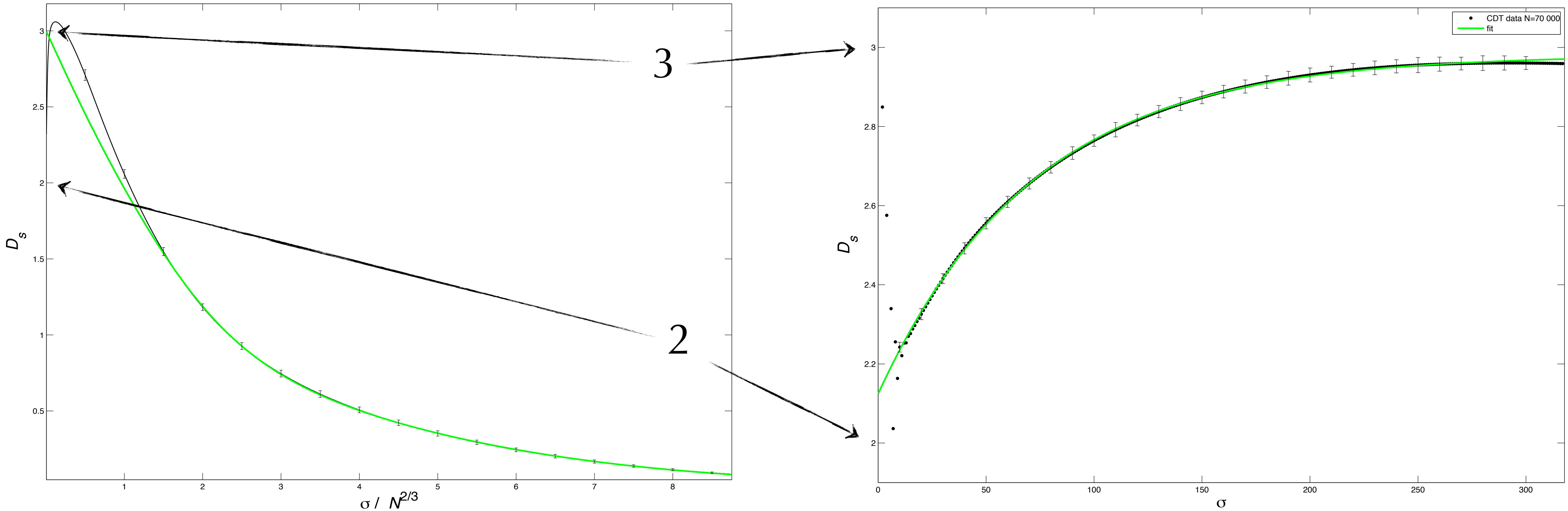
the ‘diffusion’ process ‘happens’ on the whole of (Euclidean) spacetime

→ IR limit probes global geometry, intermediate scales probe local (flat) geometry, UV limit probes quantum spacetime properties

most QG theories find running spectral dimension in the UV ($s \to 0$):

$$d_S^{(QG)}(s \to 0) \neq d$$

example (3d CDT)



IR behavior

[D. Benedetti and J. Henson PRD 2009]

UV behavior

From dispersion relation to spectral dimension

dispersion relation is the momentum space representation of the Laplacian

$$\omega^2 = f(k^2) \qquad \longleftrightarrow \qquad D_L = -\partial_t^2 - f(-\nabla^2)$$

spectral dimension is probed by a fictitious diffusion process governed by the "Wick rotated" Laplacian operator (in flat ST)

$$\left[\frac{\partial}{\partial s} + \left(-\partial_t^2 + f(-\nabla^2)\right)\right] K(\xi_0, \xi, s) = 0$$

the return probability can be written as $\quad P(s) = \int \frac{d^D k \, d\omega}{(2\pi)^{D+1}} e^{-s\left(\omega^2 + f(k^2)\right)}$

and the spectral dimension

$$d_S(s) = 2s \frac{\int d^D k \, d\omega \left[\omega^2 + f(k^2)\right] e^{-s\left(\omega^2 + f(k^2)\right)}}{\int d^D k \, d\omega \, e^{-s(\omega^2 + f(k^2))}}$$

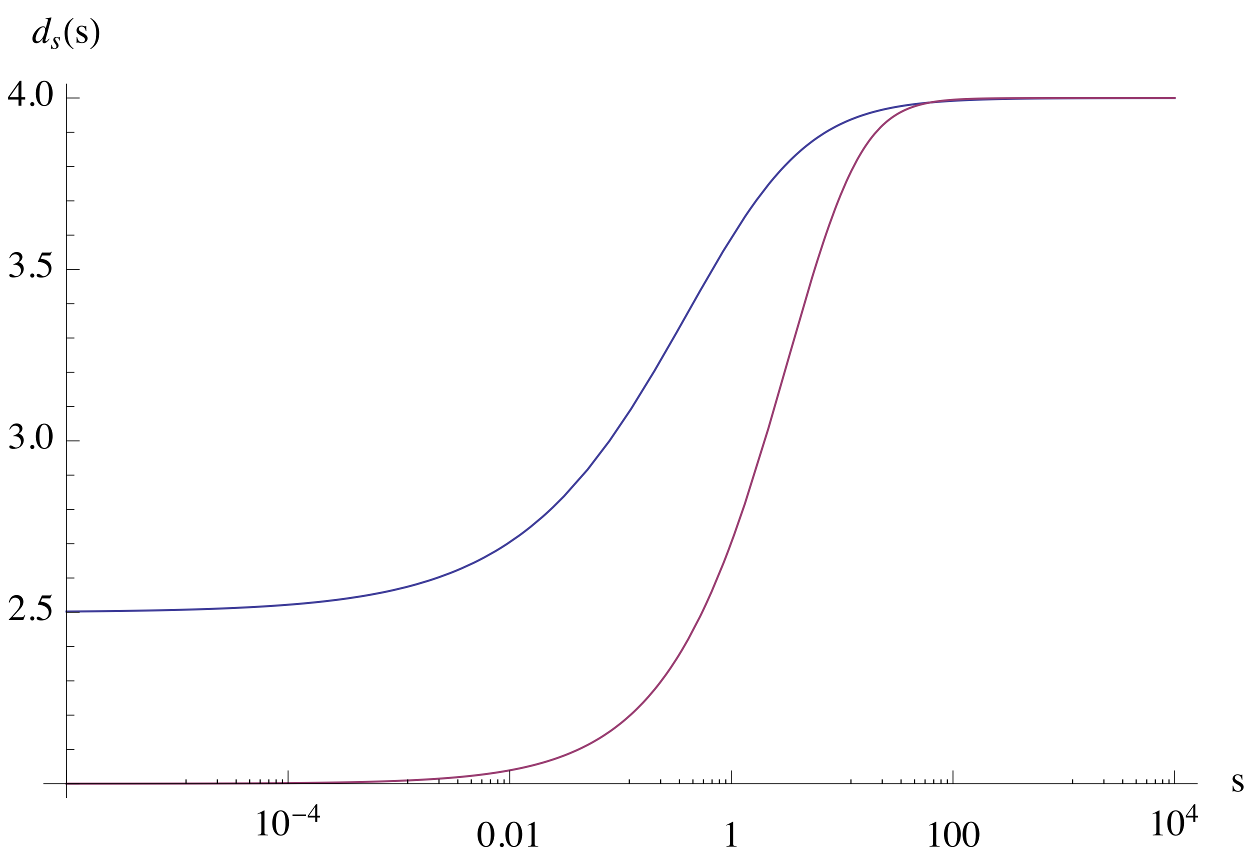
[T. Sotiriou, M. Visser, S. Weinfurtner PRD 2011]

From MDR to RSD - an example

the ansatz (Euclidean MDR): $$\omega^2 + p^2\left(1 + (\lambda p)^{2\gamma}\right) = 0$$

[$1/\lambda$ is the energy scale where QG effects become relevant]

gives the general result $$d_S(0) = 1 + \frac{D}{1+\gamma}$$ for D spatial dimensions



spectral dimension for D=3

blue: $\gamma = 1$

purple: $\gamma = 2$

running to $d_S(0) = 2$ for $\gamma = 2$ (corresponds to HL gravity with z=3)

Momentum space dimensional reduction

in the definition of the spectral dimension we can change momentum space variables:

$$d_S(s) = 2s \frac{\int d^D k \, d\omega \, \left[\omega^2 + f(k^2)\right] e^{-s\left(\omega^2 + f(k^2)\right)}}{\int d^D k \, d\omega \, e^{-s(\omega^2 + f(k^2))}}$$

$$\rightarrow \qquad d_S(s) = 2s \frac{\int \boxed{d\mu(\tilde{k}, \tilde{\omega})} [\tilde{\omega}^2 + \tilde{k}^2] e^{-s(\tilde{\omega}^2 + \tilde{k}^2)}}{\int d\mu(\tilde{k}, \tilde{\omega}) e^{-s(\tilde{\omega}^2 + \tilde{k}^2)}}$$

in the new variables only the measure of momentum space is non-trivial

Momentum space dimensional reduction

example:

$$\omega^2 + p^2\left(1 + (\lambda p)^{2\gamma}\right) = 0 \longrightarrow \tilde{p} = p\sqrt{1 + (\lambda p)^{2\gamma}}$$

all non-trivial effects are transferred to momentum space measure

$$p^2 dp \to \tilde{p}^{\frac{2-\gamma}{1+\gamma}} d\tilde{p}$$ (in the UV)

$\longrightarrow$ energy-momentum space (Haussdorf) dimension is effectively modified in the UV:

$$d_{E,\tilde{p}} = 1 + \frac{3}{1+\gamma}$$

for γ=2 momentum space is 2-dimensional

(this matches the spectral dimension of spacetime in the UV but not in the intermediate regime)

Momentum space running dimension

the momentum space/spectral dimension UV correspondence holds also for general MDRs of the form:

$$\Omega(\omega,p)\equiv\omega^2+p^2+\ell_t^{2\gamma_t}\omega^{2(1+\gamma_t)}+\ell_x^{2\gamma_x}p^{2(1+\gamma_x)}=0$$

the return probability reads: $P(s)=\int\frac{d^Dk\,d\omega}{(2\pi)^{D+1}}e^{-s\Omega(\omega,p)}$

after changing variables (in the UV)

$$\tilde\omega\propto E^{1+\gamma_t}\qquad \tilde p\propto p^{1+\gamma_x}\quad\longrightarrow\quad P(s)\propto\int d\tilde\omega d\tilde p\;\tilde p^{\frac{D-\gamma_x-1}{\gamma_x+1}}\,\tilde\omega^{-\frac{\gamma_t}{\gamma_x+1}}\,e^{-s(\tilde\omega^2+\tilde p^2)}$$

standard Laplacian but deformed measure, from which we can read dimension

$$d_H=\frac{1}{1+\gamma_t}+\frac{D}{1+\gamma_x}$$

Dimensional reduction without a preferred frame

interplay between dispersion relation and measure can be used to build a relativistic theory

example: curved momentum space with de Sitter metric (Euclidean)

momentum space metric: $$ds^2 = dE^2 + e^{2\ell E}\sum_{j=1}^{3} dp_j^2$$

measure: $$d\mu(E,p) = \sqrt{-g}\,dEd^3p = e^{3\ell E}dEd^3p$$

→ the isometries of de Sitter momentum space define the deformed relativistic symmetries

Laplacian: $\mathcal{C}_\ell(1+\ell^{2\gamma}\mathcal{C}_\ell^\gamma)$

where $\mathcal{C}_\ell = \frac{4}{\ell^2}\sinh^2\left(\frac{\ell E}{2}\right) + e^{\ell E}|\vec{p}|^2$ is *invariant* under the *deformed* relativistic symmetries

$$\longrightarrow \quad P(s) \sim \int dEdp\, p^2 e^{3\ell E} e^{-s\mathcal{C}_\ell\left(1+\ell^{2\gamma}\mathcal{C}_\ell^\gamma\right)}$$

Dimensional reduction without a preferred frame

change of variables to make the dispersion relation trivial in the UV:

$$\tilde{E} = e^{\ell E/2}/\ell = r\cos(\theta),\ \tilde{p} = e^{\ell E/2} p = r\sin(\theta) \longrightarrow P(s) \sim \int dr r^5 e^{-sr^{2(\gamma+1)}}$$

$$\hat{r} = r^{\gamma+1} \qquad\qquad P(s) \sim \int d\hat{r}\, \hat{r}^{\frac{6}{1+\gamma}-1} e^{-s\hat{r}^2}$$

from the integration measure in the variables with standard dispersion relation one reads the UV spectral dimension
(and the UV Hausdorff momentum space dimension)

$$d_S(0) = d_H = \frac{2D}{1+\gamma}$$ (for D spatial dimensions)

$\longrightarrow$ 2 spectral/Hausdorff dimensions in the UV when D=3 and $\gamma = 2$

(spectral) dimensional reduction is a common feature of QG theories
(with favoured value of 2 in the UV)

- Causal Dynamical Triangulation in 3d and 4d [J. Ambjorn, J. Jurkiewicz and R. Loll, PRL 2005] [D. Benedetti and J. Henson PRD 2009]

- asymptotically safe gravity in 4d [D. F. Litim, PRL (2004)]

- Horava-Lifshitz gravity in 3+1 dim with critical exponent z=3 [P. Horava, PRL 2009]

- Spin Foams [L. Modesto, CQG 2009] [G. Calcvagni, D. Oriti, J. Thurigen, CQG 2014]

Running to 2 dimensions has important implications for the Early Universe and the UV regime of gravity

$\longrightarrow$ See J. Magueijo's talk

But can we find a more physical notion of dimensionality?

(e.g. one that does not require Euclideanization, does not rely on some fictitious diffusion process and that refers to something we can directly observe)

Quantization with nontrivial measure

consider a general theory, where dispersion relation is trivial and momentum space measure is modified (now Lorentzian signature!)

$$E^2 = p^2 + m^2 \qquad\qquad d\mu(E, \mathbf{p})$$

in order to perform scalar field quantization one needs the covariant on-shell measure

$$d\bar{\mu}(\mathbf{p}) = \frac{d\mu(E_p, \mathbf{p})}{2E_p}$$

($d\bar{\mu}(\mathbf{p}) = \dfrac{d^3p}{(2\pi)^3 2E_p}$ in the standard case)

then the field expansion reads $\quad \phi(x) = \int d\bar{\mu}(\mathbf{p}) \left[a(\mathbf{p}) e^{-ipx} + a^\dagger(\mathbf{p}) e^{ipx} \right]$

(usual thing but with modified measure)

and the one-particle states satisfy: $\quad < \mathbf{p}' | \mathbf{p} >= \delta_{\bar{\mu}}(\mathbf{p} - \mathbf{p}')$

with $\int d\bar{\mu}\, \delta_{\bar{\mu}} = 1$

($< \mathbf{p}' | \mathbf{p} >= (2\pi)^3 2E_p \delta^{(3)}(\mathbf{p} - \mathbf{p}')$ in the standard case)

Some more on the DSR case

k-Poincaré in bicrossproduct coordinates

$$d\mu(k) = e^{3p_0/\kappa} d^3\vec{p}\, dp_0. \qquad \mathcal{C}_\kappa \equiv 4\kappa^2 \sinh^2(p_0/2\kappa) + e^{p_0/\kappa}|\vec{p}|^2.$$

relativistic covariance requires to deform composition rule of momenta, which enters the two-point function

$$< k|k' >= 2\omega_k^{(\kappa)} \delta^{(3)}(\vec{k} \oplus (\ominus \vec{k}'))$$

$$\begin{aligned} \vec{p} \oplus \vec{q} &= \vec{p} + e^{-\omega_p^{(\kappa)}/\kappa} \vec{q} \\ \ominus \vec{p} &= -e^{\omega_p^{(\kappa)}/\kappa} \vec{p} \end{aligned}$$

$$\phi(x) = \int_B d\mu(p)\ \delta(\mathcal{C}_1(p))\, \theta(p_0)\, \tilde{\phi}(p)\, e_p(x) = \int_{M_m^{\kappa+}} \frac{d\mu(\mathbf{p})}{2\omega_\kappa(\mathbf{p})}\ \phi(\mathbf{p})\, e_{\mathbf{p}}(x)$$

$$e_{\mathbf{p}}(\mathbf{k}) \equiv 2\omega_\kappa(\mathbf{k})\ \delta^3(\mathbf{p} \oplus (\ominus \mathbf{k}))\,,$$ [M. Arzano, PRD 2011]

however, implementing on-shellness of momenta

$$< k|k' >= 2\omega_k^{(\kappa)} e^{-\omega_{k_1}^{\kappa}/\kappa} \delta^{(3)}(\vec{k} - \vec{k}')$$

Vacuum fluctuations with non trivial measure

Dimensionless cosmological perturbations power spectrum is defined as

$$< 0|\zeta^2|0 >= \int \frac{dp}{p} P_\zeta(p) \qquad \left[\zeta \sim L_p^{\frac{D-1}{2}} \phi\right]$$

related to the 'covariant' power spectrum $$< 0|\phi^2|0 >= \int d\mu(\mathbf{p}) \bar{P}_\phi(\mathbf{p})$$

via $$P_\zeta(p) \sim L_p^{D-1} p \bar{\mu}(p) \bar{P}_\phi(p)$$

$$[d\bar{\mu}(\mathbf{p}) = 4\pi \bar{\mu}(p) dp]$$

the 'covariant' power spectrum has a remarkable property:

$$\bar{P}_\phi(\mathbf{p}) = 1$$ for any measure

Power spectrum in dimensionally reduced momentum space

A d_H- dimensional (isotropic) covariant measure can generically be written as (only using dimensional arguments)

$$d\bar{\mu}(p) \sim \frac{d\mu}{E_p} \sim \frac{\lambda^{d_H-1-D} p^{d_H-2} dp}{E_p}$$

($d\bar{\mu}(p) = \frac{p^2 dp}{(2\pi)^3 2E_p}$ in the standard case for D=3)

then the power spectrum for cosmological perturbations is:

$$P_\zeta(p) \sim L_p^{D-1} p\,\bar{\mu}(p) \sim L_p^{D-1}\, p\, \lambda^{d_H-1-D} p^{d_H-3}$$

Power spectrum in 2-dimensional momentum space

The dimensionless power spectrum is thus scale invariant for any measure that runs to two dimensions in the UV

$$P_\zeta(p) \sim \left(\frac{L_p}{\lambda}\right)^{D-1} (\lambda p)^{d_H - 2} \to \left(\frac{L_p}{\lambda}\right)^2 \qquad [D = 3, \ \ d_H = 2]$$

and its amplitude is given by the hierarchy between Planck scale and dimensional reduction scale

More in general we can relate the spectral index to momentum space dimension:

$$P_\zeta(p) = A^2 \left(\frac{p}{p_0}\right)^{n_S - 1} \longrightarrow \boxed{n_S = d_H - 1}$$

physical notion of dimensionality?

What about spacetime?

the quantisation procedure is consistent with equal time commutation relations:

$$[\phi(\mathbf{x},0), \dot{\phi}(\mathbf{y},0)] = i\int d\overline{\mu}(\mathbf{p})\, 2E_{\mathbf{p}} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}.$$

from which one can define a spacetime delta

$$\delta_{\overline{\mu}_x}(\mathbf{x}) = \int d\overline{\mu}(\mathbf{p})\, 2E_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}},$$

then the spacetime measure can be inferred from:

$$\int d\overline{\mu}_x(\mathbf{x})\, \delta_{\overline{\mu}_x}(\mathbf{x}) = 1.$$

comparing with

$$\delta_{\overline{\mu}_x}(\mathbf{x}) \propto \int d(\cos\theta)\, dp\, p^{d_H-2} e^{i\mathbf{p}\cdot\mathbf{x}},$$

$$\longrightarrow \qquad \delta_{\overline{\mu}_x}(\mathbf{x}) \propto \delta^{d_H-1}(\mathbf{x}).$$