Twisted submanifolds of $\mathbb{R}^n$

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"Quantum symmetries, NCG and quantum gravity "

Symposium dedicated to J. Lukierski on his 85th birthday

24 - 25 June 2021, Wroclaw (Poland) + online

Based on:

[FW21] G. Fiore, T. Weber, LMP **111** (2021), 76.

[FFW20] G.Fiore, D.Franco, T.Weber, MPAG**23** (2020), 38

Introduction and setting

Submanifolds $N \subset M$ = paramount notion in geometry, physics:

- equipotential (hyper)surfaces N for mechanics on M;
- wavefronts N for wave equations on M;
- initial/boundary conditions on N for PDEs on M;
- Huygens', holographic principles use suitable $N \subset M$;

A geometry on M uniquely induces one on N. If M is Lorentzian

- null hypersurfaces N (lightcones, event horizons,..) important;
- ...

Noncommutative Geometry (NCG) [Connes,Woronowicz,Madore, Wess,Zumino,Lukierski,Majid,Aschieri,...] is advocated as a suitable framework for a theory of quantum spacetime allowing quantization of gravity, for unifying fundamental interactions, etc.

When is a notion of submanifold possible & fruitful in NCG?

So far many examples... Some features may depart from their commutative counterparts, e.g "thick" NC submanifolds:

1. Q.G. algebra $\mathrm{Fun}\big(U_q(n)\big)$ generated by NC $\{u^i_j\}^n_{i,j=1}$, $u^\dagger = u^{-1}$. Q. G. algebra $\mathcal{A} \equiv \mathrm{Fun}\big(SU_q(n)\big) = \mathrm{Fun}\big(U_q(n)\big)/(\det_q u - 1)$ ($\det_q u$ central and invariant); the so-called Q. G. bicovariant differential calculus on $\mathcal{A}$ (i.e. the $\mathcal{A}$-bimodule Ω of 1-forms) remains of dimension n^2 instead of n^2-1 [Woronowicz].
2. $SO_q(n)$-cov. algebra $\mathrm{Fun}\big(\mathbb{R}^n_q\big)$ generated by NC $\{x^i\}^n_{i=1}$, $SO_q(n)$-cov. algebra $\mathcal{A} \equiv \mathrm{Fun}\big(S^{n-1}_q\big) = \mathrm{Fun}\big(\mathbb{R}^n_q\big)/(r^2-1)$ (quantum euclidean space,sphere), central, invariant $r^2 \equiv x^i x_i$. $SO_q(n)$-cov. diff. calc. on $\mathcal{A}$ has dim$=n$ instead of $n-1$. We cannot set $dr^2 = 0$: $d = \left[\frac{1}{q^2-1} r^{-2} dr^2, \cdot\right]$ [Steinacker,GF].

...but little attention to a general answer (except [Masson95,...]).

Here general answer in the framework of Drinfel'd twisting of the Diff(M) Hopf algebra and its (co)module algebras $\mathcal{X}^M, \Xi_M, \Omega_M, \ldots$ [AschieriEtAl06,09]. It is also an answer in the framework of *deformation quantization* [Bayen,Flato,Fronsdal,...].

We stick to NC generalizations of embedded submanifolds of $\mathbb{R}^n$ (by Whitney theorems all smooth M can be embedded in $\mathbb{R}^n$ with n large enough). Assume $M \subset \mathbb{R}^n$ consists of points $x \in \mathbb{R}^n$ fulfilling

$$f^a(x) = 0, \qquad a = 1, 2, ..., k < n, \tag{1}$$

smooth $f \equiv (f^1, ..., f^k) : \mathbb{R}^n \to \mathbb{R}^k$ s.t. $J \equiv \partial f / \partial x$ has rank k on all $\mathbb{R}^n$; more generally, f well-defined and J of rank k on an open subset $\mathcal{D}_f \subset \mathbb{R}^n$, and M consists of the points of $\mathcal{D}_f$ fulfilling (1).

In our examples $\mathcal{E}_f := \mathbb{R}^n \setminus \mathcal{D}_f$ will be empty or of zero measure.

The k-parameter family of embedded manifolds M_c ($M_0 = M$) that are level sets of f are obtained replacing in (1) $f^a(x) \mapsto f^a_c(x) := f^a(x) - c^a$, with $c \equiv (c^1, ..., c^k) \in f(\mathcal{D}_f)$.

Embedded submfds $N \subset M$ obtained adding more equations to (1).

Thus we twist-deform $\mathbb{R}^n$ and M, N defined *globally* within $\mathbb{R}^n$; one could twist-deform $N \subset M$ also *patchwise*.

Smooth embedded submanifolds $M \subset \mathbb{R}^n$ as level sets

Introduce $*$-algebra $\mathcal{X} \equiv C^\infty(\mathcal{D}_f)$, $*$-ideal $\mathcal{C} \subset \mathcal{X}$ of fcts=0 on M.

Theorem 1 in [FW21] $\mathcal{C} = \bigoplus_{a=1}^k \mathcal{X} f^a$, i.e. $\forall h \in \mathcal{C} \;\; \exists h^a \in \mathcal{X}$:

$$h(x) = \sum_{a=1}^k h^a(x) f^a(x). \tag{2}$$

Can express $*$-algebra of smooth complex-valued functions on M as

$$\mathcal{X}^M \equiv \mathcal{X}/\mathcal{C} \equiv \left\{ [\alpha] := \alpha + \mathcal{C} \mid \alpha \in \mathcal{X} \right\}; \tag{3}$$

Similarly, $\mathcal{X}^N = \mathcal{X}^M/\mathcal{C}' = \mathcal{X}/(\mathcal{C} \oplus \mathcal{C}')$. Abbreviate $\partial_i \equiv \partial/\partial x^i$. Let

$$\begin{aligned}
&\Xi \equiv \{X = X^i \partial_i \mid X^i \in \mathcal{X}\} = \text{ Lie alg. of smooth vector flds on } \mathcal{D}_f \\
&\Xi \supset \Xi_{\mathcal{C}} \equiv \left\{ X \in \Xi \mid X(f^a) \in \mathcal{C} \right\} = \text{ Lie subalgebra \& } \mathcal{X}\text{-bimodule} \\
&\Xi_{\mathcal{C}} \supset \Xi_{\mathcal{C}\mathcal{C}} \equiv \left\{ X \in \Xi_{\mathcal{C}} \mid X(h) \in \mathcal{C} \; \forall h \in \mathcal{X} \right\} = \text{ Lie ideal \& } \mathcal{X}\text{-bimodule} \\
&\Xi_M \equiv \Xi_{\mathcal{C}}/\Xi_{\mathcal{C}\mathcal{C}} \equiv \left\{ [X] := X + \Xi_{\mathcal{C}\mathcal{C}} \mid X \in \Xi_{\mathcal{C}} \right\} = \text{ Lie algebra of}
\end{aligned} \tag{4}$$

smooth vector flds tangent to M (derivations of $\mathcal{X}^M$)

Ref. [Masson95] adopts (3-4) also to define NC submanifolds!

Algebraic submanifolds $M \subset \mathbb{R}^n$ as level sets

If instead f^a = suitable irreducible polynomials and $\mathcal{X} \equiv \mathrm{Pol}^\bullet(\mathbb{R}^n)$ = $*$-algebra of complex-valued polynomial functions on $\mathbb{R}^n$, then (3-4) hold with a different meaning:

$\mathcal{X}^M$ = $*$-algebra of complex-valued polynomial functions on M;

$\mathcal{C} \subset \mathcal{X}$ = ideal of polynomial fcts=0 on M; decomposed as in (2);

Ξ = Lie algebra of polynomial vector fields on $\mathbb{R}^n$; etc. And use

Theorem 1 in [FFW20] Assume:
i) $\mathrm{rank}(J) = k$ on an open subset $\mathcal{D}_f \subset \mathbb{R}^n$, so that (1) defines an algebraic submanifold $M \subset \mathcal{D}_f$ of dimension $n-k$.
ii) M is irreducible in $\mathbb{C}^n$; e.g. there exists a k-dimensional affine subspace $\pi \subset \mathbb{R}^n$ meeting M in $s := \prod_{a=1}^k \deg f^a$ points.

Then $\mathcal{C}$ is the complexification of the ideal generated by the f^a in $\mathbb{R}[x^1, ..., x^n]$, i.e. for all $h \in \mathcal{C}$ there exist $h^a \in \mathcal{X}$ such that

$$h(x) = \sum_{a=1}^k h^a(x) f^a(x). \tag{5}$$

Deformation quantization of submanifolds

$\mathcal{X} \rightsquigarrow \mathcal{X}_\star = \mathcal{X}[[\nu]]$ as $\mathbb{C}[[\nu]]$-modules ($\nu =$ def. param.), pointwise product $\alpha \cdot \beta$ of $\alpha, \beta \in \mathcal{X}$ is deformed into the NC associative one

$$\alpha \star \beta = \alpha \cdot \beta + \sum\nolimits_{l=1}^{\infty} \nu^l B_l(\alpha, \beta), \tag{6}$$

where B_l are suitable bidifferential operators of degree l at most.

We wish to deform $\mathcal{X}^M \rightsquigarrow \mathcal{X}^M_\star$ in the form of a quotient

$$\mathcal{X}^M_\star := \mathcal{X}_\star / \mathcal{C}_\star \equiv \big\{ \, [\alpha] := \alpha + \mathcal{C}_\star \mid \alpha \in \mathcal{X}_\star \big\}, \tag{7}$$

$\mathcal{C}_\star =$ two-sided ideal of $\mathcal{X}_\star$, and $\mathcal{X}^M_\star = \mathcal{X}^M[[\nu]]$ as $\mathbb{C}[[\nu]]$-modules.
It suffices that $\mathcal{C}_\star = \mathcal{C}[[\nu]]$; this is fulfilled if

$$\alpha \star f^a = \alpha \cdot f^a = f^a \star \alpha \tag{8}$$

for all $\alpha \in \mathcal{X}$, $a = 1, .., k$ (hence f^a are central in $\mathcal{X}_\star$).
Bonus: $(\mathcal{X}/\mathcal{C})_\star = \mathcal{X}_\star / \mathcal{C}_\star$: taking quotient, deforming $\cdot$ commute.

Deformation quantization via Drinfel'd twisting

$$\alpha \star \beta := \alpha \cdot \beta + \sum_{l=1}^{\infty} \nu^l \sum_{l_l} \overline{\mathcal{F}}_1^{l_l}(\alpha)\ \overline{\mathcal{F}}_2^{l_l}(\beta)\,, \tag{9}$$

$\overline{\mathcal{F}} = 1 \otimes 1 + \sum_{l=1}^{\infty} \nu^l \sum_{l_l} \overline{\mathcal{F}}_1^{l_l} \otimes \overline{\mathcal{F}}_2^{l_l} =$ inverse of a 2-cocycle twist

$$\mathcal{F} = 1 \otimes 1 + \sum_{l=1}^{\infty} \nu^l \sum_{l_l} \mathcal{F}_1^{l_l} \otimes \mathcal{F}_2^{l_l} \in U\mathfrak{g} \otimes U\mathfrak{g}[[\nu]]; \tag{10}$$

here $\otimes = \otimes_{\mathbb{C}[[\nu]]}$. $\mathcal{F}, \overline{\mathcal{F}}$ act on tensor products of any $U\mathfrak{g}$-modules, $U\mathfrak{g}$-module algebras. We abbreviate $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, $\overline{\mathcal{F}} = \overline{\mathcal{F}}_1 \otimes \overline{\mathcal{F}}_2$.

Ref. [AschieriEtAl06,09] use $\mathfrak{g} = \Xi_M$ [∞-dim Lie algebra of Diff(M)] to twist-deform the $U\Xi_M$-module algebra $\mathcal{X}^M = C^\infty(M)$.

We deform $\mathcal{X}, \Xi, \ldots$ as in [AschieriEtAl] via a $\mathcal{F} \in U\Xi_t \otimes U\Xi_t[[\nu]]$,

$$\Xi_{\mathcal{C}} \supset \Xi_t \equiv \{X \in \Xi \mid X(f) = 0\} = \{X \text{ tangent to } \textit{all } M_c\}, \tag{11}$$

and adopt (7), ($4_\star$) to define the NC submanifold algebra $\mathcal{X}_\star^M$; as a bonus we thus define *all* $\mathcal{X}_\star^{M_c} = \mathcal{X}^{M_c}[[\nu]] = \mathcal{X}_\star / \mathcal{C}_\star^c$, $c \in f(\mathcal{D}_f)$.

Plan

1. Introduction and setting:
   - *1.1* Smooth embedded submanifolds $M \subset \mathbb{R}^n$ as level sets
   - *1.2* Algebraic submanifolds $M \subset \mathbb{R}^n$ as level sets
   - *1.3* Deformation quantization of submanifolds...
   - *1.4* ... via Drinfel'd twisting
2. Further essential preliminaries
3. Twisted differential geometry of embedded manifolds $M \subset \mathbb{R}^n$
   - *3.1* Twisted metric, (Levi-Civita) connection, intrinsic and extrinsic curvatures on M
   - *3.2* Decompositions in bases of $\Omega_\star = \Omega_{t\star} \oplus \Omega_{\perp\star}$, $\Xi_\star = \Xi_{t\star} \oplus \Xi_{\perp\star}$; Euclidean, Minkowski metrics
4. Examples: twisted quadrics of $\mathbb{R}^n$
   - *4.1* Twisted elliptic cylinders

Further essential ingredients

Using a $\mathcal{F} \in U\Xi_M \otimes U\Xi_M[[\nu]]$ one twist-deforms [AschieriEtAl06] the Hopf $*$-algebra $U\Xi_M$ of (infinitesimal) diffeomorphisms of M into the triangular noncocommutative Hopf $*$-algebra $H^{\mathcal{F}} = (U\Xi_M[[\nu]], \cdot, \eta, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}}, \mathcal{R})$, or an isomorphic one $H_\star = (U\Xi_M[[\nu]], \star, \eta, \Delta_\star, \epsilon, S_\star, \mathcal{R}_\star)$. Apply the $\star$-product to deform all $U\Xi_M$-modules, $U\Xi_M$-module algebras, the $\star$-tensor product

$$(a \otimes_\star b) \equiv \overline{\mathcal{F}}_1 \triangleright a \otimes \overline{\mathcal{F}}_2 \triangleright b \tag{12}$$

[AschieriEtAl06] to all their tensor products, in particular to tensor fields $\mathcal{T}^{p,r} := \underbrace{\Omega \otimes \ldots \otimes \Omega}_{p \text{ times}} \otimes \underbrace{\Xi \otimes \ldots \otimes \Xi}_{r \text{ times}}$ on M, e.g.

$$T^{p,q} = T_\star{}^{\lambda_1 \ldots \lambda_q}_{\mu_1 \ldots \mu_p} \star dx^{\mu_1} \otimes_\star \ldots \otimes_\star dx^{\mu_p} \otimes_\star \partial_{\lambda_1} \otimes_\star \ldots \otimes_\star \partial_{\lambda_q}. \tag{13}$$

d remains undeformed. $\star$-Lie derivative $\mathcal{L}^\star_X(T) = (\overline{\mathcal{F}}_1 \triangleright X) \triangleright \left[\overline{\mathcal{F}}_2 \triangleright T\right]$.
$\star$-pairing $\langle\ ,\ \rangle_\star := \langle\ ,\ \rangle \circ \overline{\mathcal{F}}(\triangleright \otimes \triangleright) : \Xi_\star \otimes_{\mathbb{C}} \Omega_\star \mapsto \mathcal{X}_\star$, namely

$$(X, \omega) \quad \mapsto \quad \langle X, \omega \rangle_\star := \left\langle \overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright \omega \right\rangle, \tag{14}$$

extended to higher twisted tensor powers applying (14) to

$$\langle X_p \otimes \ldots \otimes X_1, \omega_1 \otimes \ldots \otimes \omega_p \otimes \tau \rangle := \langle X_1, \omega_1 \rangle \ldots \langle X_p, \omega_p \rangle \, \tau \tag{15}$$

The *twisted connection* $\nabla^{\mathcal{F}}$ is a collection of maps $\nabla^{\mathcal{F}} : \Xi_\star \otimes_{\mathbb{C}[[\nu]]} \mathcal{T}_\star^{p,q} \to \mathcal{T}_\star^{p,q}$, $X \otimes T \mapsto \nabla^{\mathcal{F}}_X T$ $(p, q \in \mathbb{N}_0)$ fulfilling

$$\nabla^{\mathcal{F}}_X h = \mathcal{L}^\star_X(h), \tag{16}$$

$$\nabla^{\mathcal{F}}_{h\star X + h'\star Y} T = h \star \nabla^{\mathcal{F}}_X T + h' \star \nabla^{\mathcal{F}}_Y T, \tag{17}$$

$$\nabla^{\mathcal{F}}_X (T + T') = \nabla^{\mathcal{F}}_X T + \nabla^{\mathcal{F}}_X T' \ , \tag{18}$$

$$\begin{aligned} \nabla^{\mathcal{F}}_X (T \otimes_\star T') =& (\overline{\mathcal{R}}_1 \triangleright \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2' \triangleright X} (\overline{\mathcal{R}}_2'' \triangleright T)) \otimes_\star ((\overline{\mathcal{R}}_2 \overline{\mathcal{R}}_1' \overline{\mathcal{R}}_1'') \triangleright T') \\ &+ (\overline{\mathcal{R}}_1 \triangleright T) \otimes_\star \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X} T', \end{aligned} \tag{19}$$

$$\begin{aligned} \nabla^{\mathcal{F}}_X \langle Y, \omega \rangle_\star =& \langle \overline{\mathcal{R}}_1 \triangleright (\nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2' \triangleright X} (\overline{\mathcal{R}}_2'' \triangleright Y)), (\overline{\mathcal{R}}_2 \overline{\mathcal{R}}_1' \overline{\mathcal{R}}_1'') \triangleright \omega \rangle_\star \\ &+ \langle \overline{\mathcal{R}}_1 \triangleright Y, \nabla^{\mathcal{F}}_{\overline{\mathcal{R}}_2 \triangleright X} \omega \rangle_\star \end{aligned} \tag{20}$$

for all $X, Y \in \Xi_\star \equiv \mathcal{T}_\star^{0,1}$, $h, h' \in \mathcal{X}_\star \equiv \mathcal{T}_\star^{0,0}$, $T, T' \in \mathcal{T}_\star$.

The torsion and curvature associated to $\nabla^{\mathcal{F}}$ are the left $\mathcal{X}_\star$-linear maps $\mathrm{T}^{\mathcal{F}}_\star : \Xi_\star \otimes_\star \Xi_\star \to \Xi_\star$, $\mathrm{R}^{\mathcal{F}}_\star : \Xi_\star \otimes_\star \Xi_\star \otimes_\star \Xi_\star \to \Xi_\star$ defined by

$$\mathrm{T}^{\mathcal{F}}_\star(X,Y) := \nabla^{\mathcal{F}}_X Y - \nabla^{\mathcal{F}}_{\mathcal{R}_2 \triangleright Y}(\mathcal{R}_1 \triangleright X) - [X,Y]_\star \ , \tag{21}$$

$$\mathrm{R}^{\mathcal{F}}_\star(X,Y,Z) := \nabla^{\mathcal{F}}_X \nabla^{\mathcal{F}}_Y Z - \nabla^{\mathcal{F}}_{\mathcal{R}_2 \triangleright Y} \nabla^{\mathcal{F}}_{\mathcal{R}_1 \triangleright X} Z - \nabla^{\mathcal{F}}_{[X,Y]_\star} Z \ , \tag{22}$$

for all $X, Y, Z \in \Xi_\star$. They fulfill $\mathrm{T}^{\mathcal{F}}_\star(X, h \star Y) = \mathrm{T}^{\mathcal{F}}_\star(X \star h, Y)$ (and similarly for the curvature), and the antisymmetry property

$$\mathrm{T}^{\mathcal{F}}_\star(X,Y) = -\mathrm{T}^{\mathcal{F}}_\star(\mathcal{R}_2 \triangleright Y, \mathcal{R}_1 \triangleright X) \ , \tag{23}$$

$$\mathrm{R}^{\mathcal{F}}_\star(X,Y,Z) = -\mathrm{R}^{\mathcal{F}}_\star(\mathcal{R}_2 \triangleright Y, \mathcal{R}_1 \triangleright X, Z) \ . \tag{24}$$

One can regard torsion, curvature as elements of the following $\star$-tensor spaces

$$\mathrm{T}^{\mathcal{F}} \in \Omega_\star \wedge_\star \Omega_\star \otimes_\star \Xi_\star, \qquad \mathrm{R}^{\mathcal{F}} \in \Omega_\star \otimes_\star \Omega_\star \wedge_\star \Omega_\star \otimes_\star \Xi_\star, \tag{25}$$

acting on vector fields through the twisted pairing applied to higher tensor powers. We omit the $\star$ in the subscript of the lhs(25) to distinguish them from the corresponding maps. In other words, for all $X, Y, Z \in \Xi_\star = \Xi[[\nu]]$

$$\mathrm{T}^{\mathcal{F}}_\star(X,Y) = \langle X \otimes_\star Y, \mathrm{T}^{\mathcal{F}} \rangle_\star \quad \text{and} \quad \mathrm{R}^{\mathcal{F}}_\star(X,Y,Z) = \langle X \otimes_\star Y \otimes_\star Z, \mathrm{R}^{\mathcal{F}} \rangle_\star . \tag{26}$$

A *metric* is a non-degenerate symmetric element $\boldsymbol{g} \in \Omega_\star \otimes_\star \Omega_\star = \Omega \otimes \Omega[[\nu]]$. The two decompositions $\boldsymbol{g} = \boldsymbol{g}^\alpha \otimes \boldsymbol{g}_\alpha = \boldsymbol{g}^A \otimes_\star \boldsymbol{g}_A$ are related by $\boldsymbol{g}^\alpha \otimes \boldsymbol{g}_\alpha = \overline{\mathcal{F}}_1 \triangleright \boldsymbol{g}^A \otimes \overline{\mathcal{F}}_2 \triangleright \boldsymbol{g}_A$. $\boldsymbol{g}$ determines the map $\boldsymbol{g}_\star : \Xi_\star \otimes_\star \Xi_\star \to \mathcal{X}_\star$

$$\boldsymbol{g}_\star(X, Y) := \left\langle X, \left\langle Y, \boldsymbol{g}^A \right\rangle_\star \star \boldsymbol{g}_A \right\rangle_\star ; \tag{27}$$

this fulfills $\boldsymbol{g}_\star(h \star X, Y) = h \star \boldsymbol{g}_\star(X, Y)$ (left $\mathcal{X}_\star$-linearity in X) and $\boldsymbol{g}_\star(X \star h, Y) = \boldsymbol{g}_\star(X, h \star Y)$. The twisted *Levi-Civita* (LC) connection $\nabla^{\mathcal{F}}$ is a connection fulfilling $\mathrm{T}^{\mathcal{F}} = 0$ and $\nabla^{\mathcal{F}}_X \boldsymbol{g} = 0$ for all $X \in \Xi_\star$

In the commutative case the Ricci tensor is a contraction of the curvature tensor, $\mathrm{Ric}_{jk} = \mathrm{R}_{ijk}{}^i$. Similarly, the twisted Ricci map is defined as

$$\mathrm{Ric}^{\mathcal{F}}_\star(X, Y) := \langle \theta^i, \mathrm{R}^{\mathcal{F}}_\star(e_i, X, Y) \rangle'_\star \ , \tag{28}$$

$$\text{where} \quad \langle \omega \, , \, X \rangle'_\star := \left\langle \overline{\mathcal{F}}_1 \triangleright \omega \, , \, \overline{\mathcal{F}}_2 \triangleright X \right\rangle = \langle \mathcal{R}_2 \triangleright X \, , \, \mathcal{R}_1 \triangleright \omega \rangle_\star \ , \tag{29}$$

and $\{e_i\}$, $\{\theta^i\}$ are dual bases of $\Xi_\star$, $\Omega_\star$: $\langle e_i, \theta^j \rangle_\star = \delta^j_i$. The map is $\mathcal{X}_\star$-linear:

$$\mathrm{Ric}^{\mathcal{F}}_\star(h \star X, Y) = h \star \mathrm{Ric}^{\mathcal{F}}_\star(X, Y) \ , \qquad \mathrm{Ric}^{\mathcal{F}}_\star(X, h \star Y) = \mathrm{Ric}^{\mathcal{F}}_\star(X \star h, Y) \ . \tag{30}$$

The Ricci scalar is this map evaluated on $\boldsymbol{g}^{-1} = \boldsymbol{g}^{-1\alpha} \otimes \boldsymbol{g}^{-1}{}_\alpha = \boldsymbol{g}^{-1A} \otimes_\star \boldsymbol{g}^{-1}{}_A$:

$$\mathfrak{R}^{\mathcal{F}} = \mathrm{Ric}^{\mathcal{F}}_\star \left(\boldsymbol{g}^{-1A}, \boldsymbol{g}^{-1}{}_A \right) . \tag{31}$$

Twisted differential geometry of $M \subset \mathbb{R}^n$

Proposition 9 in [FW21] If $\mathcal{F} \in U\Xi_t \otimes U\Xi_t[[\nu]]$, then $\Xi_{t\star}, \Xi_{\mathcal{C}\star}$ equipped with the $\star$-Lie bracket $[X, Y]_\star \equiv \left[\overline{\mathcal{F}}^{(1)} \triangleright X, \overline{\mathcal{F}}^{(2)} \triangleright Y\right]$ are $\star$-Lie subalgebras of $\Xi_\star$, and $\Xi_{\mathcal{C}\mathcal{C}\star}$ is an ideal of $\Xi_{\mathcal{C}\star}$. Another $\star$-Lie algebra is thus

$$\Xi_{M\star} := \Xi_{\mathcal{C}\star}/\Xi_{\mathcal{C}\mathcal{C}\star} \equiv \left\{ [X] := X + \Xi_{\mathcal{C}\mathcal{C}\star} \mid X \in \Xi_{\mathcal{C}\star} \right\}. \tag{32}$$

The $\mathcal{X}_\star$-subbimodule $\Omega_{\perp\star} \equiv \{\omega \in \Omega_\star \mid \langle \Xi_{t\star}, \omega \rangle_\star = 0\} \subset \Omega_\star$ splits

$$\Omega_{\perp\star} = \bigoplus_{a=1}^{k} \mathcal{X}_\star \star \mathrm{d}f^a = \bigoplus_{a=1}^{k} \mathrm{d}f^a \star \mathcal{X}_\star. \tag{33}$$

Moreover, $\Xi_{t\star}, \Xi_{\mathcal{C}\star}, \Xi_{\mathcal{C}\mathcal{C}\star}, \Xi_{M\star}, \Omega_{\perp\star}$ resp. coincide with $\Xi_t[[\nu]], \Xi_{\mathcal{C}}[[\nu]], \Xi_{\mathcal{C}\mathcal{C}}[[\nu]], \Xi_M[[\nu]], \Omega_\perp[[\nu]]$ as $\mathbb{C}[[\nu]]$-modules. $\Xi_{t\star}$, $\Xi_{M\star}$, $\Omega_{\perp\star}$ and the associated exterior algebras $\Xi^\bullet_{t\star} = \bigwedge^\bullet_\star \Xi_{t\star}$, $\Xi^\bullet_{M\star} = \bigwedge^\bullet_\star \Xi_M$, $\Omega^\bullet_{\perp\star} = \bigwedge^\bullet_\star \Omega_{\perp\star}$ are $U\Xi^{\mathcal{F}}_t$-equivariant $\mathcal{X}_\star$-bimodules.

Twisted metric, Levi-Civita connection, intrinsic and extrinsic curvatures

Endowing $\mathcal{D}_f$ with a (non-degenerate) metric $\boldsymbol{g}$ makes all the $M_c \subset \mathcal{D}'_f$ into (pseudo)Riemannian submanifolds; $\mathcal{D}'_f \subseteq \mathcal{D}_f$ is where the restriction

$$\boldsymbol{g}_\perp^{-1} := \boldsymbol{g}^{-1}|_{\Omega_\perp \otimes \Omega_\perp} : \Omega_\perp \otimes \Omega_\perp \to \mathcal{X}$$

is non-degenerate. If $\boldsymbol{g}$ is Riemannian, then $\mathcal{D}'_f = \mathcal{D}_f$.
We introduce the $\mathcal{X}_\star$-subbimodules

$$\Xi_{\perp\star} \equiv \{X \in \Xi_\star \mid \boldsymbol{g}_\star(X, \Xi_{t\star}) = 0\}, \quad \Omega_{t\star} \equiv \{\omega \in \Omega_\star \mid \boldsymbol{g}_\star^{-1}(\omega, \Omega_{\perp\star}) = 0\}.$$

Killing Lie subalgebra:
$\mathfrak{k} \equiv \{Z \in \Xi_t \mid \boldsymbol{g}\big(\nabla_X Z, Y\big) + \boldsymbol{g}\big(X, \nabla_Y Z\big) = 0\}$

Proposition 10 in [FW21] If $\mathcal{F} \in (U\mathfrak{k} \otimes U\mathfrak{k})[[\nu]]$ $\Xi_\star$, $\Omega_\star$ split into

$$\Xi_\star = \Xi_{t\star} \oplus \Xi_{\perp\star}, \qquad \Omega_\star = \Omega_{t\star} \oplus \Omega_{\perp\star}, \tag{34}$$

direct sums of $\mathcal{X}_\star$-subbimodules orthogonal resp. w.r.t. $\boldsymbol{g}_\star$, $\boldsymbol{g}_\star^{-1}$.

$$\begin{aligned} \boldsymbol{g}_{\perp\star}^{-1} &\equiv \boldsymbol{g}_\star^{-1}|_{\Omega_{\perp\star}\otimes_\star\Omega_{\perp\star}}, & \boldsymbol{g}_{t\star}^{-1} &\equiv \boldsymbol{g}_\star^{-1}|_{\Omega_{t\star}\otimes_\star\Omega_{t\star}}, \\ \boldsymbol{g}_{\perp\star} &\equiv \boldsymbol{g}_\star|_{\Xi_{\perp\star}\otimes_\star\Xi_{\perp\star}}, & \boldsymbol{g}_{t\star} &\equiv \boldsymbol{g}_\star|_{\Xi_{t\star}\otimes_\star\Xi_{t\star}}, \end{aligned} \tag{35}$$

are non-degenerate restrictions of $\boldsymbol{g}_\star^{-1}, \boldsymbol{g}$. $\langle \Xi_{\perp\star}, \Omega_{t\star}\rangle_\star = \{0\}$. $\Xi_{t\star}, \Omega_{\perp\star}, \Xi_{\perp\star}, \Omega_{t\star}$ coincide with $\Xi_t[[\nu]], \Omega_\perp[[\nu]], \Xi_\perp[[\nu]], \Omega_t[[\nu]]$ as $\mathbb{C}[[\nu]]$-modules. Similarly for $\star$-tensor powers of the former.
The orthogonal projections $\mathrm{pr}_{\perp\star}: \Xi_\star \to \Xi_{\perp\star}$, $\mathrm{pr}_{t\star}: \Xi_\star \to \Xi_{t\star}$, $\mathrm{pr}_{\perp\star}: \Omega_\star \to \Omega_{\perp\star}$, $\mathrm{pr}_{t\star}: \Omega_\star \to \Omega_{t\star}$ are uniquely extended to

$$\begin{aligned} \mathrm{pr}_{\perp\star} &: \Omega_\star^p \to \Omega_{\perp\star}^p, & \mathrm{pr}_{t\star} &: \Omega_\star^p \to \Omega_{t\star}^p, \\ \mathrm{pr}_{\perp\star} &: \textstyle\bigwedge_\star^p \Xi_\star \to \bigwedge_\star^p \Xi_{\perp\star}, & \mathrm{pr}_{t\star} &: \textstyle\bigwedge_\star^p \Xi_\star \to \bigwedge_\star^p \Xi_{t\star} \end{aligned} \tag{36}$$

via $\mathrm{pr}_{\perp\star}(\omega \star \alpha) = \mathrm{pr}_{\perp\star}(\omega) \star \mathrm{pr}_{\perp\star}(\alpha)$, $\mathrm{pr}_{t\star}(\omega \star \alpha) = \mathrm{pr}_{t\star}(\omega) \star \mathrm{pr}_{t\star}(\alpha)$, ... $\mathbb{C}[[\nu]]$-linearly extending $\mathrm{pr}_\perp, \mathrm{pr}_t$. $\Xi_{t\star}, \Xi_{\perp\star}, \Omega_{t\star}, \Omega_{\perp\star}$, their $\star$-exterior powers and $\mathrm{pr}_{\perp\star}, \mathrm{pr}_{t\star}$ are $U\mathfrak{k}^{\mathcal{F}}$-equivariant.

Proposition 11 in [FW21] *$X \in \Xi_{\mathcal{C}\star}$, $\omega \in \Omega_{\mathcal{C}\star}$ $\Rightarrow$ $X_{t\star} \equiv \mathrm{pr}_{t\star}(X)$, $\omega_{t\star} \equiv \mathrm{pr}_{t\star}(\omega)$ respectively belong to $[X] \in \Xi_{M\star}$, $[\omega] \in \Omega_{M\star}$.*

Hence we can represent all $X \in \Xi_{M\star}$ by $X_{t\star} \in \Xi_{t\star}$ (resp. $\omega \in \Omega_{M\star}$ by $\omega_{t\star} \in \Omega_{t\star}$). Similarly for multivector fields, higher rank forms.

Twisted 1st and *2nd fundamental form* on the submanifolds M_c:

$$\begin{aligned} \boldsymbol{g}_{t\star} &:= \boldsymbol{g}_\star|_{\Xi_{t\star}\otimes_\star\Xi_{t\star}} : \Xi_{t\star}\otimes_\star\Xi_{t\star} \to \mathcal{X}_\star, \\ II^{\mathcal{F}}_\star &:= \mathrm{pr}_{\perp\star} \circ \nabla^{\mathcal{F}}|_{\Xi_{t\star}\otimes_\star\Xi_{t\star}} : \Xi_{t\star} \otimes_\star \Xi_{t\star} \to \Xi_{\perp\star}, \end{aligned} \tag{37}$$

Twisted projected Levi-Civita connection on the submanifolds M_c:

$$\nabla^{\mathcal{F}}_t := \mathrm{pr}_{t\star} \circ \nabla^{\mathcal{F}}|_{\Xi_{t\star}\otimes\Xi_{t\star}} : \Xi_{t\star}\otimes\Xi_{t\star} \to \Xi_{t\star}. \tag{38}$$

Proposition 12 in [FW21] *$\boldsymbol{g}_{t\star}$ is a metric on all M_c's, with associated twisted Levi-Civita connection $\nabla^{\mathcal{F}}_t$. $\nabla^{\mathcal{F}}_t, \boldsymbol{g}_{t\star}, II^{\mathcal{F}}_\star$ are $U\mathfrak{k}^{\mathcal{F}}$-equivariant, related to the undeformed counterparts by*

$$\boldsymbol{g}_{t\star}(X, Y) = \boldsymbol{g}_t(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y), \tag{39}$$

$$II^{\mathcal{F}}_\star(X, Y) = II(\overline{\mathcal{F}}_1 \triangleright X, \overline{\mathcal{F}}_2 \triangleright Y), \tag{40}$$

$$\nabla^{\mathcal{F}}_{t,X} Y = \nabla_{t,\overline{F}_1\triangleright X}(\overline{\mathcal{F}}_2 \triangleright Y) \tag{41}$$

Furthermore $\nabla^{\mathcal{F}}_X = \nabla^{\mathcal{F}}_t + II^{\mathcal{F}}_\star : \Xi_{t\star}\otimes\Xi_{t\star} \to \Xi_\star$.

$II^{\mathcal{F}}_{\star}$ yields the **twisted extrinsic curvature** of $M_\star$.
The **twisted intrinsic curvature** $\mathrm{R}^{\mathcal{F}}_{t\star}$ is related to the twisted curvature $\mathrm{R}^{\mathcal{F}}_{\star}$ of $\nabla^{\mathcal{F}}$ on $\mathbb{R}^n_\star$ by

Proposition 12 in [FW21] *For all $X, Y, Z, W \in \Xi_{t\star}$ the following* **twisted Gauss equation** *holds:*

$$\begin{aligned} \boldsymbol{g}_\star(\mathrm{R}^{\mathcal{F}}_{\star}(X,Y)Z,W) &= \boldsymbol{g}_\star(\mathrm{R}^{\mathcal{F}}_{t\star}(X,Y)Z,W) \\ &+ \boldsymbol{g}_\star(II^{\mathcal{F}}_{\star}(X,\overline{\mathcal{R}}_1 \triangleright Z), II^{\mathcal{F}}_{\star}(\overline{\mathcal{R}}_2 \triangleright Y, W)) \qquad (42) \\ -\boldsymbol{g}_\star(II^{\mathcal{F}}_{\star}(\overline{\mathcal{R}}_{1(\widehat{1})} \triangleright Y, \overline{\mathcal{R}}_{1(\widehat{2})} \triangleright Z), &II^{\mathcal{F}}_{\star}(\overline{\mathcal{R}}_2 \triangleright X, W)). \end{aligned}$$

The twisted 1st, 2nd fundamental forms, Levi-Civita connection, curvature tensor, Ricci tensor, Ricci scalar *on* $M_\star$ are finally obtained from the above objects by applying the further projection $\mathcal{X}_\star \to \mathcal{X}^M_\star$, i.e. selecting $M_\star$ out of the family $M_{c\star}$ by setting $c=0$.

Decompositions in vector fields, 1-forms bases (or complete sets) of $\Xi_\star, \Omega_\star$

The sets of globally defined vector fields

$$\Theta_t \equiv \{\vartheta^j\}_{j=1}^n, \quad S_W \equiv \{W_j\}_{j=1}^n, \qquad \vartheta^j \equiv \mathrm{pr}_t(\xi^j), \;\; W_j \equiv \mathrm{pr}_t(\partial_j), \tag{43}$$

are respectively complete in $\Omega_\star$, $\Xi_\star$, but are not bases, because

$$\vartheta^j \star f_j^a = 0, \qquad f^{aj} \star W_j = 0, \qquad a = 1, ..., k. \tag{44}$$

Such a redundancy is unavoidable if M is not parallelizable.
Alternative complete set S_L in $\Xi_\star$ *not depending on* $\boldsymbol{g}$: for $k = 1$,

$$S_L \equiv \{L_{ij}\}_{i,j=1}^n, \qquad L_{ij} \equiv f_i \partial_j - f_j \partial_i = -L_{ji}. \tag{45}$$

As S_W, also S_L is of rank $n-1$ over $\mathcal{X}$, because $f_{[i} L_{jk]} = 0$. The L_{ij} are antihermitean under the $*$-structure $L_{ij}^* = -L_{ij}$, and

$$[L_{ij}, L_{hk}] \;\; = \;\; f_{jh} L_{ik} - f_{ih} L_{jk} - f_{jk} L_{ih} + f_{ik} L_{jh}. \tag{46}$$

where $f_{ij} \equiv \partial_i \partial_j f$. If $f \equiv x^i x^i - 1$, $M =$ sphere, $L_{ij} = x^i \partial_j - x^j \partial_i$ span $so(n)$. If $k \geq 1$ S_L consists of the glob.def. antihermitean v.f.

$$L_{i_1 i_2 \ldots i_{k+1}} := f^1_{[i_1} f^2_{i_2} \ldots f^k_{i_k} \, \partial_{i_{k+1}]}. \tag{47}$$

Cartesian coordinates $x \equiv (x^1, ..., x^n)$ of $\mathbb{R}^n$; lower, raise indices: $dx_i = g_{ij} dx^j$, $Y_i = g_{ij} Y^j$, $\partial^i = g^{ij} \partial_j$, where $g_{ij} := \boldsymbol{g}(\partial_i, \partial_j)$, $g^{ij} = g^{-1}{}_{ij} = \boldsymbol{g}^{-1}(dx^i, dx^j)$. The metric and its dual take the form

$$\boldsymbol{g} = dx^i \otimes dx_i, \qquad \boldsymbol{g}^{-1} = \partial^i \otimes \partial_i, \tag{48}$$

whence $X = X^i \partial_i$, $Y = Y^i \partial_i$, $\alpha = \alpha_i dx^i$, $\omega = \omega_i dx^i$,

$$\boldsymbol{g}(X, Y) = X^i Y_i, \qquad \boldsymbol{g}^{-1}(\alpha, \omega) = \alpha_i \omega^i. \tag{49}$$

Let $\mathsf{E}^{ab} \equiv \boldsymbol{g}_\perp^{-1}(df^a, df^b)$. On $\mathcal{D}'_f \subseteq \mathcal{D}_f$ the matrix E is symmetric and $K \equiv \mathsf{E}^{-1}$ exists. ($\mathsf{E} > 0$ on $\mathcal{D}'_f = \mathcal{D}_f$ if $\boldsymbol{g}$ is *Riemannian*.) Let

$$N^a_\perp := K^{ab}\, \boldsymbol{g}^{-1}(df^b, dx^i)\, \partial_i = K^{ab} f^{bi} \partial_i \tag{50}$$

Propositions 8, 14 in [FW21] $\mathcal{N}_\perp := \{N^a_\perp\}^k_{a=1}$, $\mathcal{B}_\perp := \{df^a\}^k_{a=1}$ *are globally defined bases resp. of* $\Xi_{\perp\star}, \Omega_{\perp\star}$ *dual to each other:*

$$\langle N^a_\perp, df^b \rangle_{sr} = N^a_\perp(f^b) = \delta^{ab}, \qquad a, b \in \{1, ..., k\}. \tag{51}$$

$\boldsymbol{g}(N^a_\perp, N^b_\perp) = K^{ab}$. *All* $df^a, N^a_\perp, \mathsf{E}^{ab}, K^{ab}$ *are* $U\mathfrak{k}^{\mathcal{F}}$*-invariant. In these bases the action of* $\mathrm{pr}_{\perp\star}, \mathrm{pr}_{t\star}$ *on* $\Xi_\star$, $\Omega_\star$ *explicitly read [...]*

If $\boldsymbol{g}$ is Euclidean ($g_{ij} = \delta_{ij}$) or Minkowski $[g_{ij} = g^{ij} = \mathrm{diag}(1,..,1,-1)]$ the associated Levi-Civita connection on $\mathbb{R}^n$ is

$$\nabla = dx^i \otimes \mathcal{L}_{\partial_i}, \qquad \text{e.g.} \qquad \nabla_X Y = X^i \partial_i (Y^j) \partial_j. \tag{52}$$

We endow $M \subset \mathbb{R}^n$ with the induced metric $\boldsymbol{g}_t$. Using $X, Y, Z \in \Xi_t$ as representatives of elements of Ξ_M, the Levi-Civita connection on $(M, \boldsymbol{g}_t)$ is $\nabla_{t,X} Y := (\nabla_X Y)_t$.
Abbreviating $f^a_{ij} := \partial_i\big(\partial_j(f^a)\big)$, the second fundamental form $II(X,Y) := (\nabla_X Y)_\perp$ takes the explicit form

$$II(X,Y) \;=\; X^i \partial_i (Y^j) f^a_j N^a_\perp = -X^i Y^j f^a_{ij} N^a_\perp. \tag{53}$$

Replacing this result and $\mathrm{R} = 0$ in the Gauss equation, we find for the intrinsic curvature, on all $X, Y, Z, W \in \Xi_t$,

$$[\mathrm{R}_t(X,Y)Z)]^m \, W^m = f^a_{ij} K^{ab} f^b_{lm} (Y^i X^l - X^i Y^l) Z^j W^m. \tag{54}$$

Finally, $Z \in \Xi_t$ is a Killing vector field on $(M, \boldsymbol{g}_t)$ if

$$Z\Big(\boldsymbol{g}(X,Y)\Big) - \boldsymbol{g}\big([Z,X],Y\big) - \boldsymbol{g}\big(X,[Z,Y]\big) = X^h Y^i (\partial_h Z_i + \partial_i Z_h) = 0$$

In fact, this guarantees that $Z \in \mathfrak{k}$ is Killing on $(M_c, \boldsymbol{g}_t)$ for all c.
$\mathfrak{k} \subset \Xi_t$ is a Lie subalgebra of the Euclidean Lie algebra (spanned by rotations and translations of $\mathbb{R}^n$) if $\boldsymbol{g}$ is the Euclidean metric;
$\mathfrak{k} \subset \Xi_t$ is a Lie subalgebra of the Poincaré Lie algebra (spanned by rotations, boosts and translations of $\mathbb{R}^n$) if $\boldsymbol{g}$ is the Minkowski metric.

These decompositions hold also after twist-deformation.

Examples: twisted quadrics $M \subset \mathbb{R}^n$; $n = 3$.

$$f(x) \equiv \frac{1}{2} a_{ij} x^i x^j + a_{0i} x^i + \frac{1}{2} a_{00} = 0 \tag{56}$$

$a_{\mu\lambda} = a_{\lambda\mu} \in \mathbb{R}$ $(\mu, \lambda = 0, 1, ..., n)$, $f_i = a_{ij} x^j + a_{i0}$, $f_{ij} = a_{ij}$,

$$[L_{ij}, L_{hk}] = a_{jh} L_{ik} - a_{ih} L_{jk} - a_{jk} L_{ih} + a_{ik} L_{jh}; \tag{57}$$

L_{ij} span $\mathfrak{g} \subset \mathrm{aff}(n)$: $L_{ih} \triangleright$ act as lin. transfs of the coordinates x^k:

$$L_{ij} \triangleright x^h = (a_{ik} x^k + a_{0i}) \delta^h_j - (a_{jx} x^k + a_{j0}) \delta^h_i. \tag{58}$$

Making the x^i canonical for M by an Euclidean transf. we get $a_{ij} = a_i \delta_{ij}$, $b_i \equiv a_{0i} = 0$ if $a_i \neq 0$. In [FFW20] we classify $\mathfrak{g}$ and derive general results for $\mathcal{X} = \mathrm{Pol}^\bullet(\mathbb{R}^n)$ from $\mathcal{F} \in U\mathfrak{g} \otimes U\mathfrak{g}[[\nu]]$: The *global* description of differential geometry on $\mathbb{R}^n, M_c$ via generators and relations extends to twist deformations, so as to preserve the subspaces of the differential calculus algebras consisting of polynomials of any fixed degrees in x^i, dx^i, ∂_i (or other indep. vector fields). Twisted $M \subset \mathbb{R}^3$ analyzed in detail.

Quadrics $M \subset \mathbb{R}^3$

a_1	a_2	a_3	a_{03}	a_{00}	r	quadric	$\mathfrak{g} \simeq$	Abelian	Jordanian
$+$	0	0	$-$		3	parab. cylinder	$\mathfrak{h}(1)$	Yes	No
$+$	$+$	0	$-$		4	ellip. paraboloid	$\mathfrak{so}(2) \ltimes \mathbb{R}^2$	Yes	No
$+$	$+$	0	0	$-$	3	elliptic cylinder	$\mathfrak{so}(2) \ltimes \mathbb{R}^2$ $\mathfrak{so}(2) \times \mathbb{R}$	Yes Yes	No No
$+$	$-$	0	$-$		4	hyp. paraboloid	$\mathfrak{so}(1,1) \ltimes \mathbb{R}^2$	Yes	Yes
$+$	$-$	0	0	$-$	3	hyp. cylinder	$\mathfrak{so}(1,1) \ltimes \mathbb{R}^2$ $\mathfrak{so}(1,1) \times \mathbb{R}$	Yes Yes	Yes No
$+$	$+$	$-$	0	$-$	4	1-s. hyperboloid	$\mathfrak{so}(2,1)$	No	Yes
$+$	$+$	$-$	0	$+$	4	2-s. hyperboloid	$\mathfrak{so}(2,1)$	No	Yes
$+$	$+$	$-$	0	0	3	elliptic cone	$\mathfrak{so}(2,1) \times \mathbb{R}$	Yes†	Yes
$+$	$+$	$+$	0	$-$	4	ellipsoid	$\mathfrak{so}(3)$	No	No

Figure: Signs of the coefficients of the canonical eqs, rank r, symmetry Lie algebra $\mathfrak{g}$, type of twist; $\mathfrak{h}(1)$ = Heisenberg algebra. For fixed a_i: classes (f), (g), (h) altogether give a single family M_c parametrized by c; each remaining class gives another family (7 families of submanifolds). We can make $a_1 = 1$ rescaling of f. Cone (e) consists of two open components separated by the apex (singular point).

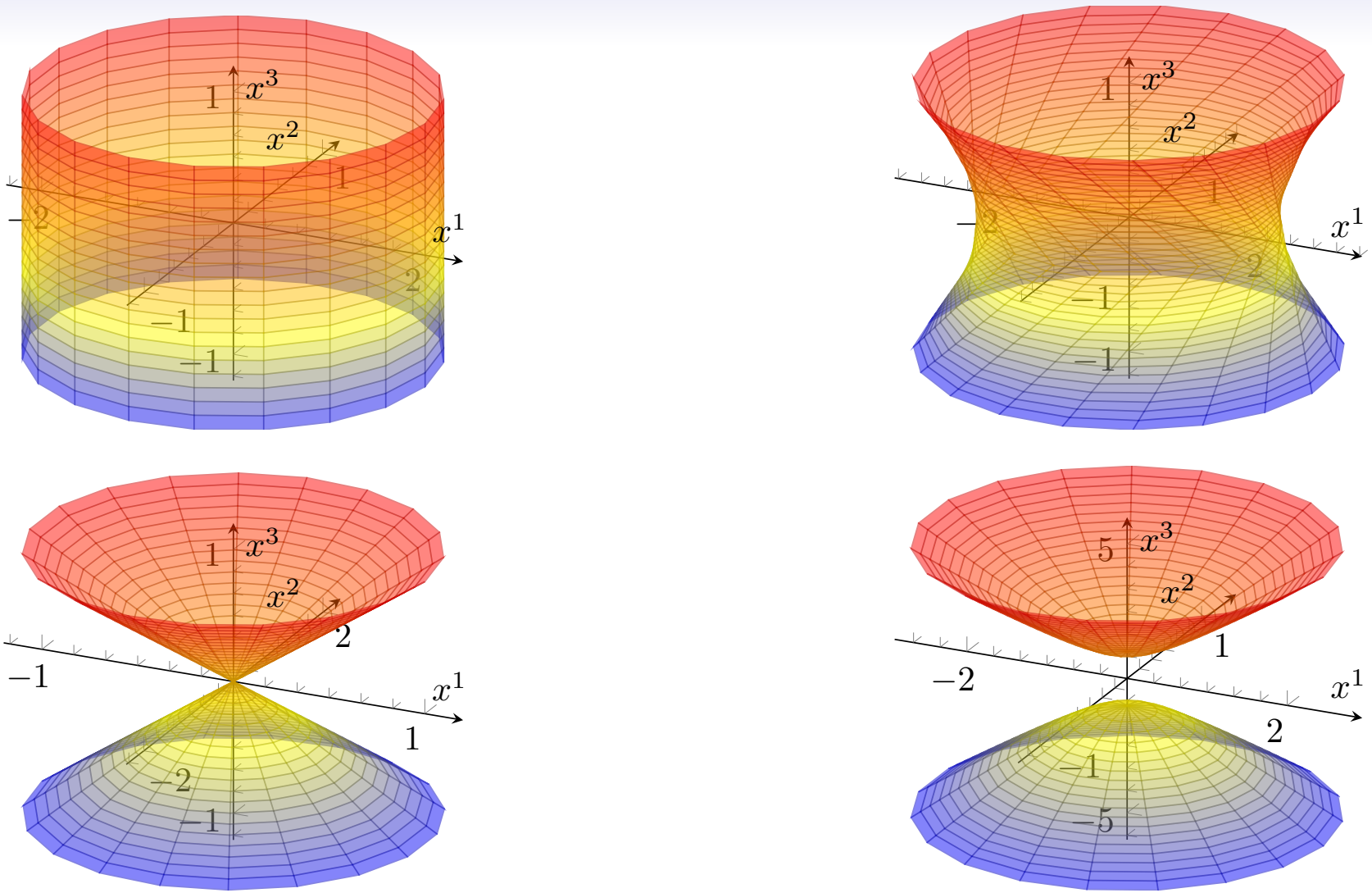

Figure: The twisted irreducible quadric surfaces of $\mathbb{R}^3$ that we illustrate here.

Twisted elliptic cylinders

Propositions 15, 16 in [FW21] *The unitary abelian twists $\mathcal{F} = \exp(i\nu L_{13} \otimes L_{23})$, $\mathcal{F} = \exp(i\nu \partial_3 \otimes L_{12})$ induce twist deformations of $U\mathfrak{g}$, $U\Xi$, of $\mathcal{Q}^\bullet$ on $\mathbb{R}^3$ and of $\mathcal{Q}^\bullet_{M_c}$ on the elliptic cylinders. In both cases the basic relations characterizing the $U\mathfrak{g}^{\mathcal{F}}$-module $*$-algebra $\mathcal{Q}^\bullet_{M_c\star}$ keep the same form, in particular*

$$f_c \equiv \frac{x^1 \star x^1 + a x^2 \star x^2}{2} - c = 0, \quad df_c \equiv \xi^1 \star x^1 + a\, \xi^2 \star x^2 = 0, \quad \epsilon^{ijk} f_i \star L_{jk} = 0.$$

Other relations differ and are reported in detail in [FFW20].
If $a_1 = a_2 = 1$ (circular cylinder of radius $R = \sqrt{2c}$) $S_t \equiv \{L, \partial_3\}$, $S_\perp \equiv \{N_\perp\}$ are resp. orthonormal bases of Ξ_t, $\Xi_\perp$; here $L \equiv L_{12}/R$, $N_\perp = f^i \partial_i / R = (x^1 \partial_1 + x^2 \partial_2)/R$. $\mathfrak{k}$ is abelian and spanned by S_t. $\nabla_{\partial_i} \partial_j = 0$, whereas $\nabla_X Y \neq 0$ with the following $X, Y \in_t \cup S_\perp$:

$$\nabla_L L = -\frac{N_\perp}{R}, \quad \nabla_L N_\perp = \frac{L}{R}, \quad \nabla_{N_\perp} L = \frac{L}{R}, \quad \nabla_{N_\perp} N_\perp = \frac{N_\perp}{R}.$$

Intrinsic, extrinsic curvatures (Gauss and mean) keep undeformed.