Dynamical systems, coadjoint orbits and space-time symmetries

Symmetry group G

↓

Ureps of G
(irreps - elementary systems)
Space of states

↓

Generators (+ elements of envelopping algebra)
- observables (via Noether theorem)

↓

Interactions: tensor products of ureps, intertwining operators ets

Classical limit $\hbar \to 0$ (?)

"Naive" quantization

Symmetry group G

Hamiltonian dynamical systems (phase space, dynamical variables, PB's, Hamiltonian); G acts through canonical transformations

Darboux variables (?); canonical quantization

Construction of generators of G (proper symmetrization of canonical variables; reconstruction of ureps of G

Elementary systems: transitive action of G on phase space

$$\boxed{\text{Transitive action of } G}$$

$\downarrow$

Phase space(s): coadjoint orbits
Symplectic structure: Kirillov-Kostant-Souriau symplectic form

$$\langle Ad^*_g \mu, X\rangle = \langle \mu, Ad_{g^{-1}} X\rangle$$
$$\langle ad^*_X \mu, Y\rangle = -\langle \mu, [X, Y]\rangle$$
$$\omega(ad^*_X \mu, ad^*_Y \mu) = \langle \mu, [X, Y]\rangle$$

Coadjoint orbits

$$O_\mu = Ad^*_G \mu$$
$$O_\mu \sim G/G_\mu, \quad G_\mu \subset G - \text{stability group of } \mu$$

$$\omega \sim d\eta$$

$\omega \to$ KKS form, $d\eta \to$ Cartan form

Generic orbits

Independent Casimir functions fixed
((compact semisimple : $\dim \mathcal{O} = \dim G - \operatorname{rank} G$
etc.)

Degenerate orbits

Casimir functions + additional conditions (or/and specific values of C.f.)

Poincare group (10-dimensional)

Two Casimir functions : mass squared & Pauli-Lubanski fourvector squared

$$(P_\mu, M_{\mu\nu}) \longrightarrow \underbrace{(\xi_\mu, \xi_{\mu\nu})}_{\text{dual space coordinates}}$$

Generic orbits: $m \neq 0$, $s \neq 0$

$$\underbrace{\underline{\xi}_\mu = (m, \vec{0}), \quad \underline{\xi}_{ij} = s\,\varepsilon_{3ij}, \quad \underline{\xi}_{i0} = -\underline{\xi}_{0i} = 0}_{\text{Canonical Point}}$$

Stability group:

$$G_{\underline{\xi}} = SO(2) \times \mathbb{R}$$

$$\dim O_{\underline{\xi}} = 8 = \underbrace{3 \text{ coordinates} + 3 \text{ momenta}}_{\text{global Darboux coordinates}} + 2\text{-dim. spin phase space}$$

Dynamics on Poincare group

$$S \sim \int \eta$$

Symmetry: global left Poincare action + local right $G_{\underline{\xi}}$ action

Dim. of phase space: 20
Second class constraints: 8
First class constraints: 2
Gauge degrees of freedom: 2

Degenerate orbits:

$m \neq 0,\ s = 0$ – massive spinless particle (6-dimensional phase space)

$m=0$ – massless particles
(6-dimensional phase space)

Constraints

$\zeta_\mu \zeta^\mu = 0$ (mass squared Casimir)

$\lambda \zeta^\mu - \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} \zeta_\nu \zeta_{\alpha\beta} = 0$ ("spin enslavement" in terminology of Souriau, Duval, Horvathy)

$\zeta_\mu \Delta^\mu \equiv \lambda \zeta_\mu \zeta^\mu \Rightarrow$ 4 independent conditions

Darboux (?) coordinates: $\vec{x}, \vec{p}$

$\{p_i, p_k\} = 0$
$\{x_i, p_k\} = \delta_{ik}$

Two possibilities:

either x_i transform nonlinearly under rotations and

$\{x_i, x_k\} = 0$

or x_i transform linearly (as threevector) and

$$\{x_i, x_k\} = \lambda\, \varepsilon_{ikl} \frac{p_l}{|\vec{p}|^3}$$

Poincare generators:

$$\zeta_i = p_i$$
$$\zeta_0 = |\vec{p}|$$
$$\zeta_{ij} = x_i p_j - x_j p_i + \lambda\, \varepsilon_{ijk} \frac{p_k}{|\vec{p}|}$$
$$\zeta_{0i} = -|\vec{p}| x_i + t p_i$$

Canonical quantization

$$\hat{p}_i \to p_i$$
$$\hat{x}_i \to i D_i$$ (covariant derivative in monopole field)

Generators - ok. except boosts

$$\hat{\zeta}_{0i} = -\frac{1}{2}\left(|\hat{\vec{p}}|\, \hat{x}_i + \hat{x}_i\, |\hat{\vec{p}}|\right)$$

Δ^μ generate ideal wrt Pb's

$$\{(\ldots), \Delta^\mu\} = (\ldots)^\mu{}_\nu \Delta^\nu$$

In particular – linear representations

The conformal group $SO(4,2)$

$$A, B = 0, 1, 2, 3, 5, 6$$

$$g_{AB} = \mathrm{diag}(+----+)$$

$$\{J_{AB}, J_{CD}\} = g_{AD} J_{BC} + g_{BC} J_{AD} - g_{AC} J_{BD} - g_{BD} J_{AC}$$

3 Casimir functions

Generic orbits – 12-dimensional

Degenerate orbits – 6-, 8- and 10-dim.

6-dimensional orbits

$$\Delta_{AB} \equiv g^{CD} J_{CA} J_{DB} - \lambda^2 g_{AB} = 0$$

$$\Sigma^{AB} \equiv J^{AB} + 8\lambda\, \varepsilon^{ABCDEF} J_{CD} J_{EF} = 0$$

9 functionally independent relations:

6 independent variables $\vec{x}, \vec{p}$

$$\zeta_{65} \to D$$

$$\left.\begin{array}{l} \zeta_{\mu 6} - \zeta_{\mu 5} \to P_\mu \\ \zeta_{\mu 6} + \zeta_{\mu 5} \to K_\mu \end{array}\right\} \quad \mu = 0, 1, 2, 3$$

Poincare generators – as above

$$D = x_k p_k$$

$$K_0 = -|\vec{p}|\,\vec{x}^{\,2} - \frac{\lambda^2}{|\vec{p}|}$$

$$K_i = -\,p_i \vec{x}^{\,2} + 2\,x_i x_k p_k - 2\lambda\,\varepsilon_{ikl}\frac{x_k p_l}{|\vec{p}|} + \frac{\lambda^2 p_i}{|\vec{p}|^2}$$

Quantization

Poincare generators – as above

Remaining generators – ordering plus quantum corrections; for example,

$$\hat{K}_0 = -(iD_k)\,|\vec{p}|\,(iD_k) + \left(\frac{3}{4} - \lambda^2\right)\frac{1}{|\vec{p}|}$$

etc.

8-dimensional orbit

$$C \equiv \zeta_{AB}\zeta^{AB} - 2\tau^2 = 0$$

$$\square^{AB} \equiv \varepsilon^{ABCDEF}\zeta_{CD}\zeta_{EF} = 0$$

7 independent relations

Darboux variables: $\vec{x}, \vec{p}, y, m$

$$\{x_i, p_k\} = \delta_{ik}$$

$$\{y, m\} = 1$$

$$P_i = p_i \quad , \quad P_0 = \sqrt{\vec{p}^2 + m^2}$$

$$M_{i0} = x_i \sqrt{\vec{p}^2 + m^2}$$

$$M_{ij} = x_i p_j - x_j p_i$$

$$D = x_i p_i + y m$$

$$K_0 = P_0 \left(\frac{\tau^2}{m^2} + \vec{x}^2 + y^2 \right)$$

$$K_i = \frac{K_0 p_i}{\sqrt{\vec{p}^2 + m^2}} - 2 D x_i$$